On the zeros of certain modular forms

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Dedicated to Professor Yasutaka Ihara on the occasion of his 60th birthday.

The aim of this short note is to list several families of modular forms and functions which exhibit certain common properties of zeros, both in characteristic 0 and in characteristic p. Each of these families already appeared and was studied in various places in the literature. However, the author believes it is worth recording the collection here with an awareness of the issues which will be discussed in this paper, in order to attract wider attention to a phenomenon which is not well understood. In the sequel, we exclusively deal with the forms/functions on the modular group $\Gamma = PSL_2(\mathbb{Z})$, and so we often make no reference to the group.

For an even integer $k \geq 4$, let M_k be the space of holomorphic modular forms of weight k (on Γ). The Eisenstein series

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=0}^{\infty}(\sum_{d|n} d^{k-1})q^n, \quad (q = e^{2\pi i\tau},\ B_k = k\text{-th Bernoulli number})$$

is a standard example of an element in M_k. Also fundamental are the discriminant function

$$\Delta(\tau) = \frac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2)$$

of weight 12 and the elliptic modular invariant

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

of weight 0 (having pole at ∞).

Since the automorphic factor $(c\tau + d)^k = f\left(\frac{a\tau+b}{c\tau+d}\right)/f(\tau)$ never vanishes on the upper half-plane $\mathfrak{H}$, we can speak of Γ- equivalence classes of zeros in $\mathfrak{H}$ of $f(\tau) \in M_k$, or equivalently, speak of zeros of $f(\tau)$ in the standard fundamental domain of Γ. This also amounts to looking at the value of $j(\tau)$ at each zero because the j-function maps bijectively the Γ- equivalence classes of points in $\mathfrak{H}$ to $\mathbb{C}$. Namely, we can associate to each modular form

$f(\tau) \in M_k$ a polynomial $\Omega_f(j)$ in j whose roots are exactly the values of $j(\tau)$ at the zeros of f. For this, write k (uniquely) in the form

$$k = 12m + 4\delta + 6\varepsilon \qquad \text{with} \quad m \in \mathbb{Z}_{\geq 0}, \quad \delta \in \{0,1,2\}, \quad \varepsilon \in \{0,1\}\,.$$

Then $f(\tau)$ is written uniquely as

$$f(\tau) = \Delta(\tau)^m\, E_4(\tau)^\delta\, E_6(\tau)^\varepsilon\, \widetilde{f}\big(j(\tau)\big)$$

for some polynomial $\widetilde{f}$ of degree $\leq m$ in $j(\tau)$ (because $f(\tau)/(\Delta(\tau)^m\, E_4(\tau)^\delta\, E_6(\tau)^\varepsilon)$ is of weight 0 and holomorphic in $\mathfrak{H}$), the coefficient of j^m in $\widetilde{f}$ being equal to the constant term of the Fourier expansion of $f(\tau)$. With this we put

$$\Omega_f(j) := j^\delta (j - 1728)^\varepsilon \widetilde{f}(j).$$

1. The first theorem we mention is the following result in 1970 due to F. Rankin and Swinnerton-Dyer ([5]).

Theorem 1 (F. Rankin and Swinnerton-Dyer). *All the zeros of* $\Omega_{E_k}(j)$ *are real and lie in the interval* $[0, 1728]$.

In other words, all the zeros of $E_k(\tau)$ in the standard fundamental domain lie on the unit circle. (Recall the values $j(e^{\frac{2\pi\sqrt{-1}}{3}}) = 0$ and $j(\sqrt{-1}) = 1728$.) Their proof is, as Atkin put it, "embarrassingly simple" using only intermediate value theorem and elementary trigonometry. On the other hand, the reduction modulo p of these zeros have nice arithmetical meaning, which was noticed by Deligne in the 70s (see [6]).

Theorem 2 (Deligne). *Let* $p \geq 5$ *be a prime. Then the polynomial* $\Omega_{E_{p-1}}(j)$ *has* p*-integral rational coefficients and the roots in* $\bar{\mathbb{F}}_p$ *of its reduction modulo* p *are exactly the* j*-invariants of all the supersingular elliptic curves in characteristic* p.

The key fact in his proof, which requires certain amount of knowledge of algebraic geometry, is that the Fourier series of $E_{p-1}(\tau)$ reduces to 1 modulo p, thanks to the theorem of von-Staudt and Clausen. We can also give an elementary proof using the same fact ([3]).

2. We have found another series of modular forms which satisfies both of the properties in Theorems 1 and 2. To define them, recall that the graded ring of modular forms on Γ has a unique (up to constant multiple) derivation of degree 2 which sends cusp forms to cusp forms. Specifically, the system of operators $\{\partial_k\}_{k\geq 4}$ defined by

$$\partial_k(f)(\tau) := \frac{1}{2\pi i}\frac{df}{d\tau}(\tau) - \frac{k}{12}E_2(\tau)f(\tau)$$

for $f(\tau) \in M_k$ has the required property. Here, $E_2(\tau) = \frac{1}{2\pi i}\frac{d}{d\tau}\log\Delta(\tau)$ is the "nearly modular" Eisenstein series of weight 2. If $k+4 \not\equiv 0 \pmod 3$, then every element of M_{k+4} is divisible by $E_4(\tau)$, so we have an endomorphism ϕ_k of M_k defined by $\phi_k(f) = E_4(\tau)^{-1}\partial_{k+2}(\partial_k(f))$. It can be shown that this endomorphism ϕ_k has a unique (up to a constant multiple) eigenvector which is *not* a cusp form (and the other eigenvectors come from lower weights, multiplied by powers of Δ). We choose certain normalization of this eigenvector and denote it by $F_k(\tau)$.

Theorem 3 (Zagier-Kaneko). *i) All the zeros of* $\Omega_{F_k}(j)$ *are real and lie in the interval* $[0, 1728]$.
ii) Let $p \geq 5$ *be a prime. Then the polynomial* $\Omega_{F_{p-1}}(j)$ *has* p*-integral rational coefficients and the roots in* $\bar{\mathbb{F}}_p$ *of its reduction modulo* p *are exactly the* j*-invariants of all the supersingular elliptic curves in characteristic* p.

Details of this result are fully described in [3]. (In fact, i) is not discussed in [3], but it is deduced from the description in [3, §8] of $F_k(\tau)$ in terms of orthogonal polynomials.)

3. Thirdly, we introduce Atkin's beautiful orthogonal polynomials which he discovered in the mid 80s. He defines, on the space of all modular functions (weight 0) holomorphic on $\mathfrak{H}$ and having pole at ∞, an inner product with respect to which the Hecke operators are Hermitian. This inner product, being non-degenerate, gives us a set of orthogonal polynomials $\{A_n(j)\}_{n\geq 0}$ (normalized to be monic) in $j = j(\tau)$ because the space under consideration is identified with the polynomial ring $\mathbb{C}[j]$. Specifically, this inner product is defined as

$$(f(\tau), g(\tau)) = \text{constant term of } f(\tau)g(\tau)E_2(\tau) \text{ as a Laurent series in } q$$

and the first few of $A_n(j)$ are

$$\begin{aligned}
A_0(j) &= 1\,,\\
A_1(j) &= j - 720\,,\\
A_2(j) &= j^2 - 1640j + 269280\,,\\
A_3(j) &= j^3 - \frac{12576}{5}j^2 + 1526958j - 107765856\,,\\
A_4(j) &= j^4 - 3384j^3 + 3528552j^2 - 1133263680j + 44184000960\,.
\end{aligned}$$

Atkin's theorem is then stated as

Theorem 4 (Atkin). *Let p be any prime and n_p be the number of (isomorphism classes of) supersingular elliptic curves in characteristic p. Then the polynomial $A_{n_p}(j)$ has p-integral rational coefficients and the roots in $\bar{\mathbb{F}}_p$ of its reduction modulo p are exactly the j-invariants of all the supersingular elliptic curves in characteristic p.*

In [3], we gave two proofs which are "modular" and "hypergeometric" in nature. However, it still seems, at least to the author, unclear why this theorem should hold. While the existence of the Atkin polynomials solely depends on that of the Hecke operators of weight 0 on $\mathbb{C}[j]$, the arithmetic-geometric meaning of the Hecke operators of weight 0 is not as much expounded as that of positive weight, and this may be the cause of opacity. Even strange is that the Atkin polynomials also have the same property of zeros in characteristic 0 as $E_k(\tau)$ and $F_k(\tau)$.

Theorem 5. *All the zeros of $A_n(j)$ are real and lie in the interval* $(0, 1728)$.

Although we did not explicitly mention this fact in [3], it follows immediately from another description of Atkin's inner product as an integration on the interval $[0, 1728]$ (hence the inner product is positive definite on $\mathbb{R}[j]$) and the standard theory of orthogonal polynomials.

4. Finally, we quote a theorem from [1] which supplies another example of our zero-property theorems, and propose a couple of questions. Consider the polynomials $\{H_n(j)\}_{n\geq 1}$ in $j = j(\tau)$ obtained from $j - 720\,(= A_1(j))$ by applying the Hecke operators:

$$H_n(j(\tau)) = n\,\left(j(\tau) - 720\right)|_0 T(n) \quad (n = 1, 2, 3, \ldots),$$

where $|_0 T(n)$ stands for the action of the n-th Hecke operator of weight 0.

Theorem 6 (Asai-Ninomiya-Kaneko). *All the zeros of $H_n(j)$ are real and lie in the interval* $(0, 1728)$.

We would like to stress that the theorem was proven again by an ad hoc manner (although natural in a sense, it does not apply to, say $A_n(j)$). Fundamental problem is then

Problem 1. *Find a unified approach to theorems 1-6.*

We remark here that, in an earlier work by R. Rankin [4], it was essentially shown that if the Atkin inner product is *not* positive definite on $\mathbb{R}[j]$, then the Eisenstein series $E_k(\tau)$ fails to have the property in Theorem 1 for all sufficiently large weight k. Does this suggest a "unified approach"?

The following is a more specific question.

Problem 2. *What is the value of the Atkin inner product $(H_m(j), A_n(j))$? In particular, is the value always non-negative?*

An explanation may be needed. The Atkin inner product can be regarded as a weight 0 counterpart of the Petersson inner product. The Atkin polynomials are then the "Hecke eigen forms" in weight 0 and especially $A_n(j)$ for $n \geq 1$ are "cusp forms" whereas $A_0(j) = 1$ is regarded as the normalized Eisenstein series of weight 0. On the other hand, the polynomials $\{H_n(j)\}_{n\geq 1}$ (which are in the space of "cusp forms" spanned by $\{A_n(j)\}_{n\geq 1}$) are considered to be the Poincaré series (see *e.g.* [2]). In the classical case of positive weight, corresponding values of Petersson inner products of Hecke eigen cusp form and Poincaré series are always positive, which seems to be the case also for weight 0 by a numerical check.

Of course, we may pursue generalizations of all the material discussed so far to the case of any congruence subgroups.

Problem 3. *Find any generalizations of the above theorems to the case of congruence subgroups.*

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