The Akiyama-Tanigawa algorithm for Bernoulli numbers

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Abstract

A direct proof is given for Akiyama and Tanigawa’s algorithm for computing Bernoulli numbers. The proof uses a closed formula for Bernoulli numbers expressed in terms of Stirling numbers. The outcome of the same algorithm with different initial values is also briefly discussed.

1 The Algorithm

In their study of values at non-positive integer arguments of multiple zeta functions, S. Akiyama and Y. Tanigawa found as a special case an amusing algorithm for computing Bernoulli numbers in a manner similar to “Pascal’s triangle” for binomial coefficients.

Their algorithm reads as follows: Start with the 0-th row 1, 1, 1, 1, 1, ... and define the first row by 1 · (1 − 1), 2 · (1 − 3), 3 · (1 − 4), ... which produces the sequence 1, 1, 1, 1, ... Then define the next row by 1 · (1 − 3), 2 · (1 − 3), 3 · (1 − 5), ..., thus giving 1, 1, 3, 6, ... as the second row. In general, denoting the m-th (m = 0, 1, 2, ...) number in the n-th (n = 0, 1, 2, ...) row by a_{n,m}, the m-th number in the (n + 1)-st row a_{n+1,m} is determined recursively by

\[ a_{n+1,m} = (m + 1) \cdot (a_{n,m} - a_{n,m+1}). \]
Then the claim is that the 0-th component $a_{n,0}$ of each row (the "leading diagonal") is just the $n$-th Bernoulli numbers $B_n$, where

$$
\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{xe^x}{e^x - 1} \left( = \frac{x}{e^x - 1} + x \right).
$$

Note that we are using the definition of the Bernoulli numbers in which $B_1 = \frac{1}{2}$. This is the definition used by Bernoulli (and independently Seki, published one year prior to Bernoulli). Incidentally, this is more appropriate for the Euler formula $\zeta(1-k) = -B_k/k$ ($k = 1, 2, 3, \ldots$) for the values of the Riemann zeta function.

Figure 1: Akiyama-Tanigawa triangle
2 Proof

The proof is based on the following identity for Bernoulli numbers, a variant of which goes as far back as Kronecker (see \[4\]). Here we denote by \( \binom{n}{m} \) the Stirling number of the second kind:

\[
x^n = \sum_{m=0}^{n} \binom{n}{m} x^m,
\]

where \( x^m = x(x-1)\cdots(x-m+1) \) for \( m \geq 1 \) and \( x^0 = 1 \). (We use Knuth’s notation \[7\]. For the Stirling number identities that we shall need, the reader is referred for example to \[5\].)

**Theorem 1**

\[
B_n = \sum_{m=0}^{n} \frac{(-1)^m m! \binom{n+1}{m+1}}{m+1}, \quad \forall n \geq 0.
\]

We shall give later a proof of this identity for the sake of completeness. Once we have this, the next proposition ensures the validity of our algorithm.

**Proposition 2** Given an initial sequence \( a_{0,m} \ (m = 0, 1, 2, \ldots) \), define the sequences \( a_{n,m} \ (n \geq 1) \) recursively by

\[
a_{n,m} = (m + 1) \cdot (a_{n-1,m} - a_{n-1,m+1}) \quad (n \geq 1, m \geq 0).
\]

Then

\[
a_{n,0} = \sum_{m=0}^{n} (-1)^m m! \binom{n+1}{m+1} a_{0,m}.
\]

**Proof.** Put

\[
g_n(t) = \sum_{m=0}^{\infty} a_{n,m} t^m.
\]

By the recursion (\[1\]) we have for \( n \geq 1 \)

\[
g_n(t) = \sum_{m=0}^{\infty} (m + 1)(a_{n-1,m} - a_{n-1,m+1}) t^m
\]

\[
= \frac{d}{dt} \left( \sum_{m=0}^{\infty} a_{n-1,m} t^{m+1} \right) - \frac{d}{dt} \left( \sum_{m=0}^{\infty} a_{n-1,m+1} t^{m+1} \right)
\]

\[
= \frac{d}{dt} (tg_{n-1}(t)) - \frac{d}{dt} (g_{n-1}(t) - a_{n-1,0})
\]

\[
= g_{n-1}(t) + (t - 1) \frac{d}{dt} (g_{n-1}(t))
\]

\[
= \frac{d}{dt} ((t - 1)g_{n-1}(t)).
\]
Hence, by putting \((t - 1)g_n(t) = h_n(t)\) we obtain

\[
h_n(t) = (t - 1)\frac{d}{dt}(h_{n-1}(t)) \quad (n \geq 1),
\]

and thus

\[
h_n(t) = \left( (t - 1)\frac{d}{dt} \right)^n (h_0(t)).
\]

Applying the formula (cf. [5, Ch. 6.7 Exer. 13])

\[
\left( x\frac{d}{dx} \right)^n = \sum_{m=0}^{n} \binom{n}{m} x^m \left( \frac{d}{dx} \right)^m
\]

for \(x = t - 1\), we have

\[
h_n(t) = \sum_{m=0}^{n} \binom{n}{m} (t - 1)^m \left( \frac{d}{dt} \right)^m h_0(t).
\]

Putting \(t = 0\) we obtain

\[
-a_{n,0} = \sum_{m=0}^{n} \binom{n}{m} (-1)^m m!(a_{0,m-1} - a_{0,m})
\]

\[
= \sum_{m=0}^{n-1} \binom{n}{m+1} (-1)^{m+1} (m + 1)!a_{0,m} - \sum_{m=0}^{n} \binom{n}{m} (-1)^m m!a_{0,m}
\]

\[
= -\sum_{m=0}^{n} (-1)^m m!a_{0,m} \left( (m + 1) \left\{ \binom{n}{m} + \binom{n}{m+1} \right\} - \sum_{m=0}^{n} \binom{n+1}{m+1} a_{0,m}. \right)
\]

(We have used the recursion \(\binom{n+1}{m+1} = (m + 1)\binom{n}{m+1} + \binom{n}{m}\).) This proves the proposition.

**Proof of Theorem 1.** We show the generating series of the right hand side coincide with that of \(B_n\). To do this, we use the identity

\[
\frac{e^x(e^x - 1)^m}{m!} = \sum_{n=m}^{\infty} \binom{n+1}{m+1} \frac{x^n}{n!}
\]

which results from the well-known generating series for the Stirling numbers (cf. [1] (7.49))

\[
\frac{(e^x - 1)^m}{m!} = \sum_{n=m}^{\infty} \binom{n}{m} \frac{x^n}{n!}
\]
by replacing $m$ with $m + 1$ and differentiating with respect to $x$. With this, we have
\[
\sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \frac{(-1)^m m! \binom{n+1}{m+1}}{m+1} \right) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^m m! e^x (e^x - 1)^m}{m+1} x^n \frac{1}{n!} = \sum_{m=0}^{\infty} \frac{(-1)^m m! e^x (e^x - 1)^m}{m+1} x^n \frac{1}{n!}.
\]
This proves Theorem 1.

\textbf{Remark.} A referee suggested the following interpretation of the algorithm using generating function:
Suppose the first row is $a_0, a_1, a_2, \ldots$, with ordinary generating function
\[
A(x) = \sum_{n=0}^{\infty} a_n x^n.
\]
Let the leading diagonal be $b_0 = a_0, b_1, b_2, \ldots$, with exponential generating function
\[
\mathbb{B}(x) = \sum_{n=0}^{\infty} b_n x^n.
\]
Then we have
\[
\mathbb{B}(x) = e^x A(1 - e^x).
\]
This follows from (7) and (6), the calculation being parallel to that of the proof of Theorem 1. To get the Bernoulli numbers we take $a_0 = 1, a_1 = \frac{1}{2}, a_2 = \frac{1}{6}, \ldots$ with $A(x) = -\log(1 - x)/x$, and find $\mathbb{B}(x) = xe^x/(e^x - 1)$.

\section{Poly-Bernoulli numbers}

If we replace the initial sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ by $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and apply the same algorithm, the resulting sequence is $(-1)^n D^{(k)}_n (n = 0, 1, 2, \ldots)$, where $D^{(k)}_n$ is a variant of “poly-Bernoulli numbers” (7), (6), (6)): For any integer $k$, we define a sequence of numbers $D^{(k)}_n$ by
\[
\frac{Li_k(1 - e^{-x})}{e^x - 1} = \sum_{n=0}^{\infty} D^{(k)}_n \frac{x^n}{n!},
\]
where $Li_k(t) = \sum_{m=1}^{\infty} \frac{t^m}{m^k}$ ($k$-th polylogarithm when $k \geq 1$). The above assertion is then a consequence of the following (or, is just a special case of the preceding remark)
Proposition 3 For any $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$D_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! {n+1 \choose m+1}}{(m+1)^k}.$$ 

Proof. The proof can be given completely in the same way as the proof of Theorem 1 using generating series, and hence will be omitted.

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References


(The Bernoulli numbers are A027641/A027642. The table in Figure 1 yields sequences A051714/A051715. Other sequences which mention this paper are A000367, A002445, A026741, A045896, A051712, A051713, A051716, A051717, A051718, A051719, A051720, A051721, A051722, A051723.)

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