On a kind of duality of multiple zeta-star values

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1 Main result

In this note, we prove a certain duality-type result for height 1 multiple zeta-star values and discuss its possible generalization.

For an index set \((k_1, k_2, \ldots, k_n)\) of positive integers with \(k_1 > 1\), the multiple zeta-star value \(\zeta^*(k_1, k_2, \ldots, k_n)\) is defined by

\[
\zeta^*(k_1, k_2, \ldots, k_n) := \sum_{m_1 \geq m_2 \geq \cdots \geq m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.
\]

If we remove the equality signs in the summation, we obtain the usual multiple zeta value:

\[
\zeta(k_1, k_2, \ldots, k_n) := \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.
\]

The height of the multiple zeta or zeta-star value is the number of \(k_i\) in the index set which is greater than 1. The following theorem can be regarded as a kind of duality for multiple zeta-star values of height 1.

**Theorem 1** For any integers \(k, n \geq 1\), we have

\[
(-1)^k \zeta^*(k + 1, 1, \ldots, 1) - (-1)^n \zeta^*(n + 1, 1, \ldots, 1) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \ldots],
\]

the right-hand side being the algebra over \(\mathbb{Q}\) generated by the values of the Riemann zeta function at positive integer arguments (> 1).

**Remark** For multiple zeta values, there is a well-known duality formula [9], and the height 1 case of the formula reads as

\[
\zeta(k + 1, 1, \ldots, 1)_{n-1} = \zeta(n + 1, 1, \ldots, 1)_{k-1}
\]

for \(k, n \geq 1\). No such simple formula has been known for multiple zeta-star values. It should be noted that the pair of indices

\[
(k + 1, 1, \ldots, 1)_n \leftrightarrow (n + 1, 1, \ldots, 1)_k
\]
in Theorem 1. We can also compute the generating function of the quantity
\[ (-1)^k \zeta^*(k + 1, 1, \ldots, 1) - (-1)^n \zeta^*(n + 1, 1, \ldots, 1) \]
in Theorem 1.

**Theorem 2** We have
\[
\sum_{k, n \geq 1} \left( (-1)^k \zeta^*(k + 1, 1, \ldots, 1) - (-1)^n \zeta^*(n + 1, 1, \ldots, 1) \right) x^k y^n = \psi(x) - \psi(y) + \pi \left( \cot(\pi x) - \cot(\pi y) \right) \frac{\Gamma(1 - x)\Gamma(1 - y)}{\Gamma(1 - x - y)}.
\]
Here, \( \psi(x) = \Gamma'(x)/\Gamma(x) \) is the digamma function, the logarithmic derivative of the gamma function.

## 2 Proof of Theorems
We prove the following basic identity, from which follow both Theorem 1 and Theorem 2.

**Proposition** For \( k, n \geq 1 \), we have
\[
(-1)^k \zeta^*(k + 1, 1, \ldots, 1) - (-1)^n \zeta^*(n + 1, 1, \ldots, 1) = k \zeta(k + 2, 1, \ldots, 1) - n \zeta(n + 2, 1, \ldots, 1) + (-1)^k \sum_{j=0}^{k-2} (-1)^j \zeta(k - j) \zeta(n + 1, 1, \ldots, 1) - (-1)^n \sum_{j=0}^{n-2} (-1)^j \zeta(n - j) \zeta(k + 1, 1, \ldots, 1),
\]
where we understand an empty sum to be 0.

**Proof.** We use two formulas for the special value of the function \( \xi_k(s) \) defined for \( k \geq 1 \) by
\[
\xi_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} Li_k(1 - e^{-t}) dt. \tag{1}
\]

1Recently, C. Yamazaki ([8]) gave another proof of them. It uses a generating function of certain sums of multiple zeta-star values which was introduced in [1].
In [3], we studied this function and obtained among others the formula
\[ \xi_k(n + 1) = (-1)^{k-1} \left[ \zeta(n + 1, 2, 1, \ldots) + \zeta(n + 1, 1, 2, 1, \ldots) + \cdots \right. \]
\[ \left. + \zeta(n + 1, 1, 1, 2) + (n + 1) \cdot \zeta(n + 2, 1, 1, \ldots) \right] \]
\[ + \sum_{j=0}^{k-2} (-1)^j \zeta(k - j) \cdot \zeta(n + 1, 1, \ldots), \]  
(2)

where \( k, n \) are integers \( \geq 1 \).

On the other hand, we showed in [6] that the value \( \xi_k(n) \) is nothing but the multiple zeta-star value of height 1, i.e., we have the formula
\[ \xi_k(n + 1) = \zeta^*(n + 1, 1, \ldots). \]  
(3)

Since the index sets \( (k + 1, 1, \ldots) \) and \( (n + 1, 1, \ldots) \) are dual (in the context of multiple zeta values) with each other, the main theorem in [6] applied to these index sets with \( l = 1 \) gives the identity
\[ \zeta(k + 1, 2, 1, \ldots) + \zeta(k + 1, 2, 1, \ldots) + \zeta(k + 1, 1, 2, 1, \ldots) + \cdots \]
\[ \cdots + \zeta(k + 1, 1, 1, 2) \]
\[ = \zeta(n + 2, 1, \ldots) + \zeta(n + 1, 2, 1, \ldots) + \zeta(n + 1, 1, 2, 1, \ldots) + \cdots \]
\[ \cdots + \zeta(n + 1, 1, \ldots, 1, 2). \]  
(4)

Combining (2), (3) and (4), we obtain the proposition.

**Proof of Theorems 1 and 2.** Recall the formula of Aomoto [2] and Drinfeld [4]
\[ \sum_{k,n \geq 1} \zeta(k + 1, 1, \ldots, 1) x^k y^n = 1 - \Gamma(1-x)\Gamma(1-y) \Gamma(1-x-y). \]  
(5)

This together with the standard Taylor expansion of the (logarithm of) gamma function
\[ \Gamma(1 + x) = \exp \left( -x + \sum_{n=0}^{\infty} \frac{\zeta(n)}{n} x^n \right), \quad (|x| < 1, \gamma : Euler's constant) \]  
(6)

shows that all multiple zeta values of height 1 (= of type \( \zeta(m, 1, \ldots) \)) can be expressed as polynomials over \( \mathbb{Q} \) in the Riemann zeta values. Theorem 1 therefore follows from the formula in Proposition.
As for the generating series, we start with the formula (5). Replace \( k \) with \( k + 1 \) in (5) and divide the both-hand sides out by \( xy \), and then differentiate with respect to \( x \) and multiply \( xy \). Then we obtain

\[
\sum_{k,n \geq 1} k \zeta(k+2,1,\ldots,1)x^ky^n = -\frac{1}{x} + \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \left( \frac{1}{x} + \psi(1-x) - \psi(1-x-y) \right),
\]

and hence by interchanging \( x \) and \( y \) and subtracting, we have

\[
\sum_{k,n \geq 1} \left( k \zeta(k+2,1,\ldots,1) - n \zeta(n+2,1,\ldots,1) \right) x^ky^n = -\frac{1}{x} + \frac{1}{y} + \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \left( \frac{1}{x} + \psi(1-x) - \frac{1}{y} - \psi(1-y) \right).
\]

Next, by the formula

\[
\sum_{i=2}^{\infty} (-1)^i \zeta(i)x^{i-1} = \psi(1+x) + \gamma
\]

(take the logarithmic derivative of (6)) and by (5), we have

\[
\sum_{k,n \geq 1} (-1)^k \sum_{j=0}^{k-2} (-1)^j \zeta(k-j)\zeta(n+1,1,\ldots,1)x^jy^n
\]

\[
= \sum_{i \geq 2, j, n \geq 1} (-1)^i \zeta(i)\zeta(n+1,1,\ldots,1)x^{i+j-1}y^n
\]

\[
= \left( \sum_{i \geq 2} (-1)^i \zeta(i)x^{i-1} \right) \left( \sum_{j,n \geq 1} \zeta(n+1,1,\ldots,1)x^jy^n \right)
\]

\[
= (\psi(1+x) + \gamma) \left( 1 - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \right),
\]

and thus we obtain

\[
\sum_{k,n \geq 1} (-1)^k \sum_{j=0}^{k-2} (-1)^j \zeta(k-j)\zeta(n+1,1,\ldots,1)
\]

\[
- (-1)^n \sum_{j=0}^{n-2} (-1)^j \zeta(n-j)\zeta(k+1,1,\ldots,1)
\]

\[
x^ky^n
\]

\[
= \left( 1 - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \right) (\psi(1+x) - \psi(1+y)).
\]

\[(8)\]
By Proposition, Theorem 2 follows from (7), (8), and the standard identities
\[ \psi(1 + x) = \frac{1}{x} + \psi(x) \quad \text{and} \quad \pi \cot(\pi x) = \frac{1}{x} + \psi(1 - x) - \psi(1 + x). \]

3 Possible generalization

In this section, we propose a possible generalization of Theorem 1 for arbitrary heights.

First, we recall a few notations which are used in [1]. The weight and the depth of multiple zeta-star values \( \zeta^*(k_1, k_2, \ldots, k_n) \) are the sum \( k_1 + k_2 + \cdots + k_n \) and the length \( n \) of its index, respectively. We denote by \( X_0(k, n, s) \) the sum of all multiple zeta-star values of weight \( k \), depth \( n \) and height \( s \), for \( k \geq n + s \) and \( n \geq s \geq 1 \).

Based on the numerical experiments up to weight 11, we conjecture the following.

**Conjecture**  For any integers \( k, n \geq s \geq 1 \), we have
\[
(-1)^k X_0(k+n+1, n+1, s) - (-1)^n X_0(k+n+1, k+1, s) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \ldots].
\]

**Remark**  Theorem 1 is nothing but the case when \( s = 1 \) of the above conjecture.

**Examples**  When the weight is 8 and the height is 2 or 3, we can show (using the double shuffle relations of multiple zeta values) the following identities, which are in favor of the conjecture.

\[
\begin{align*}
X_0(8, 3, 2) + X_0(8, 6, 2) &= \frac{876}{175} \zeta(2)^4 - \zeta(2)\zeta(3)^2 - 3\zeta(3)\zeta(5) \\
X_0(8, 4, 2) + X_0(8, 5, 2) &= \frac{1083}{280} \zeta(2)^4 + \zeta(2)\zeta(3)^2 + 2\zeta(3)\zeta(5) \\
X_0(8, 4, 3) + X_0(8, 5, 3) &= \frac{1349}{280} \zeta(2)^4 - \frac{1}{2}\zeta(2)\zeta(3)^2 - \zeta(3)\zeta(5)
\end{align*}
\]

References


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