Vector-valued modular forms and quasimodular forms with applications to the theory of vertex operator algebras

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Abstract

In this note we show that every vector-valued holomorphic modular form that has a quasimodular form of weight $k+n$ and depth at most n on $\Gamma = SL_2(\mathbb{Z})$ as an entry defines an $(n+1)$-dimensional symmetric tensor representation of the group Γ. We also prove that conversely every $(n+1)$-dimensional symmetric tensor representation of Γ gives a vector-valued modular form of which an entry is a quasimodular form with depth n. As applications we propose several criteria that a vertex operator algebra is not rational.

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Introduction

In this note we study vector-valued modular forms on $\Gamma = SL_2(\mathbb{Z})$ associated with a representation $\rho : \Gamma \to GL_n(\mathbb{C})$ that has a quasimodular form as an entry. Each entry of a vector-valued modular form is a modular form on the kernel of ρ if the kernel is non-trivial. Then the question is what kind of entries appear if the kernel is trivial. We demonstrate that a quasimodular form on Γ naturally appears as an entry of a vector-valued modular form associated with the $(n+1)$-dimensional symmetric tensor representation $\rho^{(n)}$ of Γ (The kernel of $\rho^{(n)}$ is trivial.).

The concept of vector-valued modular forms of integral weight (in particular, weight 0) naturally appears in 2-dimensional conformal field theories (2DCFT) as well as the theory of vertex operator algebras. Suppose that our 2DCFT is rational with a finiteness condition (C_2-cofiniteness, for instance). Then, it is known (by [7]) that the space of one-point functions

on a family of elliptic curves evaluated at a primary vector with weight k is finite-dimensional, and is invariant with respect to the slash operator $|_k\gamma$ for each $\gamma \in \Gamma$. In particular, every set of formal characters of simple modules for a rational vertex operator algebra gives rise to a vector-valued modular form with weight 0.

One of significant differences between rational theory and non-rational theory is that formal characters of a rational theory do not involve logarithmic terms of $q = e^{2\pi i\tau}$ ($\tau \in \mathbb{H}$), where $\mathbb{H}$ is the complex upper half-plane, while there is a non-rational theory of which formal characters (called pseudo-characters) have logarithmic terms (cf. [1], [2]). At the present moment, a non-rational theory without logarithmic terms is not known.

We now roughly explain our main theorems. Let $\mathcal{F}$ be the space of holomorphic functions on $\mathbb{H}$. Let V be an $(n+1)$-dimensional vector subspace of $\mathcal{F}$, which is invariant under the $|_k$-action of Γ and contains a quasimoduar form of weight $n+k$ and depth at most n. Then we show that there is a vector-valued modular form of weight k such that the associated representation is a symmetric tensor representation $(V, \rho^{(n)})$ of Γ (Theorem 4). Conversely, let V be a Γ-invariant $(n+1)$-dimensional vector space of $\mathcal{F}$. Let $\mathbb{F} = {}^t(f_0, f_1, \ldots, f_n)$ be a vector-valued modular form on Γ with respect to the symmetric tensor representation $(V, \rho^{(n)})$, of which entries are bounded as $\mathrm{Im}(\tau) \to \infty$. Then it is shown (as one of assertions) that f_0 is a quasimodular form (Theorem 5). In consequences we give several criteria if a C_2-cofinite (see § 3) vertex operator algebra is not rational.

In § 1 the concepts of vector-valued modular forms and quasimodular forms are reviewed. Main theorems (Theorem 4 and Theorem 5) are stated and proved in § 2. Properties of several vector-valued modular forms are also discussed here. The final section (§ 3) gives several applications of the theorems to the theory of C_2-cofinite vertex operator algebras. One of main issues here is to give criteria if a vertex operator algebra is not rational.

1 Vector-valued modular forms and quasimodular forms

In this section we recall the concepts of vector-valued modular forms (according to [5]) and quasimodular forms ([4]).

Let $\mathcal{F}$ be the space of holomorphic functions on the complex upper half-plane $\mathbb{H}$, and let $\mathbb{F} = {}^t(f_1, \ldots, f_n)$ be a column vector of n holomorphic functions on $\mathbb{H}$. The action of the group $\Gamma = SL_2(\mathbb{Z})$ on each $\mathbb{F}$ is given by

$$\mathbb{F}|_k\gamma \;=\; {}^t(f_1|_k\gamma, \ldots, f_n|_k\gamma)$$

for every $\gamma \in \Gamma$ and $k \in \mathbb{Z}$. Here the action $|_k$ of Γ on $\mathcal{F}$ is defined by

$$f\,|_k\gamma \;=\; (c\tau + d)^{-k} f(\gamma(\tau))\,, \quad \gamma \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

for each $f \in \mathcal{F}$. Let $\rho : \Gamma \to GL_n(\mathbb{C})$ be an n-dimensional complex representation of the group Γ. Then there is a right action of Γ on $\mathcal{F}^n$

$$\gamma : \mathbb{F} \mapsto \rho(\gamma)^{-1}\mathbb{F}|_k\gamma\,, \quad \gamma \in \Gamma\,. \tag{1}$$

Definition. Suppose that $\mathbb{F} = {}^t(f_1, \ldots, f_n)$ is Γ-invariant with respect to the action (1). Then $\mathbb{F}$ is called a *meromorphic vector-valued modular form* of weight k if each entry f_j has a q-expansion, convergent in a neighborhood of the infinity ($i\infty$);

$$f_j(\tau) \;=\; q^{\lambda_j} \sum_{n=0}^{\infty} a_{n,\,j} q^n \;(\lambda_j \in \mathbb{R})\,, \quad q = e^{2\pi i \tau}\;(\tau \in \mathbb{H})\,.$$

If λ_j is nonnegative we say that f_j is *holomorphic at infinity*, and if each f_j satisfies this condition then $\mathbb{F}$ is called a (*holomorphic*) *vector-valued modular form*.

Let ρ be a representation of Γ on $GL_n(\mathbb{C})$. Let $M_k(\rho)$ and $H_k(\rho)$ be the associated spaces of meromorphic and holomorphic modular forms of weight k respectively, and

$$M(\rho) \;=\; \bigoplus_{k\in\mathbb{Z}} M_k(\rho)\,, \quad H(\rho) \;=\; \bigoplus_{k\in\mathbb{Z}} H_k(\rho)\,.$$

We now recall the concept of quasimodular forms and its several properties. A quasimodular form is a "nearly modular" holomorphic function on $\mathbb{H}$; one of basic examples is the (quasi) Eisenstein series E_2 of weight 2;

$$E_2(\tau) \;=\; 1 - 24 \sum_{n=1}^{\infty} \Big(\sum_{d|n} d \Big) q^n$$

on which Γ acts as

$$(c\tau + d)^{-2} E_2(\gamma(\tau)) \;=\; E_2(\tau) + \frac{\mu c}{c\tau + d} \text{ for every } \gamma \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,,$$

where $\mu = 6/\pi i$. We do not give the precise definition of the quasimodular form here (since it is not necessary in this note), and may refer the reader to [4]. For the modular group Γ, the graded ring of quasimodular forms is identified with $\mathbb{C}[E_2, E_4, E_6]$, where E_4 and E_6 are the (normalized) Eisenstein series with weight 4 and 6 respectively;

$$E_4(\tau) \;=\; 1 + 240 \sum_{n=1}^{\infty} \big(\sum_{d|n} d^3 \big) q^n\,, \quad E_6(\tau) \;=\; 1 - 504 \sum_{n=1}^{\infty} \big(\sum_{d|n} d^5 \big) q^n\,.$$

Let f be a quasimodular form of weight k. We call f has *depth* n if f is written as $f = \sum_{i=0}^{n} f_i \cdot (E_2)^i$ where each f_i is a (classical) holomorphic modular form of weight $k - 2i$ and $f_n \neq 0$. We denote the derivation ∂_{E_2} on $\mathbb{C}[E_2, E_4, E_6]$ simply by ∂, and often use notation $\partial^{(n)} = \partial^n / n!$ for short.

2 Vector-valued modular forms arising from quasimodular forms

2.1 Pleminaries

In this section we first recall the concept of the symmetric tensor representation of $\Gamma = SL_2(\mathbb{Z})$ and establish several relations between vector-valued modular forms and quasimodular forms.

Definition. Let n be a nonnegative integer. For every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we define a square matrix $\rho^{(n)}(\gamma) = \left(\rho_{ij}^{(n)}(\gamma)\right)$ of degree $n+1$ by

$$\begin{pmatrix} (c\tau+d)^n \\ (a\tau+b)(c\tau+d)^{n-1} \\ (a\tau+b)^2(c\tau+d)^{n-2} \\ \vdots \\ (a\tau+b)^n \end{pmatrix} = \rho^{(n)}(\gamma) \begin{pmatrix} 1 \\ \tau \\ \tau^2 \\ \vdots \\ \tau^n \end{pmatrix}.$$

The Γ-module $\left(V_n, {}^t\rho^{(n)}\right)$, where V_n is the space of polynomials of τ of degree at most n, is called the *symmetric tensor representation* of the group Γ.

It readily follows that the kernel of the symmetric tensor representation of Γ is trivial for an odd n and is $\{I, -I\}$ for an even n (I is the identity matrix of degree n).

We now state several lemmas which are necessary in discussions of § 2.2. We often omit τ if it does not cause any confusions.

Lemma 1. *Let F be a quasimodular form of weight $k+n$ and depth at most n. Then for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, the identity*

$$\partial^{(\ell)} F\left(\gamma(\tau)\right) = (c\tau+d)^{k+n-2\ell} \sum_{i=\ell}^{n} \binom{i}{\ell} \partial^{(i)} F \cdot \left(\frac{\mu c}{c\tau+d}\right)^{i-\ell}$$

holds for every nonnegative integer ℓ.

Proof. The quasimodular form F takes the form $F = \sum_{i=0}^{n} f_i \cdot (E_2)^i$ as a polynomial in E_2 with modular form coefficients. It then follows from the Taylor series expansion that

$$\sum_{i=0}^{n} f_i \cdot (E_2 + x)^i = \sum_{i=0}^{n} \partial^{(i)} F \cdot x^i, \tag{2}$$

where x is a formal variable. By applying the operator $(\partial/\partial x)^\ell/\ell!$ to both sides of (2) we have

$$\sum_{i=\ell}^{n} \binom{i}{\ell} f_i \cdot (E_2 + x)^{i-\ell} = \sum_{i=\ell}^{n} \binom{i}{\ell} \partial^{(i)} F \cdot x^{i-\ell}. \tag{3}$$

Since $\partial^{(\ell)} F = \sum_{i=\ell}^{n} f_i \cdot \binom{i}{\ell} (E_2)^{i-\ell}$ it follows that

$$\begin{aligned} \left(\partial^{(\ell)} F\right)\left(\gamma(\tau)\right) &= \sum_{i=\ell}^{n} \binom{i}{\ell} f_i\left(\gamma(\tau)\right) E_2\left(\gamma(\tau)\right)^{i-\ell} \\ &= (c\tau+d)^{k+n-2\ell} \sum_{i=\ell}^{n} \binom{i}{\ell} f_i(\tau) \left(E_2(\tau) + \frac{\mu c}{c\tau+d}\right)^{i-\ell} \\ &= (c\tau+d)^{k+n-2\ell} \sum_{i=\ell}^{n} \binom{i}{\ell} \partial^{(i)} F \cdot \left(\frac{\mu c}{c\tau+d}\right)^{i-\ell}, \end{aligned}$$

where the last equality follows from (3). This completes the proof of the lemma. □

Lemma 2. *For any nonnegative integers i, m and n, we have*

$$\sum_{\ell=0}^{m} \binom{n}{\ell}^{-1} \binom{i}{\ell} \binom{m}{\ell} c^{m-\ell}(a\tau+b)^{i-\ell}(c\tau+d)^{n-i-m}$$
$$= \frac{1}{m!}\binom{n}{m}^{-1} \left(\frac{\partial}{\partial\tau}\right)^{m} \left\{(a\tau+b)^{i}(c\tau+d)^{n-i}\right\} .$$

Proof. Since $\binom{n}{m}\binom{n}{\ell}^{-1}\binom{m}{\ell} = \binom{n-\ell}{m-\ell}$ for each $m \geq \ell \geq 0$ it is enough to show

$$\sum_{\ell=0}^{m} \binom{i}{\ell} \binom{n-\ell}{m-\ell} c^{m-\ell}(a\tau+b)^{i-\ell}(c\tau+d)^{n-i-m} = \frac{1}{m!}\left(\frac{\partial}{\partial\tau}\right)^{m} \left\{(a\tau+b)^{i}(c\tau+d)^{n-i}\right\} . \quad (4)$$

By Leibniz's rule applied to the right-hand side of (4) it takes the form

$$\sum_{j=0}^{m} \binom{i}{j} \binom{n-i}{m-j} a^{j} c^{m-j}(a\tau+b)^{i-j}(c\tau+d)^{n-i-m+j} .$$

Therefore, by multiplying $(a\tau+b)^{m-i}(c\tau+d)^{m+i-n}$ to both sides of (4), we see that proving (4) is equivalent to showing

$$\sum_{\ell=0}^{m} \binom{i}{\ell} \binom{n-\ell}{m-\ell} c^{m-\ell}(a\tau+b)^{m-\ell} = \sum_{j=0}^{m} \binom{i}{j} \binom{n-i}{m-j} a^{j} c^{m-j}(a\tau+b)^{m-j}(c\tau+d)^{j} . \quad (5)$$

Since $a(c\tau+d) = 1 + c(a\tau+b)$, the right-hand side of (5) becomes

$$\sum_{j=0}^{m} \binom{i}{j} \binom{n-i}{m-j} c^{m-j}(a\tau+b)^{m-j}\left(1+c(a\tau+b)\right)^{j}$$
$$= \sum_{j=0}^{m} \binom{i}{j} \binom{n-i}{m-j} c^{m-j}(a\tau+b)^{m-j} \sum_{\ell=0}^{j} \binom{j}{\ell} c^{j-\ell}(a\tau+b)^{j-\ell}$$
$$= \sum_{\ell=0}^{m}\sum_{j=\ell}^{m} \binom{i}{j} \binom{n-i}{m-j} \binom{j}{\ell} c^{m-\ell}(a\tau+b)^{m-\ell}$$
$$= \sum_{\ell=0}^{m}\sum_{j=\ell}^{m} \binom{i-\ell}{j-\ell} \binom{n-i}{m-j} \binom{i}{\ell} c^{m-\ell}(a\tau+b)^{m-\ell} .$$

Now the relation $(1+x)^{n-\ell} = (1+x)^{i-\ell}(1+x)^{n-i}$ implies

$$\sum_{j=\ell}^{m} \binom{i-\ell}{j-\ell} \binom{n-i}{m-j} = \binom{n-\ell}{m-\ell} .$$

This completes the proof of the lemma. □

The following lemma shows the linear independence of a set $\{\tau^{i}\}$ over the ring of 1-periodic functions.

Lemma 3. *Suppose that each element $f_i \in \mathcal{F}$ $(0 \le i \le n)$ satisfies $f_i(\tau+1) = f_i(\tau)$ and $\sum_{i=0}^{n} f_i(\tau)\tau^i = 0$. Then $f_i = 0$ for each i.*

Proof. Suppose that $\sum_{i=0}^{n} f_i(\tau)\tau^i = 0$. Since $f_i(\tau+1) = f_i(\tau)$ it follows that

$$\sum_{i=0}^{n} f_i(\tau)(\tau+\ell)^i \;=\; 0 \tag{6}$$

for every nonnegative integer ℓ. Then (6) with $\ell = 0, 1, \ldots, n$ imply

$$(f_0, f_1, \ldots, f_n)\begin{pmatrix} 1 & 1 & \ldots & 1 \\ \tau & \tau+1 & \ldots & \tau+n \\ \tau^2 & (\tau+1)^2 & \ldots & (\tau+n)^2 \\ \vdots & \vdots & & \vdots \\ \tau^n & (\tau+1)^n & \ldots & (\tau+n)^n \end{pmatrix} \;=\; (0, 0, \ldots, 0)\,. \tag{7}$$

The matrix $A = (a_{ij})$ appeared in the left-hand side of (7) is now the Vandermonde matrix with entries $a_{ij} = (\tau+j-1)^{i-1}$ and its determinant is

$$\prod_{0\le i<j\le n} \Big\{(\tau+j)-(\tau+i)\Big\} \;=\; \prod_{j=1}^{n} j\,! \neq 0\,,$$

which shows $(f_0, f_1, \ldots, f_n) = 0$. This completes the proof of the lemma. □

2.2 Theorems

Let V be a linear subspace of $\mathcal{F}$ that is invariant with respect to the $|_k$-action of the group $\Gamma = SL_2(\mathbb{Z})$ and $F \in V$ a quasimodular form of weight $n+k$ and depth at most n. Then we can obtain a vector-valued modular form of weight $(n+1)$ with respect to a symmetric tensor representation.

Theorem 4. *Let V be an $(n+1)$-dimensional subspace of $\mathcal{F}$. Suppose that V is invariant with respect to the $|_k$-action of the group $\Gamma = SL_2(\mathbb{Z})$ and that there is a nonzero quasimodular form $F \in V$ of weight $k+n$ and depth at most n. Then the function*

$$F_i(\tau) \;=\; \sum_{\ell=0}^{i} \binom{n}{\ell}^{-1}\binom{i}{\ell}\mu^{\ell}\tau^{i-\ell}\partial^{(\ell)}F \quad (\mu \;=\; \frac{6}{\pi i}) \tag{8}$$

is an element of V for each $0 \le i \le n$ and the set $\{F_0 = F, F_1, \ldots, F_n\}$ is a basis of V. Further the $|_k$-action of Γ with respect to this basis is given by

$$\mathbb{F}|_k\,\gamma \;=\; \rho^{(n)}(\gamma)\,\mathbb{F} \text{ for every } \gamma \in \Gamma\,,$$

where $\mathbb{F} = {}^t(F_0, F_1, \ldots, F_n)$.

Proof. By definition, the element F takes the form $F = \sum_{i=0}^{n} f_i \cdot (E_2)^i$ where f_i is a modular form of weight $k+n-2i$ on Γ. The action $|_k\gamma$ of $\gamma \in \Gamma$ on F gives

$$(c\tau+d)^{-k} F\left(\gamma(\tau)\right) \;=\; (c\tau+d)^n \sum_{i=0}^{n} f_i \cdot \Big(E_2 + \frac{\mu c}{c\tau+d}\Big)^i, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{9}$$

since $f_i\left(\gamma(\tau)\right) = (c\tau+d)^{k+n-2i} f_i(\tau)$ and $(c\tau+d)^{-2} E_2\left(\gamma(\tau)\right) = E_2(\tau) + \mu c/(c\tau+d)$. Then it follows by (2) and (9) that

$$\begin{aligned}
F|_k\,\gamma \;&=\; (c\tau+d)^n \sum_{\ell=0}^{n} \Big(\frac{\mu c}{c\tau+d}\Big)^{\ell} \partial^{(\ell)} F \\
&= \sum_{\ell=0}^{n} (\mu c)^{\ell} (c\tau+d)^{n-\ell} \partial^{(\ell)} F \\
&= \sum_{\ell=0}^{n} \mu^{\ell} \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} c^{\ell+j} d^{n-\ell-j} \tau^j \partial^{(\ell)} F \\
&= \sum_{i=0}^{n} \sum_{\ell=0}^{i} \mu^{\ell} \binom{n-\ell}{i-\ell} c^i d^{n-i} \tau^{i-\ell} \partial^{(\ell)} F \\
&= \sum_{i=0}^{n} \binom{n}{i} c^i d^{n-i} \binom{n}{i}^{-1} \sum_{\ell=0}^{i} \binom{n-\ell}{i-\ell} \mu^{\ell} \tau^{i-\ell} \partial^{(\ell)} F \\
&= \sum_{i=0}^{n} \binom{n}{i} c^i d^{n-i} \sum_{\ell=0}^{i} \binom{n}{\ell}^{-1} \binom{i}{\ell} \mu^{\ell} \tau^{i-\ell} \partial^{(\ell)} F \\
&= \sum_{i=0}^{n} \binom{n}{i} c^i d^{n-i} F_i(\tau) \\
&= d^n \sum_{i=0}^{n} \binom{n}{i} \left(\frac{c}{d}\right)^i F_i(\tau)\,.
\end{aligned}$$

Now, since c and d are any coprime integers, a rational number c/d can be arbitrary. It then follows that

$$\sum_{i=0}^{n} \binom{n}{i} r^i F_i(\tau) \in V \tag{10}$$

for each rational number r. We now take mutually different $n+1$ rational numbers. Then (10) (by using the Vandermond determinant) yields that $F_i \in V$ for each $0 \le i \le n$.

We now determine the action $|_k\gamma$ $\left(\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma\right)$ on each F_i. By Lemma 1, it follows

that

$$
\begin{aligned}
&F_i|_k\,\gamma \\
&= (c\tau+d)^{-k}F_i\left(\gamma(\tau)\right) \\
&= (c\tau+d)^{-k}\sum_{\ell=0}^{i}\binom{n}{\ell}^{-1}\binom{i}{\ell}\mu^{\ell}\gamma(\tau)^{i-\ell}(c\tau+d)^{k+n-2\ell}\sum_{m=\ell}^{n}\binom{m}{\ell}\left(\frac{\mu c}{c\tau+d}\right)^{m-\ell}\partial^{(m)}F \\
&= \sum_{m=0}^{n}\mu^m\partial^{(m)}F\sum_{\ell=0}^{i}\binom{n}{\ell}^{-1}\binom{i}{\ell}\binom{m}{\ell}c^{m-\ell}(a\tau+b)^{i-\ell}(c\tau+d)^{n-i-m}\,.
\end{aligned}
$$

Because the factor $\binom{i}{\ell}\binom{m}{\ell}$ vanishes if $\ell > m$ we can take the sum for $0 \le \ell \le m$. Hence Lemma 2 implies

$$
F_i|_k\,\gamma \;=\; \sum_{m=0}^{n}\mu^m\partial^{(m)}F\binom{n}{m}^{-1}\frac{1}{m!}\left(\frac{\partial}{\partial\tau}\right)^m\left\{(a\tau+b)^i(c\tau+d)^{n-i}\right\}\,.
$$

By definition, we know that

$$
(a\tau+b)^i(c\tau+d)^{n-i} \;=\; \sum_{j=0}^{n}\rho_{ij}^{(n)}(\gamma)\tau^j
$$

which implies

$$
\frac{1}{m!}\left(\frac{\partial}{\partial\tau}\right)^m\left\{(a\tau+b)^i(c\tau+d)^{n-i}\right\} \;=\; \sum_{j=m}^{n}\rho_{ij}^{(n)}(\gamma)\binom{j}{m}\tau^{j-m}\,.
$$

It then follows that

$$
\begin{aligned}
F_i|_k\,\gamma \;&=\; \sum_{m=0}^{n}\mu^m\partial^{(m)}F\binom{n}{m}^{-1}\sum_{j=m}^{n}\rho_{ij}^{(n)}(\gamma)\binom{j}{m}\tau^{j-m} \\
&=\; \sum_{j=0}^{n}\rho_{ij}^{(n)}(\gamma)\Big(\sum_{m=0}^{j}\binom{n}{m}^{-1}\binom{j}{m}\mu^m\tau^{j-m}\partial^{(m)}F\Big) \qquad (11)\\
&=\; \sum_{j=0}^{n}\rho_{ij}^{(n)}(\gamma)F_j\,.
\end{aligned}
$$

Equation (11) shows that the action of $|_k\gamma$ ($\gamma \in \Gamma$) on V coincides with the symmetric tensor representation $\rho^{(n)}(\gamma)$.

Finally we show that $\{F_i\}$ is a linearly independent set. Recall from (8) that F_i takes the form

$$
F_i(\tau) \;=\; \sum_{\ell=0}^{i}\binom{n}{\ell}^{-1}\binom{i}{\ell}\mu^{\ell}\tau^{i-\ell}\partial^{(\ell)}F \;=\; \tau^iF + \frac{i}{n}\mu\tau^{i-1}\partial F + \cdots
$$

and that every $\partial^{(\ell)}F$ is 1-periodic. Now suppose that $\sum_{i=0}^{n}c_iF_i = 0$ for some complex numbers c_i. Then by Lemma 3, it follows that $c_i = 0$ for each i since the coefficients of F is τ^i. This completes the proof of the theorem. □

We here give two examples of vector-valued modular forms of which representation is the symmetric tensor representation $\rho^{(n)}$.

Examples. (1) Recall from [3, Theorem 2.1 (1) $n = 1$, $k = 6$] that $F = E_2E_4 - E_6$ is a quasimodular solution of the Kaneko-Zagier equation:

$$f'' - \frac{k}{6}E_2(\tau)f' + \frac{k(k-1)}{12}E_2'(\tau)f \;=\; 0\,, \quad {}' \;=\; q\frac{d}{dq}\,.$$

Let $F_0(\tau) = F(\tau)$ and $F_1(\tau) = \tau F_0 + \mu E_4(\tau)$ where $\mu = 6/\pi i$. Then a short calculation shows that

$$F_0|_5\gamma \;=\; dF_0 + cF_1, \quad F_1|_5\gamma \;=\; bF_0 + aF_1, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

and hence

$$\begin{pmatrix} F_0 \\ F_1 \end{pmatrix}\Bigg|_5 \gamma \;=\; \begin{pmatrix} d & c \\ b & a \end{pmatrix}\begin{pmatrix} F_0 \\ F_1 \end{pmatrix}\,. \tag{12}$$

(2) By $n = 2$ and $k = 12$ in [3, Theorem 2.1 (1)] one has $F = 12E_2E_4E_6 - 5E_4^3 - 7E_6^2$ and hence $F_0 = F$, $F_1 = \tau F_0 + 12\mu E_4E_6$. Since this is the case $n = 1$ and $k = 11$ in Theorem 4 it follows that

$$F_0|_{11}\gamma \;=\; dF_0 + cF_1, \quad F_1|_{11}\gamma \;=\; bF_0 + aF_1, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

which is equivalent to

$$\begin{pmatrix} F_0 \\ F_1 \end{pmatrix}\Bigg|_{11} \gamma \;=\; \begin{pmatrix} d & c \\ b & a \end{pmatrix}\begin{pmatrix} F_0 \\ F_1 \end{pmatrix}\,.$$

Remark. One of important aspects of a symmetric tensor representation is that its kernel is trivial. Then one might expect that any vector-valued modular forms on Γ with respect to a representation $\Gamma \to GL_n(\mathbb{C})$ that has a trivial kernel contains a quasimodular form as an entry. However, it is not true: In fact, we have a counter example arising from symplectic fermionic vertex algebras ([1]).

Let us define holomorphic functions ϕ_1, ϕ_2, ϕ_3 on $\mathbb{H}$ by

$$\phi_1(\tau) \;=\; \frac{\eta(\tau)^2}{\eta(2\tau)\eta(\tau/2)}, \quad \phi_2(\tau) = \frac{\eta(\tau/2)}{\eta(\tau)}, \quad \phi_3(\tau) \;=\; \sqrt{2}\,\frac{\eta(2\tau)}{\eta(\tau)}\,.$$

Then formal characters of simple modules for $d = 1$ symplectic fermionic vertex algebra χ_0, χ_1, χ_2 and χ_3 are

$$\begin{aligned} \chi_0 \;&=\; \frac{1}{2}\left(\frac{\phi_3(\tau)^2}{2} + \eta(\tau)^2\right), \quad \chi_1 \;=\; \frac{1}{2}\left(\frac{\phi_3(\tau)^2}{2} - \eta(\tau)^2\right), \\ \chi_2 \;&=\; \frac{1}{2}\Big(\phi_1(\tau)^2 + \phi_2(\tau)^2\Big), \quad \chi_3 \;=\; \frac{1}{2}\Big(\phi_1(\tau)^2 - \phi_2(\tau)^2\Big). \end{aligned}$$

Then it is not difficult to see that

$$\begin{aligned} \chi_0(\tau+1) \;&=\; e^{\pi i/6}\chi_0(\tau), \quad \chi_1(\tau+1) \;=\; e^{\pi i/6}\chi_1(\tau)\,, \\ \chi_2(\tau+1) \;&=\; e^{\pi i/12}\chi_2(\tau), \quad \chi_3(\tau+1) \;=\; e^{\pi i/12}\chi_3(\tau) \end{aligned}$$

and

$$\begin{aligned}
\chi_0(-1/\tau) &= \frac{1}{4}(\chi_2(\tau)-\chi_3(\tau)) + \frac{(-i\tau)}{2}(\chi_0(\tau)-\chi_1(\tau))\,,\\
\chi_1(-1/\tau) &= \frac{1}{4}(\chi_2(\tau)-\chi_3(\tau)) - \frac{(-i\tau)}{2}(\chi_0(\tau)-\chi_1(\tau))\,,\\
\chi_2(-1/\tau) &= \frac{1}{2}(\chi_2(\tau)+\chi_3(\tau)) + (\chi_0(\tau)-\chi_1(\tau))\,,\\
\chi_3(-1/\tau) &= \frac{1}{2}(\chi_2(\tau)+\chi_3(\tau)) - (\chi_0(\tau)-\chi_1(\tau))\,.
\end{aligned}$$

Further $\chi_0(-1/\tau)-\chi_1(-1/\tau)$ gives $\tau(\chi_0(\tau)-\chi_1(\tau))$. The general theory of vertex operator algebras says that the following 5 holomorphic functions on $\mathbb{H}$ form a vector-valued modular form on Γ (cf. [7]). Let

$$\begin{aligned}
f_1 &:= \chi_0+\chi_1 = \frac{\eta(2\tau)^2}{\eta(\tau)^2}\,, & f_2 &:= \chi_0-\chi_1 = \eta(\tau)^2,\\
f_3 &:= \chi_2+\chi_3 = \frac{\eta(\tau)^4}{\eta(2\tau)^2\eta(\tau/2)^2}\,, & f_4 &:= \chi_2-\chi_3 = \frac{\eta(\tau/2)^2}{\eta(\tau)^2}\\
f_5 &:= \tau(\chi_0-\chi_1) = \tau\eta(\tau)^2\,.
\end{aligned}$$

Then $\mathbb{F} = {}^t(f_1,\, f_2,\, f_3,\, f_4,\, f_5)$ is a vector-valued modular form with weight 0 on Γ with respect to a representation $\rho:\Gamma\to GL_5(\mathbb{C})$. Since

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)^2 = \varepsilon(\gamma)(c\tau+d)\eta(\tau)^2\,,\quad \gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma\,,$$

where $\varepsilon(\gamma)^{24}=1$. It follows that

$$f_2\left(\gamma(\tau)\right) = d\,\varepsilon(\gamma)f_2(\tau) - c\,\varepsilon(\gamma)f_5(\tau)\,,\quad f_5(\gamma(\tau)) = -b\,\varepsilon(\gamma)f_2(\tau) - a\,\varepsilon(\gamma)f_5(\tau)$$

and that each $\gamma(\tau)$ ($\gamma\in\mathrm{Ker}(\rho)$) satisfies $b=c=0$. Therefore we see that $\mathrm{Ker}(\rho)=\{I,\,-I\}$.

□

We show that every vector-valued modular form with weight k on Γ with respect to a symmetric tensor representation has a quasimodular form as an entry (It is the inverse of the implied statement of Theorem 4.).

Theorem 5. *Let V be an $SL_2(\mathbb{Z})$-invariant (under the $|_k$-action) $(n+1)$-dimensional vector space consisting of functions in $\mathcal{F}$ which are bounded as $\mathrm{Im}(\tau)\to\infty$. Suppose there is a basis ${}^t(f_0,\, f_1,\, \ldots,\, f_n)$ of which $(n+1)\times(n+1)$ matrix representing $|_k$-action of $SL_2(\mathbb{Z})$ is the symmetric tensor representation $\rho^{(n)}$ on V. Define g_i and h_i $(0\le i\le n)$ by*

$$\begin{aligned}
{}^t(g_0,\, g_1,\, \ldots,\, g_n) &= \left((-1)^j\binom{i}{j}\tau^{i-j}\right)_{0\le i,\,j\le n}{}^t(f_0,\, f_1,\, \ldots,\, f_n)\,,\\
{}^t(h_0,\, h_1,\, \ldots,\, h_n) &= \left(\binom{n-i}{n-j}\left(\frac{1}{\mu}E_2\right)^{j-i}\right)_{0\le i,\,j\le n}{}^t(g_0,\, g_1,\, \ldots,\, g_n)\,. \qquad (13)
\end{aligned}$$

Then every h_i $(0 \le i \le n)$ is a modular form of weight $k+n-2i$ and there is an inversion formula of (13)

$$ {}^t(g_0,\, g_1,\, \ldots,\, g_n) \;=\; \left((-1)^{i+j} \binom{n-i}{n-j} \left(\frac{1}{\mu} E_2 \right)^{j-i} \right)_{0 \le i,\, j \le n} {}^t(h_0,\, h_1,\, \ldots,\, h_n)\,, $$

and each g_i $(0 \le i \le n)$ is a quasimodular form of weight $k+n-2i$ and depth $n-i$. In particular, $f_0 = g_0$ is a quasimodular form of weight $k+n$ and depth n.

Proof. We first show that $h_i(\tau+1) = h_i(\tau)$ and $\tau^{-k-n+2i} h_i(-1/\tau) = h_i(\tau)$ for each $0 \le i \le n$. Introduce $(n+1) \times (n+1)$ matrices

$$ A(\tau) \;=\; \left((-1)^j \binom{i}{j} \tau^{i-j} \right)_{0 \le i,\, j \le n} \quad \text{and} \quad B(\tau) \;=\; \left(\binom{n-i}{n-j} \left(\frac{1}{\mu} E_2 \right)^{j-i} \right)_{0 \le i,\, j \le n}. $$

Since (obviously) $B(\tau+1) = B(\tau)$ it follows that proving $h_i(\tau+1) = h_i(\tau)$ is equivalent to showing $g_i(\tau+1) = g_i(\tau)$. By definition,

$$ \rho^{(n)} \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \;=\; \left(\binom{i}{j} \right)_{0 \le i,\, j \le n}. $$

It then suffices to show that

$$ A(\tau+1) \rho^{(n)} \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \;=\; A(\tau)\,. \tag{14} $$

Now, the left-hand side of (14) becomes

$$ \left((-1)^j \binom{i}{j} (\tau+1)^{i-j} \right) \cdot \left(\binom{i}{j} \right) \;=\; \left(\sum_{\ell=0}^{n} (-1)^\ell \binom{i}{\ell} \binom{\ell}{j} (\tau+1)^{i-\ell} \right)_{0 \le i,\, j \le n} $$

and each entry is computed as

$$ \begin{aligned} \sum_{\ell=0}^{n} (-1)^\ell \binom{i}{\ell} \binom{\ell}{j} (\tau+1)^{i-\ell} \;&=\; \sum_{\ell=0}^{n} (-1)^\ell \binom{i}{\ell} \binom{\ell}{j} \sum_{m=0}^{i-\ell} \binom{i-\ell}{i-\ell-m} \tau^{i-\ell-m} \\ &=\; \sum_{m=0}^{i} \sum_{\ell=0}^{i} (-1)^\ell \binom{i}{\ell} \binom{\ell}{j} \binom{i-\ell}{i-m} \tau^{i-m} \\ &=\; \sum_{m=0}^{i} \sum_{\ell=0}^{i} (-1)^\ell \binom{i}{m} \binom{\ell}{j} \binom{m}{\ell} \tau^{i-m}\,. \end{aligned} \tag{15} $$

The coefficient of τ^{i-m} in (15) divided by $\binom{i}{m}$ is

$$\begin{aligned}
\sum_{\ell=0}^{i}(-1)^{\ell}\binom{\ell}{j}\binom{m}{\ell} &= \begin{cases} 0 & \text{if } i<j\,, \\ \displaystyle\sum_{\ell=j}^{i}(-1)^{\ell}\binom{\ell}{j}\binom{m}{\ell} & \text{if } i\geq j \end{cases} \\
&= \sum_{\ell=j}^{i}(-1)^{\ell}\binom{m-j}{\ell-j}\binom{m}{j} \quad \text{if } i\geq j \\
&= \binom{m}{j}\sum_{\ell=j}^{m}(-1)^{\ell}\binom{m-j}{\ell-j} \\
&= (-1)^{j}\binom{m}{j}\sum_{\ell=j}^{m}(-1)^{\ell-j}\binom{m-j}{\ell-j} \\
&= (-1)^{m}\binom{m}{j}\delta_{mj}\,.
\end{aligned}$$

Hence, the right-hand side of (15) becomes $(-1)^j\binom{i}{j}\tau^{i-j}$. Thus, we have proved (14).

We now verify the transformation of each $h_i(\tau)$ $(0\leq i\leq n)$ under $\tau\to-1/\tau$. Since

$$B(\tau)A(\tau) = \left((-1)^{j}\sum_{\ell=0}^{n}\binom{n-i}{n-\ell}\left(\frac{1}{\mu}E_2\right)^{\ell-i}\binom{\ell}{j}\tau^{\ell-j}\right),$$

it follows that

$$\begin{aligned}
h_i(\tau) &= \sum_{j=0}^{n}\sum_{\ell=0}^{n}(-1)^{j}\binom{n-i}{n-\ell}\binom{\ell}{j}\left(\frac{1}{\mu}E_2\right)^{\ell-i}\tau^{\ell-j}f_j(\tau) \\
&= \sum_{j=0}^{n}\sum_{\ell=0}^{n}(-1)^{n-j}\binom{n-i}{n-\ell}\binom{\ell}{n-j}\left(\frac{1}{\mu}E_2\right)^{\ell-i}\tau^{\ell+j-n}f_{n-j}(\tau)\,.
\end{aligned}$$

Now, by the definition of Γ-module $(V,\rho^{(n)})$, we have $f_{n-j}(-1/\tau)=(-1)^{n-j}\tau^k f_j(\tau)$ and hence

$$h_i(-1/\tau) = \sum_{j=0}^{n}\sum_{\ell=0}^{n}(-1)^{n+\ell+j}\binom{n-i}{n-\ell}\binom{\ell}{n-j}\tau^{2\ell-2i}\left(\frac{E_2}{\mu}+\frac{1}{\tau}\right)^{\ell-i}\tau^{-\ell+n-j}\tau^{k}f_j\,,$$

i.e.

$$\tau^{-k-n+2i}h_i(-1/\tau) = \sum_{j=0}^{n}(-1)^{j}\tau^{-j}f_j\sum_{\ell=0}^{n}(-1)^{\ell+n}\binom{n-i}{n-\ell}\binom{\ell}{n-j}\left(\frac{E_2}{\mu}+\frac{1}{\tau}\right)^{\ell-i}\tau^{\ell}\,, \quad (16)$$

where the second sum on the right-hand side becomes

$$\begin{aligned}
&\sum_{\ell=0}^{n}(-1)^{\ell-n}\binom{n-i}{n-\ell}\binom{\ell}{n-j}\sum_{m=0}^{\ell-i}\binom{\ell-i}{m}\left(\frac{1}{\mu}E_2\right)^{\ell-i-m}\tau^{\ell-m}\\
&\qquad=\sum_{\ell=i}^{n}(-1)^{\ell-n}\binom{n-i}{n-\ell}\binom{\ell}{n-j}\sum_{m=i}^{n}\binom{\ell-i}{\ell-m}\tau^{m}\left(\frac{1}{\mu}E_2\right)^{m-i}\\
&\qquad=\sum_{m=i}^{n}\tau^{m}\left(\frac{1}{\mu}E_2\right)^{m-i}\sum_{\ell=i}^{n}(-1)^{\ell-n}\binom{n-i}{n-\ell}\binom{\ell}{n-j}\binom{\ell-i}{\ell-m}\\
&\qquad=\sum_{m=i}^{n}\tau^{m}\left(\frac{1}{\mu}E_2\right)^{m-i}\binom{n-i}{n-m}\sum_{\ell=i}^{n}(-1)^{\ell-n}\binom{n-m}{n-\ell}\binom{\ell}{n-j}.
\end{aligned}$$

The second sum on the right of the last equality is

$$\begin{aligned}
\sum_{\ell=i}^{n}(-1)^{\ell-n}\binom{n-m}{n-\ell}\binom{\ell}{n-j} &= \sum_{\ell=0}^{n}(-1)^{\ell-n}\binom{n-m}{n-\ell}\binom{\ell}{n-j}\\
&= \sum_{\ell=0}^{j}(-1)^{\ell}\binom{n-m}{\ell}\binom{n-\ell}{j-\ell}\\
&= \binom{m}{j},
\end{aligned}$$

where the second sum of first equality may run from $\ell = 0$ to n because if $\ell < i$ and $n-i \geq n-m$, we have $n-\ell > n-m$ and then $\binom{n-m}{n-\ell} = 0$. Therefore, it follows from (16) that

$$\begin{aligned}
\tau^{-k-n+2i}h_i(-1/\tau) &= \sum_{j=0}^{n}(-1)^{j}\tau^{-j}f_j(\tau)\sum_{m=i}^{n}\tau^{m}\left(\frac{1}{\mu}E_2\right)^{m-i}\binom{n-i}{n-m}\binom{m}{j}\\
&= \sum_{j=0}^{n}\sum_{m=0}^{n}(-1)^{j}\binom{n-i}{n-m}\binom{m}{j}\left(\frac{1}{\mu}E_2\right)^{m-i}\tau^{m-j}f_j(\tau)\\
&= h_i(\tau)\,.
\end{aligned}$$

Finally we show the inversion formula. It suffices to show that

$$\left((-1)^{i+j}\binom{n-i}{n-j}\left(\frac{1}{\mu}E_2\right)^{j-i}\right)\left(\binom{n-i}{n-j}\left(\frac{1}{\mu}E_2\right)^{j-i}\right) = I_{n+1}\,.$$

The $(i,\,j)$ entry of the left hand-side of the last equation is

$$\begin{aligned}
\left(\frac{1}{\mu}E_2\right)^{j-i}\sum_{k=0}^{n}(-1)^{i+k}\binom{n-i}{n-k}\binom{n-k}{n-j} &= \left(\frac{1}{\mu}E_2\right)^{j-i}\sum_{k=0}^{n}(-1)^{i+n-k}\binom{n-i}{k}\binom{k}{n-j}\\
&= \left(\frac{1}{\mu}E_2\right)^{j-i}\binom{n-i}{n-j}\delta_{n-i,n-j}\\
&= \left(\frac{1}{\mu}E_2\right)^{j-i}\delta_{ij} = \delta_{ij}\,.
\end{aligned}$$

This completes the proof of the theorem. □

2.3 A criterion for determining quasimodular forms

In this short (but important) section we show a way to determine the exact form (in terms of E_2, E_4 and E_6) of a quasimodular form of given weight on $\Gamma = SL_2(\mathbb{Z})$.

Recall that the space of quasimodular form on Γ is a graded ring $\mathbb{C}[E_2, E_4, E_6] = \bigoplus_{k\geq 0} \widetilde{M}_k$ where

$$\widetilde{M}_k = \sum_{\substack{a,\,b,\,c\geq 0\\ 2a+4b+6c=k}} \mathbb{C}\cdot E_2^a E_4^b E_6^c\,.$$

Let f be a quasimodular form of weight k with a q-expansion $f = a_0 + a_1 q + \cdots + c_n q^n + \cdots$. Suppose that

$$E_2^a E_4^b E_6^c = c_0^{(a,\,b,\,c)} + c_1^{(a,\,b,\,c)} q + \cdots + c_n^{(a,\,b,\,c)} q^n + \cdots\,.$$

Then we have a system of linear equations

$$\sum_{(a,b,c)} c_i^{(a,\,b,\,c)} x^{(a,\,b,\,c)} = a_i \quad (0 \leq i \leq d_k - 1)\,, \tag{17}$$

where $d_k = \dim_{\mathbb{C}} \widetilde{M}_k$. Suppose that (17) has a solution $\left(x^{(a,\,b,\,c)}\right)$ and that the coefficient of q^{d_k} in

$$f - \sum_{\substack{a,\,b,\,c\geq 0\\ 2a+4b+6c=k}} x^{(a,\,b,\,c)} E_2^a E_4^b E_6^c \tag{18}$$

vanishes. Then we can conclude that

$$f = \sum_{\substack{a,\,b,\,c\geq 0\\ 2a+4b+6c=k}} x^{(a,\,b,\,c)} E_2^a E_4^b E_6^c\,.$$

3 Applications to the theory of vertex operator algebras

In this section we give several applications of our theorems (Theorem 4 and Theorem 5) to the theory of vertex operator algebras (or equivalently 2DCFT).

Let V be a vertex operator algebra and $\mathcal{C}(V)$ the space of one-point functions on a family of elliptic curves (one-point functions for short). Then it is known that the space $\mathcal{C}_V(\mathbf{1})$ of one-point functions evaluated at the vacuum $\mathbf{1}$ is invariant with respect to the slash $|_0$-action of $\Gamma = SL_2(\mathbb{Z})$ and then each basis of $\mathcal{C}_V(\mathbf{1})$ is a (meromorphic) vector-valued modular form of weight 0. This aspect is one of motivations of our research on vector-valued modular forms. One of difficulties of study on one-point functions is that it is rather hard to find a basis of the space $\mathcal{C}(V)$. In fact, there are only few examples of which basis are known in (especially) non-rational theory (cf. [2]). Each element of the space $\mathcal{C}_V(\mathbf{1})$ is called *pseudo-characters* of V. Of course, determining a pseudo-character is not easy in general, however,

formal characters of a vertex operator algebra and its simple modules, which are a typical examples of pseudo-characters is mostly computable (at least several terms of q-expansion and therefore one can know if they are quasimodular forms on Γ). By a short calculation we have

Lemma 6. *Let f be a quasimodular form of weight k and depth d (at most n) on $SL_2(\mathbb{Z})$. Then*

$$f(-1/\tau) \;=\; \tau^k g(\tau) E_2(\tau)^d + \Big(\tau^k g_1(\tau) + d\mu\tau^{k-1} g_0(\tau)\Big)\, E_2^{d-1} + \cdots\,,$$

where g_i $(i = 1,\, 2)$ is a modular form of weight $k - 2d + 2i$ on $SL_2(\mathbb{Z})$.

Definition. Let V be a vertex operator algebra and $C_2(V)$ the vector subspace of V that is linearly generated by $a_{(-2)}b$ for all $a\,, b \in V$. If the quotient space $V/C(V)$ is finite-dimensional, then V is called a C_2-*cofinite* vertex operator algebra.

Proposition 7. *Let V be a C_2-cofinite vertex operator algebra and suppose that there is a quasimodular pseudo-chracter. Then V is not rational.*

Proof. Suppose that there is a quasimodular pseudo-chracter. Then by Lemma 6 there is a pseudo-chracter with logarithmic terms of q. Therefore V is not rational since every element of $\mathcal{C}(\mathbf{1})$ has no logarithmic form if V is rational and C_2-cofinite. □

Corollary. Let V be a C_2-cofinite vertex operator algebra and suppose that the formal character of V is a quasimodular form. Then V is not rational.

It is well known that if V is C_2-cofinite then the space of pseudo-characters is finite-dimensional, and central charge and conformal weights are rational numbers (see [6] and [7]). Then the following theorem is useful under most circumstances.

Theorem 8. *Let V be a C_2-cofinite vertex operator algebra and $\dim \mathcal{C}_V(\mathbf{1}) = n+1$ where n is a nonnegative integer. Suppose that there is a pseudo-character f which is a quasimodular form of weight n on $\Gamma = SL_2(\mathbb{Z})$. Then*

$$f_i(\tau) \;=\; \sum_{\ell=0}^{i} \binom{n}{\ell}^{-1} \binom{i}{\ell} \mu^\ell \tau^{i-\ell} \partial^{(\ell)} f\,, \quad \mu = \frac{6}{\pi i}$$

is an element of V for each $0 \le i \le n$ and the set $\{f_0,\, f_1,\, \ldots,\, f_n\}$ is a basis of V. Moreover, the $|_0$-action of Γ with respect to this basis is given by

$$\mathbb{F}|_0\,\gamma = \rho^{(n)}\,(\gamma)\,\mathbb{F} \text{ for every } \gamma \in \Gamma\,,$$

where $\mathbb{F} = {}^t(f_0,\, f_1,\, \ldots,\, f_n)$.

Proof. Since the space $\mathcal{C}_V(\mathbf{1})$ is invariant with respect to the $|_0$-action of Γ, it follows from Theorem 4 that there are holomorphic functions f_i $(0 \le i \le n)$ such that $\mathbb{F}|_0\gamma = \rho^{(n)}(\gamma)\,\mathbb{F}$ for every $\gamma \in \Gamma$ where $\mathbb{F} = {}^t(f_0,\, f_1,\, \ldots,\, f_n)$ and that $\{f_0,\, f_1,\, \ldots,\, f_n\}$ is a basis of $\mathcal{C}_V(\mathbf{1})$. This completes the proof of the theorem. □

Theorem 9. *Let V be a C_2-cofinite vertex operator algebra and* $\dim \mathcal{C}_V(\mathbf{1}) = n+1$. *Suppose that each element of $\mathcal{C}_V(\mathbf{1})$ belongs to $\mathcal{F}$ and is bounded as* $\mathrm{Im}(\tau) \to \infty$, *and that there is a basis* ${}^t(f_0, f_1, \ldots, f_n)$ *of $\mathcal{C}_V(\mathbf{1})$, of which $(n+1) \times (n+1)$ matrix representing the $|_0$-action of $SL_2(\mathbb{Z})$ is the symmetric representation $\rho^{(n)}$. Then f_0 is a quasimodular pseudo-character of weight n. In particular V is not rational.*

Proof. It follows from Theorem 5 that there is a quasimodular pseudo-character of weight n (it is g_0 in Theorem 5). Then Lemma 6 completes the proof of theorem. □

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