Poly-Bernoulli numbers

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Abstract. By using polylogarithm series, we define "poly-Bernoulli numbers" which generalize classical Bernoulli numbers. We derive an explicit formula and a duality theorem for these numbers, together with a von Staudt-type theorem for di-Bernoulli numbers and another proof of a theorem of Vandiver.

For every integer $k$, we define a sequence of rational numbers $B_n^{(k)} (n = 0, 1, 2, \cdots)$, which we refer to as poly-Bernoulli numbers, by

$$
\left. \frac{1}{z} \ln_k (z) \right|_{z = 1 - e^{-z}} = \sum_{n=0}^{\infty} \frac{B_n^{(k)} x^n}{n!}.
$$

Here, for any integer $k$, $\ln_k (z)$ denotes the formal power series (for the $k$-th polylogarithm if $k \geq 1$ and a rational function if $k \leq 0$) $\sum_{m=1}^{\infty} z^m / m^k$. When $k = 1$, $B_n^{(1)}$ is the usual Bernoulli number (with $B_1^{(1)} = 1/2$):

$$
\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n^{(1)} x^n}{n!},
$$

and when $k \geq 1$, the left-hand side of (1) can be written in the form of "iterated integrals":

$$
e^x \cdot \frac{1}{e^x - 1} \int_0^x \frac{1}{e^t - 1} \int_0^t \cdots \int_0^{t_1} \frac{t}{e^{t_1} - 1} dt_1 \cdots dt_k \cdot \cdots = \sum_{n=0}^{\infty} \frac{B_n^{(k)} x^n}{n!}.
$$

Manuscrit reçu le 17 février 1994.
In this paper, we give both an explicit formula for $B_n^{(k)}$ in terms of the Stirling numbers of the second kind and a sort of duality for negative index poly-Bernoulli numbers. Both formulas are elementary, and in fact almost direct consequences of the definition and properties of the Stirling numbers. As applications, we prove a von Staudt-type theorem for di-Bernoulli numbers ($k = 2$) and give an alternative proof of a theorem due to Vandiver on a congruence for $B_n^{(1)}$.

1. Explicit formula and duality

An explicit formula for $B_n^{(k)}$ is given by the following:

**Theorem 1.**

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! S(n, m)}{(m + 1)^k} \quad (n \geq 0, \ \forall k),$$

where

$$S(n, m) = \frac{(-1)^m}{m!} \sum_{\ell=0}^{m} (-1)^\ell \binom{m}{\ell} \ell^n$$

is the Stirling number of the second kind.

**Remark.** When $k = 1$, the theorem and its many variants are classical results in the study of Bernoulli numbers (cf. [1]).

Because the Stirling numbers are integers, we see from the formula that $B_n^{(k)}$ for $k \leq 0$ is an integer (actually positive, as demonstrated in the remark at the end of this section).

**Theorem 2.** For any $n, k \geq 0$, we have

$$B_n^{(-k)} = B_k^{(-n)}.$$
\begin{equation}
S(n, m) = \frac{(-1)^m}{m!} \sum_{\ell=0}^{m} (-1)^\ell \binom{m}{\ell} \ell^n,
\end{equation}

\begin{equation}
\frac{(e^t - 1)^m}{m!} = \sum_{n=m}^{\infty} S(n, m) \frac{t^n}{n!}.
\end{equation}

For other definitions, further properties and proofs, we refer to [3].

Now Theorem 1 is readily derived from the definition (1) and the formula (4). In fact,

\[
\left. \frac{1}{z} \operatorname{Li}_k(z) \right|_{z=1-e^{-z}} = \sum_{m=0}^{\infty} \frac{(1 - e^{-z})^m}{(m + 1)^k} = \sum_{m=0}^{\infty} \frac{m!}{(m + 1)^k} \sum_{n=m}^{\infty} (-1)^m S(n, m) \frac{(-z)^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \frac{(-1)^m m! S(n, m)}{(m + 1)^k} \right) \frac{(-z)^n}{n!}.
\]

Hence the theorem follows.

To prove Theorem 2, we calculate the generating function of $B_n^{(-k)}$:

\[
\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} B_n^{(-k)} \frac{x^n y^k}{n! k!} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (1 - e^{-z})^m (m + 1)^k \frac{y^k}{k!} = \sum_{m=0}^{\infty} (1 - e^{-z})^m e^{(m+1)y} = \frac{e^z y}{e^x + e^y - e^{z+y}}.
\]

The last expression being symmetric in $x$ and $y$ yields Theorem 2.

REMARK. Since

\[
\frac{e^{x+y}}{e^x + e^y - e^{x+y}} = \frac{e^{x+y}}{1 - (e^x - 1)(e^y - 1)} = e^{x+y} (1+(e^x-1)(e^y-1)+((e^x-1)(e^y-1))^2+\cdots),
\]

the number $B_n^{(-k)}$ ($k \geq 0$) is always positive.
2. Denominators of di-Bernoulli numbers

Using Theorem 1, we can completely determine the denominator of di-Bernoulli numbers as follows.

**Theorem 3.**

(1) When $n$ is odd, $B_{n}^{(2)} = -\frac{(n-2)}{4} B_{n-1}^{(1)}$ ($n \geq 1$). (Hence the description of the denominator reduces to the classical Clausen-von Staudt theorem.)

(2) When $n$ is even ($\geq 2$), the $p$-order $\text{ord}(p,n)$ of $B_{n}^{(2)}$ for a prime number $p$ is given as follows.

(a) $\text{ord}(p,n) \geq 0$ if $p > n + 1$.

(b) For $5 \leq p \leq n + 1$, we have:

(i) $\text{ord}(p,n) = -2$ if $p - 1 \nmid n$.

(ii) If $p - 1 \nmid n$, then:

(A) $\text{ord}(p,n) \geq 0$ if $p \mid \frac{B_{n}}{n}$, or $n \equiv n' \mod p(p - 1)$ for some $1 < n' < p - 1$.

(B) $\text{ord}(p,n) = -1$ otherwise.

(c) $\text{ord}(3,n) \geq 0$ if $n \equiv 2 \mod 3$ and $n > 2$. Otherwise $\text{ord}(3,n) = -2$.

(d) $\text{ord}(2,n) \geq 0$ if $n \equiv 2 \mod 4$ and $n > 2$. $\text{ord}(2,n) = -1$ if $n \equiv 0 \mod 4$. $\text{ord}(2,2) = -2$.

Before proving the theorem, we establish the following lemma, which will be needed in the proof.

**Lemma 1.** Assume $n \geq 2$ is even and $p \geq 5$ is a prime number such that $m + 1 = 2p$. Then

$$(-1)^{m}m! S(n,m) \equiv 0 \mod p^{2},$$

and hence $(-1)^{m}m! S(n,m)/(m+1)^{2}$ is $p$-integral.

**Proof.** By (3),

$$(-1)^{m}m! S(n,m) = \sum_{\ell=1}^{2p-1} (-1)^{\ell} \binom{2p-1}{\ell} \ell^{n}$$

$$= \sum_{\ell=1}^{p-1} \left\{ (-1)^{\ell} \binom{2p-1}{\ell} \ell^{n} + (-1)^{2p-\ell} \binom{2p-1}{2p-\ell} (2p-\ell)^{n} \right\}$$

$$+ (-1)^{p} \binom{2p-1}{p} p^{n}$$

$$\equiv \sum_{\ell=1}^{p-1} \left\{ (-1)^{\ell} \binom{2p-1}{\ell} \ell^{n} + (-1)^{\ell} \binom{2p-1}{\ell-1} (-2n \ell^{n-1} + \ell^{n}) \right\} \mod p^{2}.$$
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Since
\[
\binom{2p - 1}{\ell} + \binom{2p - 1}{\ell - 1} = \frac{2p}{\ell} \binom{2p - 1}{\ell - 1},
\]
the last sum is equal to
\[
2p(1 - n) \sum_{\ell=1}^{p-1} (-1)^{\ell} \binom{2p - 1}{\ell - 1} \ell^{n-1}.
\]
Noting that
\[
\binom{2p - 1}{\ell - 1} \equiv (-1)^{\ell-1} \mod p,
\]
we see that
\[
\sum_{\ell=1}^{p-1} (-1)^{\ell} \binom{2p - 1}{\ell - 1} \ell^{n-1} \equiv - \sum_{\ell=1}^{p-1} \ell^{n-1} \equiv 0 \mod p,
\]
because \( p - 1 \nmid n - 1 \) (recall that \( n \) is even and \( p \) is odd). This proves the lemma.

**PROOF OF THEOREM 3.** 1. Let \( B_n = B_n^{(1)} \) for \( n \neq 1 \) and \( B_1 = -1/2 \). Then \( \sum_{n=0}^{\infty} B_n x^n/n! = x/(e^x - 1) \). By (2) in the introduction, we have
\[
\sum_{n=0}^{\infty} B_n^{(2)} \frac{x^n}{n!} = \frac{e^x}{e^x - 1} \int_0^\infty \sum_{\ell=0}^{\infty} B_\ell \frac{x^\ell}{\ell!} dt = \sum_{m=0}^{\infty} B_m^{(1)} \frac{x^m}{m!} \cdot \sum_{\ell=0}^{\infty} B_\ell \frac{x^\ell}{(\ell + 1)!}.
\]
From this we see that
\[
B_n^{(2)} = \sum_{\ell=0}^{n} \binom{n}{\ell} B_{n-\ell}^{(1)} \frac{B_\ell}{\ell + 1}.
\]
Since \( B_\ell^{(1)} = B_\ell = 0 \) for odd \( \ell \geq 3 \), we have for odd \( n \)
\[
B_n^{(2)} = \frac{n}{2} B_{n-1}^{(1)} B_1 + B_1^{(1)} B_{n-1} = -\frac{(n-2)}{4} B_{n-1}^{(1)}.
\]
2. We make use of Theorem 1. Part (a) is obvious because the Stirling numbers in the formula in Theorem 1 are integers. For the remainder of the proof, first we note that the expression \( m!/(m+1)^2 \) in the summand of the formula is an integer except when \( m+1 = 8, 9, \) a prime number, or \( 2 \times \) a prime number, as can be checked in an elementary way. Now, Lemma 1 tells us that any prime number \( p \geq 5 \) satisfying \( m+1 = 2p \) does not appear in the denominator of \( B_n^{(2)} \).
Our next task is to consider the case \( m + 1 = p \), where \( p \) is a prime number \( \geq 5 \). In this case

\[
(-1)^m m! S(n, m) = \sum_{\ell=1}^{p-1} (-1)^{\ell} \left( \frac{p - 1}{\ell} \right) \ell^n.
\]

The righthand side is congruent modulo \( p \) to \(-1\) if \( p - 1 \mid n \) and to 0 if \( p - 1 \nmid n \). Thus if \( p - 1 \nmid n \), the \( p \)-order of \((-1)^m m! S(n, m)/(m + 1)^2\) is \(-2\).

Since the other summands in the formula in Theorem 1 are \( p \)-integral, we have shown part (b)-i. Suppose \( p - 1 \nmid n \) and calculate modulo \( p^2 \). Using

\[
\left( \frac{p - 1}{\ell} \right) \equiv (-1)^{\ell} + (-1)^{\ell-1} p \sum_{i=1}^{\ell} \frac{1}{i} \mod p^2,
\]

we see that

\[
\sum_{\ell=1}^{p-1} (-1)^{\ell} \left( \frac{p - 1}{\ell} \right) \ell^n \equiv \sum_{\ell=1}^{p-1} \ell^n - p \sum_{i=1}^{\ell} \frac{1}{i} \mod p^2.
\]

It is known that (cf. Cor. of Prop. 15.2.2 in [2]) if \( n \) is even and \( p - 1 \nmid n \), then

\[
\sum_{\ell=1}^{p-1} \ell^n \equiv p B_n^{(1)} \mod p^2.
\]

On the other hand, when we put \( n \mod p - 1 = n', 1 < n' < p - 1 \) (since both \( n \) and \( p - 1 \) are even, \( n' \) is also even), we find

\[
\sum_{\ell=1}^{p-1} \ell^n \sum_{i=1}^{\ell} \frac{1}{i} \equiv \sum_{\ell=1}^{p-1} \ell^{n'} \sum_{i=1}^{\ell} \frac{1}{i} \mod p
\]

\[
\equiv B_{n'}^{(1)} \mod p \ (\text{see (63) of Vandiver [4] and Section 3 below}).
\]

We therefore have

\[
(-1)^m m! S(n, m) \equiv p (B_n^{(1)} - B_{n'}^{(1)}) \mod p^2,
\]

where \( m + 1 = p \) and \( n' \equiv n \mod p - 1, 1 < n' < p - 1 \). Since \( p - 1 \nmid n \), the number \( B_n^{(1)} / n \) is \( p \)-integral and \( B_{n'}^{(1)} \equiv n' B_n^{(1)} / n \mod p \) (Prop. 15.2.4 and Th.5 following it in [2]). Thus

\[
(-1)^m m! S(n, m) \equiv p (n - n') \frac{B_n^{(1)}}{n} \mod p^2.
\]

This readily gives part (b)-ii of the theorem.

The only summands in Theorem 1 which may not be 3-integral are \( 2! S(n, 2)/3^2 \), \(-5! S(n, 5)/6^2 \), and \( 8! S(n, 8)/9^2 \). By direct calculation using the formula (3), we obtain part (c). In a similar manner, we can determine the 2-order as well, but we omit the details here.
3. A theorem of Vandiver

As an application of Theorems 1 and 2, we prove the following proposition originally due to Vandiver.

**PROPOSITION.** Let $p$ be an odd prime number. For $1 \leq i \leq p - 2$,

$$B_i^{(1)} \equiv \sum_{m=1}^{p-2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right)(m+1)^i \mod p.$$  

**PROOF.** By Theorem 1 and Fermat's little theorem, we see that

$$B_i^{(1)} \equiv B_i^{(2-p)} \mod p.$$  

Theorem 2 says that the righthand side is equal to $B_p^{(-i)}$, which by Theorem 1 is equal to $-\sum_{m=0}^{p-2} (-1)^m m! S(p-2, m) (m+1)^i$.

**LEMMA 2.** Suppose $p$ is an odd prime, and $1 \leq m \leq p - 2$. Then

$$(-1)^{m-1} m! S(p - 2, m) \equiv 1 + \frac{1}{2} + \cdots + \frac{1}{m} \mod p.$$  

**PROOF.** The Stirling numbers satisfy the recurrence formula

$$S(n, m) = S(n-1, m-1) + mS(n-1, m) \ (n \geq 1) \ \text{(see [3])}.$$  

Thus if we put $(-1)^{m-1} m! S(p - 2, m) = b_m$, we get

$$(-1)^{m-1} m! S(p - 1, m) = m(-b_{m-1} + b_{m}) \ (m \geq 2).$$

But by (3),

$$(-1)^{m-1} m! S(p - 1, m) = -\sum_{\ell=1}^{m} (-1)^{\ell} \binom{m}{\ell} \ell^{p-1} = -\sum_{\ell=1}^{m} (-1)^{\ell} \binom{m}{\ell} \mod p$$

$$\equiv 1 \mod p,$$

and we thus conclude that

$$b_m \equiv b_{m-1} + \frac{1}{m} \mod p.$$  

This together with the relation $b_1 = S(p - 2, 1) = 1$ gives the lemma and hence completes the proof of the proposition.

**REMARK.** If $i > 1$, the righthand side of the proposition is congruent modulo $p$ to

$$\sum_{m=1}^{p-1} \left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right)m^i,$$
and this being congruent to $B_i$ (even when $i = 1$) is a special case of Vandiver's congruence (63) in [4].

Acknowledgement

The present paper was written during the author's stay in 1993 at the university of Cologne in Germany, as a research fellow of the Alexander von Humboldt foundation. He would like to thank the foundation and his host professor Peter Schneider for their hospitality and support.

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