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PRO-l PURE BRAID GROUPS OF RIEMANN SURFACES AND GALOIS REPRESENTATIONS[1]

To the memory of the late Professor Michio Kuga

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Introduction

Let X be a smooth irreducible algebraic curve of genus g over a field k of characteristic 0, and l be a prime number. For each $n=1, 2, \cdots$, consider the configuration space

$$Y_n = F_{0,n}X = \{(p_1, \cdots, p_n)\in X^n; p_i \neq p_j \quad \text{for} \quad i\neq j\} .$$

Then the Galois group $\mathrm{Gal}(\bar{k}/k)$ acts outerly on the pro-l fundamental group $P_n=\pi_1^{pro\text{-}l}(Y_n)$;

$$\varphi_n: \mathrm{Gal}(\bar{k}/k) \to \mathrm{Out}\, P_n .$$

The main purpose of this paper is to prove that φ_n has the same kernel for all sufficiently large $n\geq n_0=n_0(X/k, l)$ (Theorem 2, §4). For example, we can take $n_0=1$ if $g\geq 1$ and X is affine, $n_0=2$ if $g\geq 1$, and $n_0=4$ in all cases. This theorem is based on some group theoretic property of Out P_n (Theorem 1, §1).

The present work grew out of our previous work [7], [8] and [6].

1. The statement of Theorem 1

1.1. Let X^{cpt} be a compact Riemann surface of genus $g\geq 0$, and $X=X^{cpt}\backslash \{a_1, \cdots, a_r\}$ $(r\geq 0)$ be the complement of r distinct points $a_1, \cdots, a_r$ in X^{cpt}. For each integer $n\geq 1$, consider the configuration space

$$Y_n = F_{0,n}X = \{(p_1, \cdots, p_n)\in X^n; p_i \neq p_j \quad \text{for} \quad i\neq j\} ,$$

and let $\pi_1(Y_n, b)$ be its fundamental group with a base point $b=(b_1, \cdots, b_n)$. It is the pure braid group of X with n strands. For each i $(1\leq i\leq n, n\geq 2)$, the projection

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(1.1.1) $$Y_n \ni (p_1, \cdots, p_n) \to (p_1, \cdots, \check{p}_i, \cdots, p_n) \in Y_{n-1}$$

is a locally trivial topological fibering (cf. [2], §1.2).

It induces a short homotopy exact sequence

(1.1.2) $$\begin{aligned} 1 &\to \pi_1(X \setminus \{b_1, \cdots, \check{b}_i, \cdots, b_n\}, b_i) \to \pi_1(Y_n, b) \\ &\to \pi_1(Y_{n-1}, (b_1, \cdots, \check{b}_i, \cdots, b_n)) \to 1 , \end{aligned}$$

because (i) the fiber of (1.1.1) above $(b_1, \cdots, \check{b}_i, \cdots, b_n)$ can be identified with $X \setminus \{b_1, \cdots, \check{b}_i, \cdots, b_n\}$ which is connected, and (ii) $\pi_2(Y_{n-1}) = \{1\}$ ([2], Prop. 1.3).

For each i $(1 \leq i \leq n)$, the group $\pi_1(X \setminus \{b_1, \cdots, \check{b}_i, \cdots, b_n\}, b_i)$ is generated by the elements $x_j^{(i)}, y_j^{(i)}, z_k^{(i)}$ $(1 \leq j \leq g,\ 1 \leq k \leq r+n,\ k \neq r+i)$ described by the loops in Fig. 1. These generators satisfy a single defining relation

(1.1.3) $$[x_1^{(i)}, y_1^{(i)}] \cdots [x_g^{(i)}, y_g^{(i)}]\, z_{r+n}^{(i)} \cdots \check{z}_{r+i}^{(i)} \cdots z_1^{(i)} = 1 .$$

It is free of rank $2g+r+n-2$. As is well-known, these elements $x_j^{(i)}, y_j^{(i)}, z_k^{(i)}$ for all i generate $\pi_1(Y_n, b)$ (with more relations than (1.1.3) for all i).

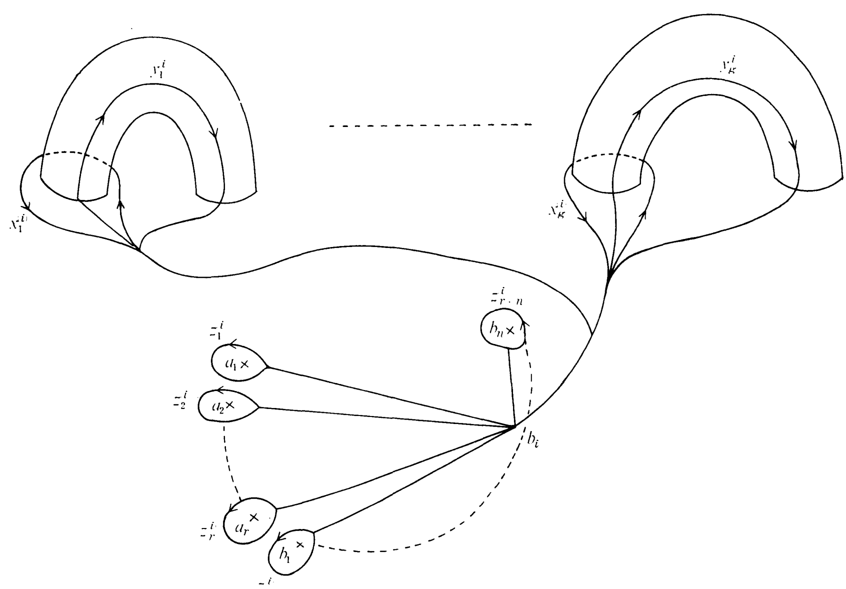

Figure 1

1.2. Now fix a prime number l, and pass to the pro-l completions. Call $N_n^{(i)}, P_n, P_{n-1}^{(i)} (\cong P_{n-1})$ the pro-l completions of the groups

(1.2.1) $\pi_1(X \setminus \{b_1, \cdots, \check{b}_i, \cdots, b_n\}, b_i),\ \pi_1(Y_n, b),\ \pi_1(Y_{n-1}, (b_1, \cdots, \check{b}_i, \cdots, b_n))$,

respectively. Then since the leftmost group of (1.2.1) is free, the exact sequence

(1.1.2) induces that of pro-l groups

$$1 \to N_n^{(i)} \to P_n \to P_n^{(i)} \to 1 \tag{1.2.2}$$

([1], Prop. 3; cf. also [6], Lemma 7.1.2). Call $N_n^{(i)}(2)$ the minimal closed normal subgroup of $N_n^{(i)}$ containing $[N_n^{(i)}, N_n^{(i)}]$ (the closure of the algebraic commutator) together with all the $z_k^{(i)}(1 \leq k \leq r+n, k \neq r+i)$. Here and in what follows, we shall use the same notation (e.g., $z_k^{(i)}$) for an element of a group and its image in the pro-l completion. The notation $N_n^{(i)}(2)$ refers to a filtration defined later (§3.2).

When $i=n$, we shall often suppress the superscript (i) and write as $N_n = N_n^{(n)}$, $x_j = x_j^{(n)}$, etc.

1.3. Now assume

$$\begin{array}{llll} n \geq 2, & \text{if} & g \geq 1 \text{ and } r \geq 1, & \text{or} \\ & & g = 0 \text{ and } r \geq 3, & \\ n \geq 3, & \text{if} & g \geq 1 \text{ and } r = 0, & \text{or} \\ & & g = 0 \text{ and } r = 2, & \\ n \geq 4, & \text{if} & g = 0 \text{ and } r = 1, & \\ n \geq 5, & \text{if} & g = r = 0\,. & \end{array} \tag{1.3.1}$$

Our first main result is the following

Theorem 1. *Let n be as in* (1.3.1), *and σ be an automorphism of P_n which stabilizes N_n and induces an inner automorphism of $P_{n-1} \simeq P_n/N_n$. If σ satisfies moreover the following conditions* (σ1), (σ2), *then σ itself is an inner automorphism.*

(σ1) $\sigma(z_k^{(i)}) \sim z_k^{(i)}$ ($\sim$: *P_n-conjugacy*) *for all i, k* $(1 \leq i \leq n,\ 1 \leq k \leq r+n,\ k \neq r+i)$,

(σ2) *σ stabilizes $N_n^{(i)}$ and acts trivially on its quotient mod $N_n^{(i)}(2)$* $(1 \leq i \leq n)$.

Remark. We do not know whether our assumption (1.3.1) for n is the best possible; especially whether the theorem is still valid when $g \geq 2$, $r=0$, $n=2$.

2. Key lemmas for the proof of Theorem 1

2.1. The element $z = z_1^{(n)}$ will play a special role in the sequel. Note that the loop with base point b_n defining z (Fig. 1) is a "trip" around a_1 if $r>0$, but if $r=0$ it is a trip around b_1. Our proof of Theorem 1 will be based on the following two key lemmas. Here and in what follows, if $g_1, \cdots, g_r$ are elements of a topological group G, $\langle g_1, \cdots, g_r \rangle$ will denote the smallest closed subgroup of G containing $g_1, \cdots, g_r$.

Lemma A. *Let C be the centralizer of z in P_n. Then* (*i*) $P_n = C \cdot N_n$, (*ii*) $C \cap N_n = \langle z \rangle$.

Thus, $C \hookrightarrow P_n$ is close to giving a splitting of the projection $P_n \to P_n/N_n$. Put

$$W = \{x_j, y_j\ (1 \leq j \leq g),\quad z_k\ (2 \leq k \leq r+n-2)\} \subset N_n .$$

Note that $W \cup \{z\}$ is a set of free generators of N_n.

Lemma B. *For each $w \in W$, there exists a subset $S = S_w \subset P_n$ such that*

(i) $S \subset N_n^{(n-1)}$,

(ii) *the centralizer of S in $N_n = N_n^{(n)}$ is $\langle w, z \rangle$.*

2.2. Proof of Lemma A. To check (i) it suffices to show that if w is one of the generators $x_j^{(i)}, y_j^{(i)}, z_k^{(i)}$ of P_n then wzw^{-1} is conjugate to z by an element of $\pi_1(X \setminus \{b_1, \cdots, b_{n-1}\}, b_n)$ $(\subset N_n)$. The following explicit formula for wzw^{-1} proves its validity.

$$\begin{aligned} &wzw^{-1} = n(w)\, zn(w)^{-1}, \quad \text{where} \\ &\quad n(x_j^{(n)}) = x_j^{(n)},\ n(y_j^{(n)}) = y_j^{(n)} \quad (1 \leq j \leq g), \\ &\quad n(z_k^{(n)}) = z_k^{(n)} \quad (1 \leq k \leq r+n-1), \\ &\quad n(z_1^{(i)}) = (z_{r+i}^{(n)}\, z_1^{(n)})^{-1}, \quad n(z_{r+n}^{(i)}) = z_{r+i}^{(n)} \quad (1 \leq i \leq n-1), \\ &\quad n(w) = 1, \quad \text{for all other } w. \end{aligned}$$

This settles the proof of (i). The statement (ii) is obvious, as z can be chosen to be one of the free generators of N_n.

2.3. Reducing Lemma B to Lemma B'. For each $w \in W$, call $\alpha(w)$

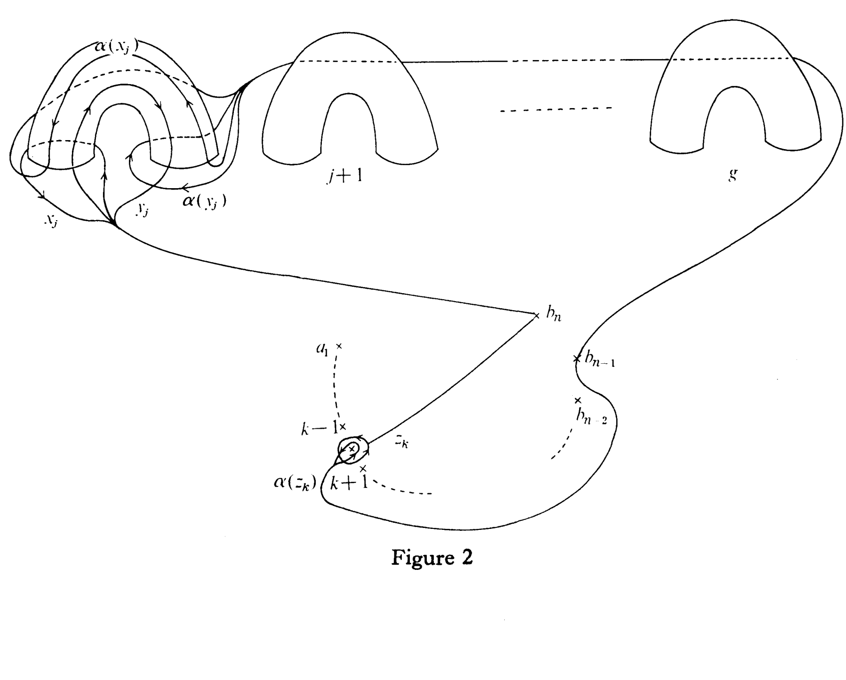

Figure 2

the element of

$$\pi_1(X\backslash\{b_1, \cdots, b_{n-2}, b_n\}, b_{n-1}) \ (\subset N_n^{(n-1)})$$

defined by the loop described in Fig. 2.

It is clear that $\alpha(w)$ commutes with z and also with any $w'\in W$, $w'\neq w$.

Lemma B'. *The centralizer of $\alpha(w)$ in $N_n=N_n^{(n)}$ is precisely $\langle W\backslash\{w\}, z\rangle$.*

We shall reduce Lemma B to Lemma B'. Assume Lemma B', and set

$$S_w = \{\alpha(w'); w'\in W, w'\neq w\} .$$

Then $S_w \subset N_n^{(n-1)}$, and

$$\text{the centralizer of } S_w \text{ in } N_n = \bigcap_{\substack{w'\in W\\ w'\neq w}} \langle W\backslash\{w'\}, z\rangle = \langle w, z\rangle ,$$

which implies Lemma B. The last equality is because $W\cup\{z\}$ is a set of free generators of N_n (see Cor. of Lemma 2.4.2, §2.4). Thus, Lemma B is reduced to Lemma B'.

2.4. Proof of Lemma B' We know that N_n is free on $W\cup\{z\}$. Let $\tau=\tau_w$ denote the automorphism of N_n defined by the outer $\alpha(w)$-conjugation

$$\tau\colon \nu \to \alpha(w)\,\nu\,\alpha(w)^{-1} \quad (\nu\in N_n) .$$

We know that

$$\tau(w') = w' , \quad w'\in W\backslash\{w\}$$
$$\tau(z) = z ,$$

and our task is to show that $N_n^\tau=\langle W\backslash\{w\}, z\rangle$ (N_n^τ: the τ-invariant elements of N_n; the inclusion $\supset$ is obvious). So, what we do is to write down $\tau(w)$ explicitly and, using the "difference" between $\tau(w)$ and w, to show that the τ-invariant elements of N_n cannot "contain" w.

First we prove the case $w=x_j$. (The case $w=y_j$ is essentially the same and will be omitted.)

The effect of τ on N_n is given by

$$\tau(x_j) = x_j\,\Delta_j\,z_{r+n-1}\,\Delta_j^{-1} \quad (\Delta_j = y_j\,x_j^{-1}\,y_j^{-1}\,[x_{j+1}, y_{j+1}]\cdots[x_g, y_g]) ,$$
$$\tau(w) = w \quad (w\in W\backslash\{x_j\}\cup\{z\}) .$$

Fix an isomorphism of the completed group algebra $\boldsymbol{Z}_l[[N_n]]$ of N_n over the ring of l-adic integers $\boldsymbol{Z}_l$ and the noncommutative power series algebra $\Lambda=\boldsymbol{Z}_l[[X_1, \cdots, X_g, Y_1, \cdots, Y_g, Z_1, \cdots, Z_{r+n-2}]]_{n.c}$ over $\boldsymbol{Z}_l$ with $2g+r+n-2$ indeterminates such that

$$x_j \leftrightarrow 1+X_j\,,\quad y_j \leftrightarrow 1+Y_j\,,\quad z_k \leftrightarrow 1+Z_k\,.$$

Here, we regard Λ as being equipped with the graduation which assigns X_j, Y_j $(1\le j\le g)$ degree 1 and $Z_k(1\le k\le r+n-2)$ degree 2. Extend τ to an automorphism of Λ. For each $m\ge 1$, let I_m denote the ideal of Λ consisting of all power series whose lowest degree is greater than or equal to m. Then the effect of τ on I_1/I_3 is

$$\begin{aligned}\tau(X_j) &= X_j-\sum_{k=1}^{g}(X_k Y_k-Y_k X_k)-\sum_{k=1}^{r+n-2} Z_k\,,\\ \tau(X_i) &= X_i\ (i\neq j)\,,\quad \tau(Y_i)= Y_i\ (1\le i\le g)\,,\\ \tau(Z_k) &= Z_k\ (1\le k\le r+n-2)\,.\end{aligned}$$

We claim that for every m

$$\{f\in I_m/I_{m+2}\,|\,\tau(f)=f\} = \left\{\begin{array}{l}\text{homogeneous elements of degree } m\\ \text{not containing } X_j\end{array}\right\} \oplus \left\{\begin{array}{l}\text{homogeneous elements}\\ \text{of degree } m+1\end{array}\right\}. \tag{2.4.1}$$

The inclusion $\supset$ is clear. Let $\{f_\mu\,|\,\mu\in M\}$ be the set of all monomials of degree m which contain X_j. (M is a finite set of indices.) It suffices to show that the elements $\tau(f_\mu)-f_\mu$ $(\mu\in M)$ are linearly independent over $\boldsymbol{Z}_l$. To show this, we proceed by double induction on the invariants $a(f_\mu)$ and $b(f_\mu)$ defined as follows. We define $a(f_\mu)$ to be the sum of degrees of indeterminates which do not lie left on the leftmost X_j in f_μ. The invariant $b(f_\mu)$ is defined to be the number of X_iY_i, Y_iX_i $(i\neq j)$ and $Z_k(1\le k\le r+n-2)$ which appear on the left of the leftmost X_j in f_μ. For example, when $j=1$ and $m=6$, $a(Y_1Y_2X_1Z_1X_1)=4$ (recall that deg $(Z_k)=2$), $a(X_2X_1^2Y_2^3)=5$, $a(X_1Z_1^2Y_1)=6$ etc., $b(Y_1Y_2X_1Z_1X_1)=0$, $b(X_2Y_2X_1Z_1Y_1)=1$, $b(Z_1X_2Y_2X_2X_1)=3$ etc. Assume that a relation

$$\sum_{\mu\in M} c_\mu(\tau(f_\mu)-f_\mu)=0\,,\quad c_\mu\in\boldsymbol{Z}_l\,,$$

holds. If $a(f_\mu)=1$ and $b(f_\mu)=0$, then $f_\mu=f'X_j$ where f' is of degree $m-1$ and does not contain X_j, X_iY_i, Y_iX_i $(i\neq j)$ nor $Z_k(1\le k\le r+n-2)$. For this we have

$$\tau(f_\mu)-f_\mu = -f'\,\{\sum_{\substack{l=1\\ i\neq j}}^{g}(X_iY_i-Y_iX_i)+\sum_{k=1}^{r+n-2}Z_k\}-f'X_jY_j+f'Y_jX_j\,.$$

Look at the term $f'Y_jX_j$. This can never be supplied by any other $\tau(f_{\mu'})-f_{\mu'}$ $(\mu'\in M)$. Hence we must have $c_\mu=0$ for such $\mu\in M$ that $a(f_\mu)=1$ and $b(f_\mu)=0$. Let $a>1$. Assume that $c_{\mu'}=0$ for all $\mu'\in M$ such that $a(f_{\mu'})<a$ and $b(f_{\mu'})=$

0. Let f_μ be an element with $a(f_\mu)=a$ and $b(f_\mu)=0$. Then we can write $f_\mu=f'X_jf''$ where f' does not contain X_j, X_iY_i, Y_iX_i $(i\neq j)$ nor $Z_k(1\leq k\leq r+n-2)$ and $\deg(f'')=a-1$. For this we have

$$\tau(f_\mu)-f_\mu=-f'\{\sum_{\substack{i=1\\ i\neq j}}^{g}(X_iY_i-Y_iX_i)+\sum_{k=1}^{r+n-2}Z_k\}f''-f'X_jY_jf''+f'Y_jX_jf''+\cdots.$$

The term $f'Y_jX_jf''$ cannot be cancelled out by any other terms in $\tau(f_\mu)-f_\mu$ itself. If $c_\mu\neq 0$, the term $c_\mu(f'Y_jX_jf'')$ should be cancelled out by some term in another $c_{\mu'}(\tau(f_{\mu'})-f_{\mu'})$ $(\mu'\neq\mu)$. But then $f_{\mu'}$ must be of the form $f'Y_jX_jf'''$ with $\deg(f''')=a-2$. By the induction hypothesis we have $c_{\mu'}=0$, hence $c_\mu=0$. Thus we conclude by induction that $c_\mu=0$ for all $\mu\in M$ such that $b(f_\mu)=0$. Let $a\geq 1$, $b>0$ and assume that $c_\mu=0$ for all $\mu\in M$ such that either

$$a(f_\mu)>a \quad\text{and}\quad b(f_\mu)=b-1$$

or

$$a(f_\mu)=a-1 \quad\text{and}\quad b(f_\mu)=b\,.$$

Let f_μ be an element such that $a(f_\mu)=a$, $b(f_\mu)=b$ and write $f_\mu=f'X_jf''$, $\deg(f'')=a-1$. Then

$$\tau(f_\mu)-f_\mu=-f'\{\sum_{\substack{i=1\\ i\neq j}}^{g}(X_iY_i-Y_iX_i)+\sum_{k=1}^{r+n-2}Z_k\}f''-f'X_jY_jf''+f'Y_jX_jf''+\cdots.$$

The term $f'Y_jX_jf''$ can appear in another $\tau(f_{\mu'})-f_{\mu'}$ only if $f_{\mu'}$ is of the form $f'Y_jX_jf''$ or $f_{\mu'}$ is such that $a(f_{\mu'})>a$ and $b(f_{\mu'})=b-1$. By the induction hypothesis, we conclude that $c_\mu=0$. This settles the proof of the claim (2.4.1).

Now if an element $\nu\in N_n$ is fixed by τ, then by the claim above we have

$$\nu-1\in \boldsymbol{Z}_l[[X_1,\cdots,\check{X}_j,\cdots,X_g,Y_1,\cdots,Y_g,Z_1,\cdots,Z_{r+n-2}]]_{n.c}\,.$$

In particular

$$\nu-1\in\Lambda(X_1-1)+\cdots+\widecheck{(\Lambda(X_j-1))}+\cdots.$$

By Lemma 2.4.2 below we conclude from this that

$$\nu\in\langle x_1,\cdots,\check{x}_j,\cdots,x_g,y_1,\cdots,y_g,z_1,\cdots,z_{r+n-2}\rangle\,.$$

Lemma 2.4.2. *Let F be a free pro-l group of rank $r\geq 2$ with free generators $x_1,\cdots,x_r$ and Λ be its completed group algebra over $\boldsymbol{Z}_l$; $\Lambda=\boldsymbol{Z}_l[[F]]$. If $g\in F$ is such that*

$$g-1\in\Lambda(x_1-1)+\Lambda(x_2-1)+\cdots+\Lambda(x_s-1)$$

for some s $(1\leq s\leq r)$, then $g\in\langle x_1,\cdots,x_s\rangle$.

Proof. Let $H=\langle x_1, \cdots, x_s\rangle$. Define $\boldsymbol{Z}_l[[F/H]]$, a topological left Λ-module as follows. For each finite quotient $F\to\bar{F}$ of F, let $\bar{H}$ denote the image of H. Consider $\boldsymbol{Z}_l[[\bar{F}/\bar{H}]]$ as a left $\bar{F}$-module, and take the limit $\boldsymbol{Z}_l[[F/H]]:=\lim \boldsymbol{Z}_l[[\bar{F}/\bar{H}]]$ which is a left Λ-module. Let v be the element of $\boldsymbol{Z}_l[[F/H]]$ corresponding to H. Then $x_i v=v$ i.e., $(x_i-1)v=0$ $(1\leq i\leq s)$. Therefore,

$$(g-1)\,v=\left(\frac{\partial g}{\partial x_1}(x_1-1)+\cdots+\frac{\partial g}{\partial x_s}(x_s-1)\right)v=0\,.$$

Therefore, $gv=v$, and hence $g\in H$.

Corollary. *Let F be as above. For $I\subset\{1, \cdots, r\}$, define $F_I=\langle x_i \mid i\in I\rangle$. Then $F_I\cap F_J=F_{I\cap J}$ $(I, J\subset\{1, \cdots, r\})$.*

This completes the proof in case of $w=x_j$.

As for $w=z_k$, we use the normal graduation of Λ, namely, every indeterminate has degree 1. The action of τ on N_n is given by

$$\tau(z)=\delta_k^{-1}z_k\,\delta_k\;(\delta_k=(z_{r+n-2}\cdots z_k)^{-1}z_{r+n-1}(z_{r+n-2}\cdots z_k))\,,$$
$$\tau(w)=w\quad(w\in W\cup\{z\}\setminus\{z_k\})\,.$$

Again extend τ to an automorphism of Λ. Let I be the augmentation ideal of Λ. Then τ keeps I^m and the effect of τ on I/I^3 is

$$\tau(X_j)=X_j\,,\quad\tau(Y_j)=Y_j\quad(1\leq j\leq g)\,,$$
$$\tau(Z_j)=Z_j\quad(j\neq k)\,,$$
$$\tau(Z_k)=Z_k+\sum_{j=1}^{r+n-2}(Z_jZ_k-Z_kZ_j)\,.$$

As before it suffices to show that for every m

$$\{f\in I^m/I^{m+2}\mid\tau(f)=f\}=\left\{\begin{array}{l}\text{homogeneous elements of degree } m\\ \text{not containing } Z_k\end{array}\right\}\oplus\left\{\begin{array}{l}\text{homogeneous elements}\\ \text{of degree } m+1\end{array}\right\}.$$

Let $\{f_\mu\mid\mu\in M\}$ be the set of all monomials of degree m which contain Z_k. We only need to show that the elements $\tau(f_\mu)-f_\mu$ $(\mu\in M)$ are linearly independent over $\boldsymbol{Z}_l$, and this will be established by single induction on the invariant $a(f_\mu)$ of f_μ defined as the number of indeterminates which do not lie left on the leftmost Z_k in f_μ. The argument is similar to that in the first step (case $b(f_\mu)=0$) of previous double induction in case $w=x_j$ and is omitted here.

3. Proof of Theorem 1

3.1. First, we need:

Claim 1. *Each inner automorphism σ of P_n satisfies $(\sigma 1)$, $(\sigma 2)$.*

Proof. It suffices to show that any inner automorphism of P_n acts trivially on $N_n^{(i)}/N_n^{(i)}(2)$. But P_n being generated by the $x_j^{(i)}, y_j^{(i)}, z_k^{(i)}$, it suffices to show that if w and w' belong to this set of generators of P_n and if $w\in N_n^{(i)}$ then $w'ww'^{-1}w^{-1}\in N_n^{(i)}(2)$. If either $w=x_j^{(i)}$ and $w'=y_j^{(k)}$ or $w=y_j^{(i)}$ and $w'=x_j^{(k)}$ $(k\neq i)$, $w'ww'^{-1}w^{-1}=[w', w]$ is given as follows and is contained in $N_n^{(i)}(2)$:

$$[y_j^{(k)}, x_j^{(i)}]=\begin{cases}(x_j^{(i)}y_j^{(i)^{-1}}z_{r+i-1}^{(i)}\cdots z_{r+k+1}^{(i)})\,z_{r+k}^{(i)^{-1}} \\ \quad\times(x_j^{(i)}y_j^{(i)^{-1}}z_{r+i-1}^{(i)}\cdots z_{r+k+1}^{(i)})^{-1} \quad (i>k)\\ (z_{r+k-1}^{(i)}\cdots z_{r+i+1}^{(i)}y_j^{(i)}x_j^{(i)^{-1}})^{-1}z_{r+k}^{(i)^{-1}} \\ \quad\times(z_{r+k-1}^{(i)}\cdots z_{r+i+1}^{(i)}y_j^{(i)}x_j^{(i)^{-1}}) \quad (i<k)\end{cases}$$

$$[x_j^{(k)}, y_j^{(i)}]=\begin{cases}[z_{r+i-1}^{(i)}\cdots z_{r+k+1}^{(i)}z_{r+k}^{(i)^{-1}}(z_{r+i-1}^{(i)}\cdots z_{r+k+1}^{(i)}), x_j^{(i)}] \\ \quad\times(y_j^{(i)}z_{r+i-1}^{(i)}\cdots z_{r+k+1}^{(i)})\,z_{r+k}^{(i)} \\ \quad\times(y_j^{(i)}z_{r+i-1}^{(i)}\cdots z_{r+k+1}^{(i)})^{-1} \quad (i>k)\\ (z_{r+k-1}^{(i)}\cdots z_{r+i+1}^{(i)}x_j^{(i)}y_j^{(i)^{-1}})\,z_{r+k}^{(i)} \\ \quad\times(z_{r+k-1}^{(i)}\cdots z_{r+i+1}^{(i)}x_j^{(i)}y_j^{(i)^{-1}})^{-1} \quad (i<k)\,.\end{cases}$$

In other cases, $w'ww'^{-1}$ is $N_n^{(i)}$-conjugate to w and hence $[w', w]\in[N_n^{(i)}, N_n^{(i)}]\subset N_n^{(i)}(2)$.

Now let g, r, n be as (1.3.1), and σ be an automorphism of P_n which stabilizes N_n, induces an inner automorphism of $P_{n-1}\cong P_n/N_n$, and satisfies the conditions $(\sigma 1)$, $(\sigma 2)$ of Theorem 1.

Claim 2. *We may assume that (i) $\sigma z=z$, (ii) σ acts trivially on P_n/N_n.*

Proof. Obvious, by $(\sigma 1)$, Claim 1 and Lemma A(i).

Let W be the subset of N_n defined in §2.1.

Claim 3. *For each $w\in W$, $\sigma w\in\langle w, z\rangle$.*

Proof. Let $S=S_w$ be the subset of $N_n^{(n-1)}$ in Lemma B. Then by Lemma B, it suffices to show that $\sigma w\in N_n$ and that σw centralizes S. As $\sigma N_n=N_n$, the first assertion is obvious. To prove the second, take any $s\in S$. By Claim 2, $\sigma z=z$ and σ acts trivially mod N_n. As $\sigma z=z$, we have $\sigma C=C$. But $S\subset C$ (Lemma B); hence $\sigma(s)\,s^{-1}\in C\cap N_n=\langle z\rangle$. On the other hand, as σ stabilizes also $N_n^{(n-1)}$, and $S\subset N_n^{(n-1)}$, we have $\sigma(s)\,s^{-1}\in N_n^{(n-1)}$. But $N_n^{(n-1)}\cap\langle z\rangle=\{1\}$, as can be checked easily by considering the geometric meaning of the projection of z on $P_n/N_n^{(n-1)}$. (This is where we need the assumption $n\geq 3$ if $r=0$, a part of (1.3.1).) Therefore, $\sigma s=s$ for all $s\in S$. Since w centralizes S, σw centralizes $\sigma S=S$. Therefore, $\sigma w\in\langle w, z\rangle$.

3.2. We shall use the invariance of the relation (1.1.3) by the action of σ, and the above Claim 3, to push σ nearer to 1. The method we employ is a pro-l Lie calculus. We shall suppress also the subscript n, and write often as

$N=N_n=N_n^{(n)}$, etc. We shall first look at the action of σ on N.

By (σ1), (σ2), we may put

$$\sigma x_j = s_j x_j, \quad \sigma y_j = t_j y_j \quad (1 \le j \le g),$$
$$\sigma z_k = u_k z_k u_k^{-1} \quad (2 \le k \le r+n-1),$$

with $s_j, t_j \in N(2)$ and $u_k \in N$ (cf. §2.2). By Claim 3,

(3.2.1) $$s_j \in \langle x_j, z \rangle, \quad t_j \in \langle y_j, z \rangle \quad (1 \le j \le g),$$

and

(3.2.2) $$u_k z_k u_k^{-1} \in \langle z_k, z \rangle \quad (2 \le k \le r+n-2).$$

From the last inclusion we shall deduce:

Claim 4.

$$u_k \in \langle z_k, z \rangle \quad (2 \le k \le r+n-2).$$

Proof. Consider the free differentiation w.r.t. the basis $x_1, \cdots, x_g, y_1, \cdots, y_g, z_1, \cdots, z_{r+n-2}$. Then for $w \in W$, $w \neq z_k$,

$$0 = \frac{\partial}{\partial w}(u_k z_k u_k^{-1}) = (1-u_k z_k u_k^{-1}) \frac{\partial u_k}{\partial w}.$$

Since the element $1-u_k z_k u_k^{-1}$ in $\boldsymbol{Z}_l[[N]]$ is not a zero divisor ([5], Lemma 3.1), we have $\frac{\partial u_k}{\partial w}=0$. From this and Lemma 2.4.2 we conclude that $u_k \in \langle z_k, z \rangle$.

Our next goal is to prove:

Claim 5. *σ acts trivially on N (In other terms, $s_j=t_j=u_k=1$, all j, k.).*

Proof. Assume first that $g>0$. Let $\{N(m)\}_{m \ge 1}$ be the central filtration of the group $N=N(1)=N_n$ which was defined and studied in [8]. It is the filtration such that

(i) the degrees of x_j and y_j $(1 \le j \le g)$ are 1 (i.e., $x_j, y_j \in N(1) \backslash N(2)$), and the degrees of $z_k (1 \le k \le r+n-1)$ are 2 ($z_j \in N(2) \backslash N(3)$),

(ii) the degree of a commutator $[x, y]$ is the sum of degrees of x and y.

We have $[N(m), N(n)] \subset N(m+n)$ and, in particular, $\mathrm{gr}^m N := N(m)/N(m+1)$ is a $\boldsymbol{Z}_l$-module. Under the commutator operation, the $\boldsymbol{Z}_l$-module

$$L := \mathrm{gr}\, N = \bigoplus_{m \ge 1} \mathrm{gr}^m N$$

has a structure of graded Lie algebra over $\boldsymbol{Z}_l$ and it was shown in [8] that L is *free* Lie algebra generated by

$$X_j = x_j \bmod N(2), \quad Y_j = y_j \bmod N(2) \quad (1 \le j \le g)$$

and

$$Z_k = z_k \bmod N(3) \quad (1 \leq k \leq r+n-2).$$

By the Magnus embedding

$$N \to \boldsymbol{Z}_l[[X_1, \cdots, X_g, Y_1, \cdots, Y_g, Z_1, \cdots, Z_{r+n-2}]]_{n.c} = \Lambda$$

of N into the non-commutative formal power series algebra ($x_j \mapsto 1+X_j$, $y_j \mapsto 1+Y_j$, $z_k \mapsto 1+Z_k$), $N(m)$ is mapped into $1+I_m$, where I_m is the ideal of Λ consisting of all power series whose lowest degree is at least m ($\deg(X_j)=\deg(Y_j)=1$, $\deg(Z_j)=2$), and $\mathrm{gr}^m N$ is identified with the $\boldsymbol{Z}_l$-module of homogeneous "Lie polynomials" of degree m. In particular $\bigcap_{m\geq 1} N(m)=1$. Hence in order to prove Claim 5, it suffices to show that the inclusions

$$(\sharp_m) \qquad s_j, t_j \in N(m+1) \ (1 \leq j \leq g), \quad u_k \in N(m) \ (2 \leq k \leq r+n-1)$$

hold for all $m \geq 1$. First, by the assumption (ii), ($\sharp_1$) holds. Suppose ($\sharp_m$) holds for some m and put

$$S_j = s_j \bmod N(m+2), \quad T_j = t_j \bmod N(m+2) \quad (1 \leq j \leq g)$$
$$U_k = u_k \bmod N(m+1) \quad (2 \leq k \leq r+n-1).$$

Then from (3.2.1) and Claim 4 we have

$$(3.2.3) \qquad \begin{aligned} &S_j \in \langle X_j, Z_1\rangle, \quad T_j \in \langle Y_j, Z_1\rangle \quad (1 \leq j \leq g) \\ &U_k \in \langle Z_k, Z_1\rangle \quad (2 \leq k \leq r+n-2). \end{aligned}$$

Here, $\langle X_j, Z_1\rangle$ (resp. $\langle Y_j, Z_1\rangle$, $\langle Z_k, Z_1\rangle$) is the Lie subalgebra of L generated by X_j (resp. Y_j, Z_k) and Z_1.
By letting σ act on the relation

$$[x_1, y_1]\cdots[x_g, y_g]\, z_{r+n-1}\cdots z_2\, z_1 = 1$$

and considering it modulo $N(m+3)$, we get the following relation in L;

$$\sum_{j=1}^{g}([S_j, Y_j]+[X_j, T_j])+\sum_{k=2}^{r+n-1}[U_k, Z_k] = 0.$$

Write V for U_{r+n-1}. Since $Z_{r+n-1}=-\sum_{j=1}^{g}[X_j, Y_j]-\sum_{k=1}^{r+n-2}Z_k$ in $\mathrm{gr}^2 N$, the above relation can be rewritten as

$$(3.2.4) \qquad \begin{aligned} &\sum_{j=1}^{g}([S_j, Y_j]+[X_j, T_j])+\sum_{k=2}^{r+n-2}[U_k, Z_k] \\ &= [V, \sum_{j=1}^{g}[X_j, Y_j]+\sum_{k=1}^{r+n-2}Z_k]. \end{aligned}$$

We first show that ($\sharp_m$) holds for some m with $m \geq 3$. Let $m=1$. Then by

(3.2.3) we have

$$S_j = a_j Z_1\,,\quad T_j = b_j Z_1 \quad (1 \leq j \leq g)\,,\quad U_k = 0 \quad (2 \leq k \leq r+n-2)\,,$$

and

$$V = \sum_{j=1}^{g} (c_j X_j + d_j Y_j) \quad \text{with} \quad a_j, b_j, c_j, d_j \in \boldsymbol{Z}_l\,.$$

Putting these into (3.2.4) and noting that the elements $[Z_1, Y_j]$, $[X_j, Z_1]$, $[X_k, [X_j, Y_j]]$, $[Y_k, [X_j, Y_j]]$ $(1 \leq j, k \leq g)$ constitute a part of a $\boldsymbol{Z}_l$-basis in $\mathrm{gr}^3 N$, we conclude that $a_j = b_j = c_j = d_j = 0$; hence $(\sharp_2)$ holds. Suppose $m=2$. This time there exist by (3.2.3) $a_j, b_j, c_k, d_k \in \boldsymbol{Z}_l$ such that

$$\begin{aligned} &S_j = a_j[Z_1, X_j]\,,\quad T_j = b_j[Z_1, Y_j] \quad (1 \leq j \leq g) \\ &U_k = c_k Z_k + d_k Z_1 \quad (2 \leq k \leq r+n-2)\,. \end{aligned}$$

Write $V = V_0 + \sum_{k=1}^{r+n-2} e_k Z_k$, where $e_k \in \boldsymbol{Z}_l$ and V_0 is a linear combinations of $[X_i, Y_k]$'s. Putting these into (3.2.4) we get

$$\begin{aligned} &\sum_{j=1}^{g} (a_j [[Z_1, X_j], Y_j] + b_j [X_j, [Z_1, Y_j]]) + \sum_{k=2}^{r+n-2} d_k [Z_1, Z_k] \\ &= [V_0, \sum_{i=1}^{g} [X_j, Y_j]] + [V_0, \sum_{k=1}^{r+n-2} Z_k] + [e_1 Z_1, \sum_{j=1}^{g} [X_j, Y_j]] \\ &\quad + [\sum_{k=2}^{r+n-2} e_k Z_k, \sum_{j=1}^{g} [X_j, Y_j]] + [\sum_{k=1}^{r+n-2} e_k Z_k, \sum_{k=1}^{r+n-2} Z_k] \end{aligned} \tag{3.2.5}$$

Since each term except $[V_0, \sum_{j=1}^{g} [X_j, Y_j]]$ contains some Z_k $(1 \leq k \leq r+n-2)$ and the elements $[[X_l, X_m], [X_j, Y_j]]$ $((l, m) \neq (j, j))$ constitute a part of $\boldsymbol{Z}_l$-basis in $\mathrm{gr}^4 N$ whose $\boldsymbol{Z}_l$-span never contains an element including Z_k, we must have $[V_0, \sum_{j=1}^{g} [X_j, Y_j]] = 0$. Hence $V_0 = f \sum_{j=1}^{g} [X_j, Y_j]$ with some $f \in \boldsymbol{Z}_l$. By replacing u_{r+n-1} by $u_{r+n-1} \cdot z_{r+n-1}^f$ $(z_{r+n-1} = ([x_1, y_1] \cdots [x_g, y_g])^{-1} (z_{r+n-2} \cdots z_1)^{-1})$ we may assume that $f=0$ (so $V_0 = 0$). Then the term $[\sum_{k=2}^{r+n-2} e_k Z_k, \sum_{j=1}^{g} [X_j, Y_j]]$ in the right hand side of (3.2.5), $[Z_k, [X_j, Y_j]]$ being a generator of $\mathrm{gr}^4 N$, must be zero and thus $e_k = 0$ for $2 \leq k \leq r+n-2$. Comparing the remaining terms, we easily conclude that

$$a_j = b_j = d_k = e_1 \quad (1 \leq j \leq g, 2 \leq k \leq r+n-2)\,.$$

Hence, by replacing σ by $\mathrm{Int}(z_1^{-e_1}) \cdot \sigma$ ($\mathrm{Int}(g)$ is the inner automorphism by an element g), we may assume $e_1 = 0$, i.e., $(\sharp_3)$ holds. When $m \geq 3$, Lemma 3.2.6 below shows that $(\sharp_{m+1})$ holds and by induction our proof of Claim 5 in case $g > 0$ is done.

Lemma 3.2.6. *Let L be a free Lie algebra over $\mathbf{Z}_l$ with free generators $X_1, \cdots, X_g$, $Y_1, \cdots, Y_g, Z_1, \cdots, Z_{r+n-2}$ equipped with a graduation such that* $\deg(X_j)=\deg(Y_j)=1 (1\leq j\leq g)$ *and* $\deg(Z_k)=2$ $(1\leq k\leq r+n-2)$. *Let* $S_j\in\langle X_j, Z_1\rangle$, $T_j\in\langle Y_j, Z_1\rangle$ $(1\leq j\leq g)$ *be homogeneous elements of degree $m+1$ and* $U_k\in\langle Z_k, Z_1\rangle$ $(2\leq k\leq r+n-2)$, $V\in L$ *be homogeneous elements of degree $m\geq 3$. Suppose that these elements satisfy the relation*

$$\sum_{j=1}^{g}([S_j, Y_j]+[X_j, T_j])+\sum_{k=2}^{r+n-2}[U_k, Z_k] = [V, \sum_{j=1}^{g}[X_j, Y_j]+\sum_{k=1}^{r+n-2}Z_k]. \tag{3.2.7}$$

Then $S_j=T_j=U_k=V=0$ $(1\leq j\leq g, 2\leq k\leq r+n-2)$.

Proof. Our proof is essentially similar to that of Lemma 4.3.2 in [6]. It is easy to see that $V=0$ implies $S_j=T_j=V_k=0$. Suppose $V\neq 0$ and decompose V as $V=\sum_{\tau} V^{(\tau)}$ with $V^{(\tau)}\in L^{(\tau)}$, where $L^{(\tau)}$ consists of homogeneous elements of multidegree $\tau=(l_j, m_j, n_k)_{1\leq j\leq g, 1\leq k\leq r+n-2}$ in $(X_j, Y_j, Z_k)_{1\leq j\leq g, 1\leq k\leq r+n-2}$. Let $V^{(\tau_0)}$ be a component whose degree in Z_1 is as large as possible. Then the term $[V^{(\tau_0)}, Z_1]$ from the RHS of (3.2.7) must be cancelled out by the term from the LHS. By the assumptions $S_j\in\langle X_j, Z_1\rangle$, $T_j\in\langle Y_j, Z_1\rangle$ and $U_k\in\langle Z_k, Z_1\rangle$, no two of the $[S_j, Y_j]$, $[X_j, T_j]$ and $[U_k, Z_k]$ have the term of same multidegree in common.

Case 1. $[V^{(\tau_0)}, Z_1]$ is cancelled out by some term from $[S_j, Y_j]$ or $[X_j, T_j]$. In this case $V^{(\tau_0)}$ belongs to the subalgebra $\langle X_j, Y_j, Z_1\rangle$ and has degree at least 1 in each X_j, Y_j and Z_1 (because $m\geq 3$). Then the term $[V^{(\tau_0)}, [X_j, Y_j]]$ $(\neq 0)$ from the RHS of (3.2.7) is of degree at least 2 both in X_j and Y_j, thus cannot appear in the LHS. Hence it must appear in $[V^{(\tau_1)}, Z_1]$ for some τ_1. $V^{(\tau_1)}$ is in $\langle X_j, Y_j, Z_1\rangle$ and of degree at least 3 both in X_j and Y_j. The degree in Z_1 of $V^{(\tau_1)}$ is less by 1 than that of $V^{(\tau_0)}$. Now consider $[V^{(\tau_1)}, [X_j, Y_j]]$ from the RHS, and so on. We finally get $V^{(\tau_k)}$ which is in $\langle X_j, Y_j\rangle$. But then $[V^{(\tau_k)}, [X_j, Y_j]]$ ($\neq 0$ because $m\geq 3$) cannot be cancelled out, contradiction.

Case 2. $[V^{(\tau_0)}, Z_1]$ is cancelled out by some term from $[U_k, Z_k]$. In this case $V^{(\tau_0)}$ belongs to $\langle Z_k, Z_1\rangle$. As the degree of U_k is greater than 2, U_k is of degree at least 2 in Z_k. Thus the term $[V^{(\tau_0)}, [X_j, Y_j]]$ from the RHS of (3.2.7) cannot be cancelled out by any term from the LHS, hence it must be cancelled out by $[V^{(\tau_1)}, Z_k]$ or $[V^{(\tau_1)}, Z_1]$ for some τ_1 from the RHS. Consider the term $[V^{(\tau_1)}, [X_j, Y_j]]$ in the RHS. This is of degree 2 both in X_j and Y_j, hence must be cancelled out by some $[V^{(\tau_2)}, Z_k]$ or $[V^{(\tau_2)}, Z_1]$ from the RHS. Continuing these arguments we are lead to a contradiction as in Case 1.
This settles the proof of Claim 5 when $g>0$.

Suppose $g=0$. Then $N=N_n$ is a free pro-l group of rank $r+n-2$ generated by $z_k (1\leq k\leq r+n-1)$, $z_{r+n-1}\cdots z_2 z_1=1$. Recall that we have put

$$\sigma z_k = u_k z_k u_k^{-1}, \quad u_k \in N \quad (2 \le k \le r+n-1)\ (\sigma z_1 = z_1)$$

and that by Claim 4 we have

$$u_k \in \langle z_k, z_1 \rangle \quad (2 \le k \le r+n-2). \tag{3.2.8}$$

In this case we use the filtrations by the lower central series of N. Let $\{N[m]\}_{m \ge 1}$ be the lower central series and put $L = \bigoplus_{m \ge 1} N[m]/N[m+1]$. Then L is a free Lie algebra over $\boldsymbol{Z}_l$ on $Z_1 = z_1 \bmod N[2], \cdots, Z_{r+n-2} = z_{r+n-2} \bmod N[2]$ (cf. [4]). Let m be a positive integer satisfying $u_k \in N[m]$ for all k $(2 \le k \le r+n-1)$ and define $U_k = u_k \bmod N[m+1]$. Then by (3.2.8) we have

$$U_k \in \langle Z_k, Z_1 \rangle \quad (2 \le k \le r+n-2). \tag{3.2.9}$$

The relation $z_{r+n-1} \cdots z_2 z_1 = 1$ applied by σ yields

$$\begin{aligned}&[Z_2, U_2] + \cdots + [Z_{r+n-2}, U_{r+n-2}] \\ &\quad = [Z_1 + Z_2 + \cdots + Z_{r+n-2}, U_{r+n-1}].\end{aligned}$$

As in the case of $g > 0$, this with (3.2.9) implies that we may assume $m \ge 2$. Then, by Lemma 4.3.2 in [6], of which proof is valid over $\boldsymbol{Z}_l$, we conclude that $u_k = 1$ for all k hence Claim 5 for $g = 0$.

Now let σ be an automorphism of P_n which satisfies the conditions of Theorem 1 and Claim 2. The final step of our proof of Theorem 1 is:

Claim 6. *σ acts trivially on P_n.*

Proof. Take any element α in P_n. First we claim that $\sigma(\alpha) \cdot \alpha^{-1}$ is conjugate in N_n to some l-adic power of z. When $\alpha \in C$, this is because $\sigma(\alpha) \cdot \alpha^{-1} \in C \cap N_n = \langle z \rangle$ (Lemma A(ii)). In general, α being written as $\alpha = nc$ with $n \in N_n$ and $c \in C$, we have $\sigma(\alpha) = n \cdot \sigma(c) = nz^k c = nz^k n^{-1} \alpha$ for some $k \in \boldsymbol{Z}_l$. Therefore, $\sigma(\alpha)\, \alpha^{-1}$ is conjugate in N_n to some l-adic power of z. Replacing z with $z_2 = z_2^{(n)}$ (this is the second place where we need the assumption $n \ge 3$ if $r = 0$ which ensure the existence of $z_2^{(n)}$) and C with the centralizer of $z_2^{(n)}$, and tracing the arguments as before under the assumption that σ acts trivially on N_n, we conclude that $\sigma(\alpha) \cdot \alpha^{-1}$ is conjugate in N_n also to some power of z_2. If $n + r > 3$, this together with the fact that z and z_2 constitute free generators of N_n implies that $\sigma(\alpha) \cdot \alpha^{-1}$ must be the identity element. If $n + r = 3$, consider the relation

$$nz^k n^{-1} = n' z_2^{k'} n'^{-1} \bmod N_n[3]\ (= [N_n, [N_n, N_n]]).$$

By writing down this relation explicitly with free generators x_j, y_j $(1 \le j \le g)$ and z $(z_2 = ([x_1, y_1] \cdots [x_g, y_g])^{-1} z^{-1})$, we readily see that we must have $k = k' = 0$. Therefore $\sigma(\alpha) \cdot \alpha^{-1}$ must be the identity element.

4. Galois representations

4.1. We shall now give some applications to Galois representations. Let X^{cpt} be any complete smooth irreducible algebraic curve over $\boldsymbol{C}$, given together with r distinct $\boldsymbol{C}$-rational points $a_1, \cdots, a_r$ $(r \geq 0)$, and put $X = X^{cpt} \backslash \{a_1, \cdots, a_r\}$. As before, consider the configuration space

$$Y = Y_n = F_{0,n} X = \{(x_1, \cdots, x_n) \in X^n;\ x_i \neq x_j\ (i \neq j)\}\ ,$$

choose a $\boldsymbol{C}$-rational point $b = (b_1, \cdots, b_n)$ of Y_n as base point, and look at the algebraic fundamental group $\boldsymbol{P} = \boldsymbol{P}_n = \hat{\pi}_1(Y_n, b)$, the profinite completion of the topological fundamental group $\pi_1(Y_n(\boldsymbol{C}), b)$. For each open subgroup $H \subset \boldsymbol{P}$, let $f_H \colon (Y_H, b_H) \to (Y, b)$ be the covering corresponding to H (unique up to $\simeq$). For each pair (H, H') of subgroups of $\boldsymbol{P}$ with finite indices, and an element $g \in \boldsymbol{P}$ with $H' \subset gHg^{-1}$, call $i_{H',H}(g)$ the unique projection $(Y_{H'}, b_{H'}) \to (Y_H, gb_H)$. Call M the union of $\boldsymbol{C}(Y_H)$ (the function field) with respect to the embeddings $i^*_{H',H}(1) \colon \boldsymbol{C}(Y_H) \hookrightarrow \boldsymbol{C}(Y'_H)$ (for $H' \subset H$), which is a Galois extension over $\boldsymbol{C}(Y)$ and for each $g \in \boldsymbol{P}_n$, call $i^*(g)$ the element of Gal $(M/\boldsymbol{C}(Y))$ defined by the system $\{i^*_{N,N}(g)\}_{N \triangleleft \boldsymbol{P}_n}$.

Proposition 4.1.1. (i) *M is a maximal Galois extension of $\boldsymbol{C}(Y) = \boldsymbol{C}((X^{cpt})^n)$ unramified outside the prime divisors*

$$(4.1.2) \qquad \begin{cases} [a_s]_i = \{(x_1, \cdots, x_n) \in (X^{cpt})^n;\ x_i = a_s\} & (1 \leq i \leq n,\ 1 \leq s \leq r) \\ \Delta_{ij} = \{(x_1, \cdots, x_n) \in (X^{cpt})^n;\ x_i = x_j\} & (1 \leq i, j \leq n,\ i \neq j)\ , \end{cases}$$

of $Y^{cpt} = (X^{cpt})^n$. (ii) *The homomorphism $i^* \colon \boldsymbol{P}_n \to \mathrm{Gal}(M/\boldsymbol{C}(y))$ is an isomorphism.*

Proof. A theorem of Grauert-Remmert on unique extendability of partial finite coverings of normal analytic spaces, and GAGA (the generalized Riemann existence theorem, and GAGA for morphisms) [3], Exp. XII.

4.2. Call $Br(Y)$ the set of all prime divisors of $Y^{cpt} = (X^{cpt})^n$ belonging to (4.1.2). For each $D \in Br(Y)$, choose a point $Q_D \in |D|$ (the support of D), an open neighborhood U_D of Q_D in $Y^{cpt}(\boldsymbol{C})$, and a biholomorphic map $u_D \colon U_D \xrightarrow{\sim} W^n$, where $W = \{w \in \boldsymbol{C},\ |w| < 1\}$. We require that $U_D \cap |D'| = \phi$ for any $D' \in Br(Y)$, $D' \neq D$, and that $U_D \cap |D|$ corresponds to $\{(w_1, \cdots, w_n);\ w_1 = 0\}$ *via* u_D. Choose any path $p_D \colon I \to Y(\boldsymbol{C})$ such that $p_D(0) = b$ and $p_D(1) = Q'_D \in U_D - |D|$ $(I = [0, 1])$. Put $u_D(Q'_D) = (w'_1, \cdots, w'_n)$, and let $c_D \colon I \to U_D - |D|$ be the loop, with base point Q'_D, defined by

$$u'_D(c_D(t)) = (w'_1 \exp(2\pi i t),\ w'_2, \cdots, w'_n) \quad (t \in I)\ .$$

Such a path p_D determines, on the one hand, an element $z_D = z_D(p_D)$ of $\boldsymbol{P}_n = \hat{\pi}_1(Y_n, b)$, and on the other hand, an extension $\tilde{v}_D = \tilde{v}_D(p_D)$ to M of the valuation v_D of $\boldsymbol{C}(Y)$ corresponding to D. Namely, z_D is the class of the loop $p_D^{-1}\circ c_D\circ p_D$, and $\tilde{v}_D$ is defined as follows. For each subgroup H of $\boldsymbol{P}_n$ with finite index, let f_H: $Y_H^{cpt}\to Y^{cpt}$ be the integral closure of Y^{cpt} in $\boldsymbol{C}(Y_H)$, and $p_{D,H}$ be the lifting of p_D to a path on $Y_H(\boldsymbol{C})$ such that $p_{D,H}(0)=b_H$. Let $V_{D,H}$ be the unique connected component of $f_H^{-1}(U_D)$ containing $p_{D,H}(1)$. Then there is a unique prime divisor D_H of Y_H^{cpt} lying above D such that $Y_H^{cpt}(\boldsymbol{C})\cap V_{D,H}\neq\phi$. It is clear that $\{D_H\}_H$ is a system of prime divisors of Y_H^{cpt} compatible with the projections and hence corresponds to an extension $\tilde{v}_D(p_D)$ of D to M. By construction, the following assertion is obvious.

Proposition 4.2.1. *$i^*(z_D(p_D))$ generates the inertia group of $\tilde{v}_D(p_D)$ in $M/\boldsymbol{C}(Y)$ in the sense of topological groups.*

From now on, we shall suppress the p_D and write as z_D, $\tilde{v}_D$.

4.3. Write $X^n=X_1\times\cdots\times X_n(X_i=X$ for $1\leq i\leq n)$, and put $\Sigma=\{1,2,\cdots,n\}$. For each finite non-empty subset $J\subset\Sigma$ with cardinality m $(1\leq m\leq n)$, call $Y_{m,J}$ the projection of Y on $\prod_{i\in J}X_i$. In particular, $Y=Y_n=Y_{n,\Sigma}$. By Fadell and Neuwirth ([2] Th 1.2), $Y(\boldsymbol{C})\to Y_{m,J}(\boldsymbol{C})$ is a locally trivial fiber space, and the fiber above $(b_{j_1},\cdots,b_{j_m})$ is

$$Z_J=F_{0,n-m}(X\backslash\{b_j\,(j\in J)\})\quad(\approx F_{r+m,n-m}(X^{cpt}))\,.$$

Since $\pi_2(Y_{m,J}(\boldsymbol{C}))=(1)$ ([2] Prop. 1.3), the above fibering induces a short homotopy exact sequence of topological fundamental groups

$$\begin{aligned}(4.3.1)\qquad 1\to\pi_1(Z_J(\boldsymbol{C}),b'')\to\pi_1(Y_n(\boldsymbol{C}),b)=P_n\\ \to\pi_1(Y_{m,J}(\boldsymbol{C}),b')=P_{m,J}\to 1\,,\end{aligned}$$

where $b'=\prod_{j\in J}b_j$, $b''=\prod_{j\notin J}b_j$, $b=(b',b'')$. In particular, when $m=n-1$ (≥ 1), the kernel group in (4.3.1) is $\pi_1(X(\boldsymbol{C})\backslash\{b_j\,(j\in J)\},b'')$, which is free of rank $2g+r+m-1$, where g is the genus of X^{cpt}.

Proposition 4.3.2. *If $(W,w)\to(Y_{m,J},b')$ is a connected finite etale covering corresponding to $H\subset P_{m,J}=\pi_1(Y_{m,J}(\boldsymbol{C}),b')$, a subgroup with finite index, then the fiber product $(W\times_{Y_{m,J}}Y_n,w\times b)$ is a connected finite etale covering corresponding to the inverse image of H in $\pi_1(Y_n(\boldsymbol{C}),b)$.*

Proof. The fiber product covering is obviously etale, and it is connected because each fiber of $Y\to Y_{n,J}$ is connected. By the definition of the fiber product, an element of $\pi_1(Y_n(\boldsymbol{C}),b)$ belongs to the image of $\pi_1((W\times_{Y_{m,J}}Y_n)(\boldsymbol{C}),w\times b)$ if and only if its projection on $\pi_1(Y_{m,J}(\boldsymbol{C}),b')$ belongs to the image of $\pi_1(W(\boldsymbol{C}),w)$ i.e., to H.

Denote by M_J the field M for $Y_{m,J}$. Then $M_J\cdot\boldsymbol{C}(Y)$ is a Galois subextension of $M/\boldsymbol{C}(Y)$.

Corollary 4.3.3. *The normal subgroup of* $\boldsymbol{P}_n$ *corresponding to* $M_J \cdot \boldsymbol{C}(Y)$ *via* i^*: $\boldsymbol{P}_n \xrightarrow{\sim} \mathrm{Gal}(M/\boldsymbol{C}(y))$ *is the kernel of* $\boldsymbol{P}_n \to \boldsymbol{P}_m$ *induced by* (4.3.1), *and* Gal $(M_J\, \boldsymbol{C}(Y)/\boldsymbol{C}(Y))$ *is canonically isomorphic* (*via* i^*) *to* $\boldsymbol{P}_m$.

4.4. Now let k be a subfield of $\boldsymbol{C}$ such that X is defined over k and the points $a_j (1 \leq j \leq r)$ are k-rational. Let $\mathrm{Aut}(\boldsymbol{C}/k)$ be the group of all automorphisms σ of $\boldsymbol{C}$ acting trivially on k. We can associate to each $n \geq 1$ a group homomorphism

$$\varphi = \varphi_n \colon \mathrm{Aut}(\boldsymbol{C}/k) \to \mathrm{Out}\, \boldsymbol{P}_n$$

($\boldsymbol{P}_n = \hat{\pi}_1(Y_n, b)$, Out: the outer automorphism group) as follows. For each $\sigma \in \mathrm{Aut}(\boldsymbol{C}/k)$, let σ' be the unique automorphism of $\boldsymbol{C}(Y)$ which extends σ and which acts trivially on $k(Y)$. Note that σ' leaves the discrete valuations v_D ($D \in Br(Y)$) invariant. By the characterization of M given in Prop. 4.1.1 (i), σ' extends to an *automorphism* $\tilde{\sigma}$ of M. Identify $\mathrm{Gal}(M/\boldsymbol{C}(Y))$ with $\boldsymbol{P}_n$ via i^* (Prop. 4.1.1 (ii)). Then $\tilde{\sigma}$ is unique up to elements of $\boldsymbol{P}_n$. The element of Out $\boldsymbol{P}_n$ represented by the automorphism $g \to \tilde{\sigma} g \tilde{\sigma}^{-1}$ of $\boldsymbol{P}_n$ is well-defined by σ, which is the definition of $\varphi_n(\sigma)$. For any non-empty subset $J \subset \Sigma = \{1, 2, \cdots, n\}$, the homomorphism $\varphi_J = \varphi_{m,J}$ is defined using $M_J/k(Y_J)$ instead of $M/k(Y)$.

$$\varphi_{m,J} \colon \mathrm{Aut}(\boldsymbol{C}/k) \to \mathrm{Out}\, \boldsymbol{P}_{m,J} \quad (m = |J|)\,.$$

We denote by $\chi \colon \mathrm{Aut}(\boldsymbol{C}/k) \to \hat{\boldsymbol{Z}}^\times$ the cyclotomic character.

Proposition 4.4.1. (i) *Let* $D \in Br(Y)$ *and* $\sigma \in \mathrm{Aut}(\boldsymbol{C}/k)$. *Then* $\varphi(\sigma)\, z_D \sim z_D^{\chi(\sigma)}$ ($\sim$: $\hat{P}_n$-*conjugacy*). (ii) *Let* $J \subset \{1, 2, \cdots, n\}$, $J \neq \phi$. *Then* $\varphi(\sigma)$ *leaves the kernel of* $\boldsymbol{P}_n \to \boldsymbol{P}_{m,J}$ *invariant, and induces on* $\boldsymbol{P}_{m,J}$ *the outer automorphism* $\varphi_{m,J}(\sigma)$.

Proof. (i) Choose any prime element π of v_D in $k(Y)$, and put $M^* = M(\pi^{1/n}; n \geq 1)$. (We cannot always choose π such that $M^* = M$.) Since M^* is a composite of M with a Galois extension of $k(Y)$, $\tilde{\sigma}$ extends to an automorphism $\tilde{\sigma}^*$ of M^*. Let $\tilde{v}_D$ be as in §4.2, and extend it to a valuation $\tilde{v}_D^*$ of M^*. Note that $M^*/\boldsymbol{C}(Y)$ is also Galois, and call I^* the inertia group of $\tilde{v}_D^*$ in $M^*/\boldsymbol{C}(Y)$. The restriction to M gives a surjective homomorphism $I^* \to I$ onto the inertia group of $\tilde{v}_D$ in $M/\boldsymbol{C}(Y)$. Moreover, both I^* and I are topologically cyclic (the residue characteristic being 0). Therefore, z_D extends to a generator z_D^* of I^*. Now the valuation $\tilde{v}_D^* \circ \tilde{\sigma}^{*^{-1}}$ of M^* is an extension of the valuation $v_D \circ \sigma^{-1} = v_D$ of $\boldsymbol{C}(Y)$. Therefore, there exists $s^* \in \mathrm{Gal}(M^*/\boldsymbol{C}(Y))$ such that $\tilde{v}_D^* \circ \tilde{\sigma}^{*^{-1}} = \tilde{v}_D^* \circ s^{*^{-1}}$. Comparison of inertia groups gives:

$$(*) \qquad \tilde{\sigma}^*\, z_D^*\, \tilde{\sigma}^{*^{-1}} = s^*\, z_D^{*^{\alpha}}\, s^{*^{-1}}$$

with some $\alpha \in \hat{\boldsymbol{Z}}^\times$. By applying the Kummer character

$$\kappa_\pi\colon \mathrm{Gal}(M^*/\boldsymbol{C}(Y)) \to \hat{\boldsymbol{Z}}(1) = \varprojlim \mu_n$$

to both sides of (*), noting that $\kappa_\pi(z_D^*)$ is a generator of $\hat{\boldsymbol{Z}}(1)$, we obtain $\chi(\sigma)=\alpha$. Therefore,

$$\tilde{\sigma} z_D \tilde{\sigma}^{-1} = s z_D^{\chi(\sigma)} s^{-1},$$

if $s \in \mathrm{Gal}(M/\boldsymbol{C}(Y)) = \boldsymbol{P}_n$ is the restriction of s^*. This settles (i). The assertion (ii) is obvious from the definitions.

4.5. Now we shall fix a prime number l and denote by P_n, $P_{m,J}$ etc. the maximal pro-l quotient of $\boldsymbol{P}_n$, $\boldsymbol{P}_{m,J}$, etc. (i.e., the pro-l completions of the corresponding topological fundamental groups). Then the passage to the pro-l quotient $\boldsymbol{P}_n \to P_n$ induces from $\boldsymbol{\varphi}_n$, $\boldsymbol{\varphi}_{m,J}$ the representations φ_n, $\varphi_{m,J}$ of $\mathrm{Aut}(\boldsymbol{C}/k)$ in Out P_n, Out $P_{m,J}$, etc.

The second main result of this paper is the following

Theorem 2. *Let X^{cpt} be a complete smooth absolutely irreducible curve of genus g over a subfield k of $\boldsymbol{C}$, and $a_1, \cdots, a_r$ be r distinct k-rational points of X^{cpt}. Let l be a prime number and $\varphi_n (n=1, 2, \cdots)$ be the representations of* $\mathrm{Aut}(\boldsymbol{C}/k)$ *in* Out P_n *defined from the data $X = X^{cpt} \backslash \{a_1, \cdots, a_r\}$, via the outer action of* $\mathrm{Aut}(\boldsymbol{C}/k)$ *on $P_n = \pi_1^{pro-l}(F_{0,n}X)$. Then*

$$\mathrm{Ker}\,\varphi_n = \mathrm{Ker}\,\varphi_{n-1},$$

if either $g \geq 1$ and $n+r \geq 3$, or $g=0$ and $n+r \geq 5$. In particular, if $g \geq 1$ and $r \geq 1$, or $g=0$ and $r \geq 3$, then

$$\mathrm{Ker}\,\varphi_n = \mathrm{Ker}\,\varphi_1.$$

Proof. Note first that $\varphi_{m,J}$ is induced from φ_n by the canonical projection $P_n \to P_{m,J}$. In particular, φ_{n-1} is a quotient representation of φ_n; hence $\mathrm{Ker}\,\varphi_n \subset \mathrm{Ker}\,\varphi_{n-1}$.

Now to prove the opposite inclusion, let σ be any element of $\mathrm{Ker}\,\varphi_{n-1}$. We shall show that $\varphi_n(\sigma) \in \mathrm{Out}\,P_n$ satisfies the assumptions of Theorem 1. Let $\chi_l\colon \mathrm{Aut}(\boldsymbol{C}/k) \to \boldsymbol{Z}_l^\times$ be the l-cyclotomic character. Then by Prop. 4.4.1 (i) we have

$$(\sharp) \qquad \varphi_n(\sigma)\, z_D \sim z_D^{\chi_l(\sigma)} \quad (\sim\colon P_n\text{-conjugacy}).$$

But since $\sigma \in \mathrm{Ker}\,\varphi_1$, σ acts trivially on the abelianization of $\pi_1^{pro-l}(X)$. If $r \geq 2$, this together with ($\sharp$) gives $\chi_l(\sigma)=1$. If $g \geq 1$, then the determinant of the action of σ on the abelianization of $\pi_1^{pro-l}(X^{cpt})$ is $\chi_l(\sigma)$; hence, again, $\chi_l(\sigma)=1$. If $g=0$ and $r \leq 1$, we may assume $n \geq 4$ and hence also that σ acts trivially on P_3, and hence also on

$$\mathrm{Ker}(P_3 \to P_2) = \pi_1^{pro-l}(X-(r+2pts)).$$

On the other hand, σ raises parabolic conjugacy classes to their $\chi_l(\sigma)$-th power.

Therefore, $\chi_l(\sigma)=1$ in all cases. Therefore, by (#), the assumption (σ1) of Theorem 1 is satisfied.

To check (σ2), we may assume $i=n$. First, by Prop. 4.4.1 (ii), $\varphi_n(\sigma)$ leaves $N_n^{(n)}$ invariant. Secondly, to see that it acts trivially on $N_n^{(n)}/N_n^{(n)}(2)$, consider the projection $P_n \to P_{1,\{n\}}$. Its restriction to $N_n^{(n)}$ is a homomorphism onto $\pi_1^{pro-l}(X, b_n)$, induced from the natural homomorphism

$$\pi_1(X\setminus\{b_1, \cdots, b_{n-1}\}, b_n) \to \pi_1(X, b_n)$$

by pro-l completion. Moreover, this homomorphism $N_n^{(n)} \to \pi_1^{pro-l}(X, b_n)$ commutes with the action of σ, and the kernel (being generated by loops around $b_1, \cdots, b_{n-1}$) is contained in $N_n^{(n)}(2)$. Since $\varphi_1(\sigma)=1$, σ acts trivially on $\pi_1^{pro-l}(X, b_n)$, and hence also on $N_n^{(n)}/N_n^{(n)}(2)$. Therefore, (σ2) is also satisfied. Therefore, by Theorem 1, $\varphi_n(\sigma)=1$.

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