Dualities in Algebraic Logic

Yde Venema
Institute for Logic, Language and Computation
Universiteit van Amsterdam
https://staff.fnwi.uva.nl/y.venema

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Overview

- Introduction
- Modal Dualities
- Subdirectly irreducible algebras and rooted structures
- Vietoris via modal logic
- Final remarks
Algebraic Logic

- aim: study logics using methods from (universal) algebra
  - examples:
    - propositional logic: Boolean algebras
    - intuitionistic logic: Heyting algebras
    - first-order logic: cylindric algebras
  - other examples:
    - interpolation: amalgamation
    - completeness: representation
  - abstract algebraic logic:
    - study Logic using methods from (universal) algebra
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  - first-order logic: cylindric algebras
- other examples:
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  - completeness: representation
- abstract algebraic logic:
  - study Logic using methods from (universal) algebra
Duality

- in mathematics: categorical dualities
Duality

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- \( C \) and \( D \) are dual(ly equivalent) if \( C \) and \( D^\circ \) are equivalent
Duality

- in mathematics: categorical dualities
- C and D are dual(ly equivalent) if C and $D^\circ$ are equivalent
  i.e. there are contravariant functors linking C and D
A Fundamental Duality

verbal

visual
A Fundamental Duality

<table>
<thead>
<tr>
<th>verbal</th>
<th>visual</th>
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<tbody>
<tr>
<td>algebra</td>
<td>geometry</td>
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A Fundamental Duality

- verbal
- algebra
- syntax

and

- visual
- geometry
- semantics

Stone duality:
A Fundamental Duality

verbal | visual
algebra | geometry
syntax | semantics

Stone duality:

\[
\text{BA} \quad \xrightarrow{S} \quad \text{Stone} \quad \xleftarrow{A}
\]
Variants of Stone duality

- Heyting algebra vs Esakia spaces
- compact regular frames vs compact Hausdorff spaces
- distributive lattices vs Priestley spaces
- modal algebras vs topological Kripke structures
- cylindric algebras vs . . .
- . . .
Variants of Stone duality

- Heyting algebra vs Esakia spaces
- compact regular frames vs compact Hausdorff spaces
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Contravariance
Variants of Stone duality

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**Contravariance** In all these examples both categories are concrete!
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Modal duality

Main characters

■ modal algebras (MA)
■ Kripke structures (KS)
■ Stone spaces (Stone)
Modal duality

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- Kripke structures (KS)
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- topological Kripke structures (TKS)
Modal duality

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- . . . , and of course their morphisms!
Modal duality

Main characters

- modal algebras (MA)
- Kripke structures (KS)
- Stone spaces (Stone)
- topological Kripke structures (TKS)
- . . . , and of course their morphisms!

Aim:

- introduce TKS
- develop duality between MA and TKS
Modal Algebras

\[ A = (A, \lor, -, \bot, \Diamond) \] is a modal algebra if:

1. \((A, \lor, -, \bot)\) is a Boolean algebra
2. \(\Diamond : A \rightarrow A\) preserves finite joins:
   \[ \Diamond \bot = \bot \text{ and } \Diamond (a \lor b) = \Diamond a \lor \Diamond b \]
Modal Algebras

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\(h : \mathbb{A}' \to \mathbb{A}\) is an MA-morphism if it preserves all operations:

- \(h(a' \lor b') = h(a') \lor h(b'), \ldots, h(\Diamond a') = \Diamond h(a')\).
Modal Algebras

- $\mathbb{A} = (A, \lor, -, \bot, \Diamond)$ is a modal algebra if
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- MA is the category of modal algebras with MA-morphisms
Modal Algebras

- \( \mathbb{A} = (A, \lor, -, \bot, \lozenge) \) is a modal algebra if
  - \( (A, \lor, -, \bot) \) is a Boolean algebra
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- MA is the category of modal algebras with MA-morphisms

- A modal logic \( L \) can be algebraized by a variety \( V_L \) of modal algebras
Modal Algebras

- A = (A, ∨, -, ⊥, ◊) is a modal algebra if
  ▶ (A, ∨, -, ⊥) is a Boolean algebra
  ▶ ◊ : A → A preserves finite joins:
    ◊⊥ = ⊥ and ◊(a ∨ b) = ◊a ∨ ◊b

- h : A' → A is an MA-morphism if it preserves all operations:
  ▶ h(a' ∨' b') = h(a') ∨ h(b'), ..., h(◊'a') = ◊h(a').

- MA is the category of modal algebras with MA-morphisms

- A modal logic L can be algebraized by a variety VL of modal algebras
- Modal algebras are (the simplest) Boolean Algebras with Operators
A Kripke structure (frame) is a pair $\mathcal{S} = (S, R)$ with $R \subseteq S \times S$.

These provide the possible-world semantics of modal logic.
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These provide the possible-world semantics of modal logic.

$f : (S', R') \rightarrow (S, R)$ is a bounded morphism if

- $R's't'$ implies $Rf(s')f(t')$
- $Rf(s')t$ implies the existence of $t'$ with $R's't'$ and $f(t') = t$. 

KS is the category of Kripke structures with bounded morphisms.
Kripke structures

- A **Kripke structure (frame)** is a pair $S = (S, R)$ with $R \subseteq S \times S$
  - these provide the possible-world semantics of modal logic
- $f : (S', R') \rightarrow (S, R)$ is a **bounded morphism** if
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- **KS** is the category of Kripke structures with bounded morphisms
Stone spaces

- A (topological) space is a pair \((S, \tau)\) where \(\tau\) is a topology on \(S\).
- A Stone space is a space \((S, \tau)\) where \(\tau\) is:
  - compact,
  - Hausdorff
  - zero-dimensional (i.e. it has a basis of clopen sets)
- Stone is the category of Stone spaces and continuous functions
Stone duality

From Stone spaces to Boolean algebras: $(\cdot)^*$

**Objects** Given $(S, \tau)$ take $(S, \tau)^* := (\text{Clp}(\tau), \cup, \sim_S, \emptyset)$

**Arrows** Given $f : (S', \tau') \to (S, \tau)$ define $f^* : \text{Clp}(\tau) \to \text{Clp}(\tau')$

$$f^*(X) := \{s' \in S' \mid fs' \in X\}$$
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**Objects** Given \(\mathbb{A} = (A, \lor, -, \bot)\) take \(A^*_\) := \((\text{Uf}(\mathbb{A}), \sigma_\mathbb{A})\), where
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**Objects** Given $\mathbb{A} = (A, \lor, -, \bot)$ take $A^* := (\text{Uf}(\mathbb{A}), \sigma_{\mathbb{A}})$, where

- $\text{Uf}(\mathbb{A})$ is the set of ultrafilters of $\mathbb{A}$ and
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From Boolean algebras to Stone spaces: $(\cdot)_*$

**Objects** Given $\mathbb{A} = (A, \lor, \neg, \bot)$ take $A_* := (\text{Uf}(\mathbb{A}), \sigma_{\mathbb{A}})$, where

- $\text{Uf}(\mathbb{A})$ is the set of ultrafilters of $\mathbb{A}$ and
- $\sigma_{\mathbb{A}}$ is generated by the basis $\{ \hat{a} \mid a \in A \}$
- with $\hat{a} := \{ u \in \text{UF}(\mathbb{A}) \mid a \in u \}$
Stone duality

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From Boolean algebras to Stone spaces: $(\cdot)_*$

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  \item with $\hat{a} := \{u \in \text{UF}(\text{A}) \mid a \in u\}$
\end{itemize}

Arrows  Given $h : A' \rightarrow A$ define $h_* : \text{Uf}(\text{A}) \rightarrow \text{Uf}(\text{A}')$ by

\[
h_*(u) := \{a' \in A' \mid ha' \in u\}
\]
Theorem
The functors \((\cdot)^{*}\) and \((\cdot)_{*}\) witness the dual equivalence of BA and Stone.
Stone duality 2

**Theorem**
The functors $(\cdot)^*$ and $(\cdot)_*$ witness the dual equivalence of BA and Stone.

This is a natural duality evolving around the schizophrenic object 2
Complex algebras

From Kripke structures to modal algebras: $(\cdot)^+$

**Objects** Given $(S, R)$ take $(S, R)^+ := (PS, \cup, \sim_S, \emptyset, \langle R \rangle)$, where
Complex algebras

From Kripke structures to modal algebras: \((\cdot)^+\)

**Objects** Given \((S, R)\) take \((S, R)^+ := (PS, \cup, \sim_S, \emptyset, \langle R\rangle)\), where

\[
\langle R\rangle(X) := \{s \in S \mid R[s] \cap X \neq \emptyset\}
\]

**Arrows** Given \(f : (S', R') \to (S, R)\) define \(f^+\) as inverse image
Complex algebras

From Kripke structures to modal algebras: $(\cdot)^+$

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- The operation $\langle R \rangle$ encodes the semantics of the modal diamond
Complex algebras

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**Arrows** Given $f : (S', R') \to (S, R)$ define $f^+$ as inverse image

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- $(S, R)^+$ is the complex algebra of $(S, R)$
Complex algebras

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- The operation \(\langle R \rangle\) encodes the semantics of the modal diamond
- \((S, R)^+\) is the complex algebra of \((S, R)\)
- Complex algebras are perfect modal algebras (PMAs):
  - complete, atomic and completely additive
Complex algebras

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- $(\cdot)^+$ is part of a discrete duality between PMA and KS
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  (with the opposite functor \((\cdot)^+\) taking the atom structure of a PMA)
Ultrafilter structures

From modal algebras to Kripke structures:
Ultrafilter structures

From modal algebras to Kripke structures:

**Objects** With $\mathbb{A} = (A, \lor, \neg, \bot, \Diamond)$ take $\mathbb{A}_\bullet := (\text{Uf}(\mathbb{A}), Q\Diamond)$, where
Ultrafilter structures

From modal algebras to Kripke structures:

Objects  With $\mathcal{A} = (A, \lor, \neg, \bot, \Diamond)$ take $\mathcal{A}^\bullet := (Uf(\mathcal{A}), Q^\Diamond)$, where

$Q^\Diamond uv$ iff $\forall a \in v. \Diamond a \in u$
From modal algebras to Kripke structures:

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**Arrows**  Given $f : \mathbb{A}' \rightarrow \mathbb{A}$ define $f_\bullet$ as inverse image

- These operations provide a functor: $\text{MA} \rightarrow \text{KS}$
Ultrafilter structures

From modal algebras to Kripke structures:

**Objects**  With $\mathcal{A} = (A, \lor, -, \bot, \lozenge)$ take $\mathcal{A}_\bullet := (Uf(A), Q\lozenge)$, where

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**Arrows**  Given $f : \mathcal{A}' \to \mathcal{A}$ define $f_\bullet$ as inverse image

- These operations provide a functor: $\text{MA} \to \text{KS}$
- $\mathcal{A}_\bullet$ is the **ultrafilter structure** or **canonical structure** of $\mathcal{A}$
Ultrafilter structures

From modal algebras to Kripke structures:

**Objects** With \( A = (A, \lor, -, \bot, \lozenge) \) take \( A_\bullet := (Uf(A), Q_\lozenge) \), where

\[ Q_\lozenge uv \iff \forall a \in v. \lozenge a \in u \]

**Arrows** Given \( f : A' \to A \) define \( f_\bullet \) as inverse image

- These operations provide a functor: \( MA \to KS \)
- \( A_\bullet \) is the ultrafilter structure or canonical structure of \( A \)
- \( A \) embeds in its canonical extension \((A_\bullet)^+\)
Ultrafilter structures

From modal algebras to Kripke structures:

**Objects** With $\mathbb{A} = (A, \lor, -, \bot, \Diamond)$ take $\mathbb{A}_\bullet := (Uf(A), Q\Diamond)$, where

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- These operations provide a functor: $\text{MA} \to \text{KS}$
- $\mathbb{A}_\bullet$ is the **ultrafilter structure** or **canonical structure** of $\mathbb{A}$
- $\mathbb{A}$ embeds in its **canonical extension** $(\mathbb{A}_\bullet)^+$
- **Open Problem** characterize the ultrafilter structures modulo isomorphism
A topological Kripke structure is a triple $(S, R, \tau)$ such that:
- $(S, R)$ is a Kripke structure
- $(S, \tau)$ is a Stone space
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- \((S, R)\) is a Kripke structure
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- \(\langle R \rangle X\) is clopen if \(X \subseteq S\) is clopen
A **topological Kripke structure** is a triple \((S, R, \tau)\) such that

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- \(R(s)\) is closed
A topological Kripke structure is a triple \((S, R, \tau)\) such that

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- \(R(s)\) is closed

**TKS** is the category with

- objects: topological Kripke structures
- arrows: continuous bounded morphism
Topological modal duality

From modal algebras to topological Kripke structures: $(\cdot)^*$

**Objects** Given $\mathbb{A} = (A, \lor, -, \bot, \Box)$ take $\mathbb{A}^* := (Uf(\mathbb{A}), Q\Diamond, \sigma_\mathbb{A})$

**Arrows** Given $h : \mathbb{A}' \to \mathbb{A}$ define $h^*$ as inverse image

From topological Kripke structures to modal algebras: $(\cdot)^*$

**Objects** Given $\mathbb{S} = (S, R, \tau)$ take $\mathbb{S}^* := (Clp(\tau), \cup, \sim_S, \emptyset, \langle R \rangle)$

**Arrows** Given $f : \mathbb{S}' \to \mathbb{S}$ define $f^*$ as inverse image

**Theorem**
The functors $(\cdot)^*$ and $(\cdot)^*$ witness the dual equivalence of MA and TKS:
Remarks

History:

■ Jónsson & Tarski, Lemmon, Scott, Esakia, Goldblatt, . . .
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- algebraic logic || modal logic
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Research Topics:
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Research Topics:

- (canonicity) Which varieties are closed under \((A \mapsto (A\bullet)^+)\)
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Research Topics:
- (canonicity) Which varieties are closed under \((A \rightarrow (A \bullet)^+)\)
- (correspondence) FO properties of \(S \sim\) equational prop's of \(S^+\)
  - e.g. \(S \models \forall v R vv\) iff \(S^+ \models x \leq \Diamond x\)
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- (canonicity & correspondence) Sahlqvist theorem
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- (completions) canonical extensions, MacNeille completions, ...
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- ...
Overview

- Introduction
- Modal Dualities
- Subdirectly irreducible algebras and rooted structures
- Vietoris via modal logic
- Final remarks
Subdirect Irreducibility

- Given an algebra $\mathbb{A}$, let $\text{Con}\mathbb{A}$ be its lattice of congruences.

- $\mathbb{A}$ is simple if $\text{Con}\mathbb{A} \cong 2$.

- $\mathbb{A}$ is subdirectly irreducible if $\text{Con}\mathbb{A}$ has a least non-identity element.

- Birkhoff: every variety is generated by its s.i. members.

Question

What is the dual of an s.i. modal algebra?

Folklore

Subdirect irreducibility is related to rootedness.
Subdirect Irreducibility

- Given an algebra $\mathbb{A}$, let $\text{Con}\mathbb{A}$ be its lattice of congruences.
- $\mathbb{A}$ is simple if $\text{Con}\mathbb{A} \cong \mathbb{2}$.
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Subdirect Irreducibility

- Given an algebra $\mathbb{A}$, let $\text{Con}\mathbb{A}$ be its lattice of congruences.
- $\mathbb{A}$ is simple if $\text{Con}\mathbb{A} \cong 2$
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- Given an algebra $\mathbb{A}$, let $\text{Con}\mathbb{A}$ be its lattice of congruences.
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- $\mathbb{A}$ is subdirectly irreducible if $\text{Con}\mathbb{A}$ has a least non-identity element.
- Birkhoff: every variety is generated by its s.i. members.

**Question** What is the dual of an s.i. modal algebra?
Subdirect Irreducibility

- Given an algebra $\mathbb{A}$, let $\text{Con}\mathbb{A}$ be its lattice of congruences.
- $\mathbb{A}$ is **simple** if $\text{Con}\mathbb{A} \cong 2$.
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**Question** What is the dual of an s.i. modal algebra?

**Folklore** Subdirect irreducibility is related to **rootedness**.
Auxiliary definitions

\[ R^\omega := \bigcup_{n>0} R^n, \]

where \( R^0 := \text{Id}_S \) and \( R^{n+1} := R \circ R^n \)
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- $R^\omega := \bigcup_{n>0} R^n$,
  - where $R^0 := \text{Id}_S$ and $R^{n+1} := R \circ R^n$
- $R(s) := \{ t \in S \mid Rst \}$

$r \in S$ is a root of $S$ if $S = R^\omega(r)$

$S$ is rooted if its collection $\mathcal{W}_S$ of roots is non-empty
Auxiliary definitions

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Subdirect Irreducibility and Rootedness

**Proposition** (folklore)
\[ W_S \neq \emptyset \quad (\text{\$ is rooted}) \quad \text{iff} \quad \mathbb{S}^+ \text{ is s.i.} \]
Proposition (folklore)
\( \mathcal{W}_S \neq \emptyset \) (\( S \) is rooted) iff \( S^+ \) is s.i.

Example (Sambin)
There are rooted TKSs of which the dual algebra is not s.i.
Subdirect Irreducibility and Rootedness

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**Example** (Sambin)
There are rooted TKSs of which the dual algebra is not s.i.

**Proposition** (Sambin)
(1) If \( \text{Int}(\mathcal{W}_{\mathbb{A}^*}) \neq \emptyset \) then \( \mathbb{A} \) is s.i.
Proposition (folklore)
$\mathcal{W}_S \neq \emptyset$ (S is rooted) iff $S^+$ is s.i.

Example (Sambin)
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Proposition (Sambin)
(1) If $Int(\mathcal{W}_{A^*}) \neq \emptyset$ then $A$ is s.i.
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(1) If $\text{Int}(W_{A^*}) \neq \emptyset$ then $A$ is s.i.
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There are simple algebras of which the dual structure has no roots.
Subdirect Irreducibility and Rootedness

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Example (Kracht)
There are simple algebras of which the dual structure has no roots.

Proposition (Rautenberg)
$A$ is s.i. iff $A^*$ has a largest nontrivial, closed hereditary subset.
Fix a modal algebra $\mathbb{A}$.

- $r$ is a root of $\mathbb{A}^*$ iff $Q^\omega_\diamondsuit(r) = Uf(\mathbb{A})$
Fix a modal algebra $\mathbb{A}$.

- $r$ is a root of $\mathbb{A}^*$ iff $Q_\omega^\ast(r) = Uf(\mathbb{A})$
- $Q_\omega^\ast uv$ iff $\exists n \in \omega \forall a \in v. \Diamond^n a \in u$
Fix a modal algebra $\mathbb{A}$.

- $r$ is a root of $\mathbb{A}^*$ iff $Q^\omega(r) = Uf(\mathbb{A})$.
- $Q^\omega(\mathbb{A})$ iff $\exists n \in \omega: \forall a \in v. \diamond^n a \in u$.
- Define $Q^\star$ by putting
  $Q^\star(\mathbb{A})$ iff $\forall a \in v: \exists n \in \omega. \diamond^n a \in u$.
Fix a modal algebra $\mathbb{A}$.

- $r$ is a root of $\mathbb{A}$ if $Q^\omega (r) = Uf(\mathbb{A})$
- $Q^\omega uv$ iff $\exists n \in \omega \forall a \in v. \Diamond^n a \in u$
- Define $Q^*$ by putting
  $Q^* uv$ iff $\forall a \in v \exists n \in \omega. \Diamond^n a \in u$
- Call $r \in Uf(\mathbb{A})$ a topo-root if $Q^\Diamond (r) = Uf(\mathbb{A})$
Fix a modal algebra $\mathbb{A}$.

- $r$ is a root of $\mathbb{A}_*$ iff $Q^{\omega}_r(r) = Uf(\mathbb{A})$
- $Q^{\omega}_r uv$ iff $\exists n \in \omega \forall a \in v. \diamond^n a \in u$
- Define $Q^*_r$ by putting $Q^*_r uv$ iff $\forall a \in v \exists n \in \omega. \diamond^n a \in u$
- Call $r \in Uf(\mathbb{A})$ a topo-root if $Q^*_r(r) = Uf(\mathbb{A})$
- Let $T_{\mathbb{A}_*}$ denote the collection of topo-roots of $\mathbb{A}_*$
Observations

**Proposition** For any modal algebra $A$:

1. $Q^*$ is transitive
2. $Q^\omega \subseteq Q^*$
3. $Q^*(u)$ is hereditary for any ultrafilter $u$
4. $Q^*(u)$ is closed for any ultrafilter $u$
5. $Q^*(u) = Q^\omega(u)$ for any ultrafilter $u$
6. $\langle Q^* \rangle$ maps opens to opens
7. If $Q$ is transitive then $Q = Q^\omega = Q^*$
Characterizations

**Theorem** For any modal algebra $\mathbb{A}$:

(1) $\mathbb{A}$ is simple iff $T_{A^*} = Uf(A)$
Characterizations

**Theorem** For any modal algebra $\mathbb{A}$:

1. $\mathbb{A}$ is simple iff $T_{\mathbb{A}^*} = Uf(\mathbb{A})$
2. $\mathbb{A}$ is s.i. iff $\text{Int}(T_{\mathbb{A}^*}) \neq \emptyset$

Note: Earlier results follow from this.

**Theorem (Birchall)** Similar results for distributive modal algebras (based on distr. lattices).

**Suggestion** Develop the modal theory of $\mathbb{Q}^\star$.
Characterizations

**Theorem** For any modal algebra $\mathbb{A}$:

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Characterizations

**Theorem** For any modal algebra $\mathbb{A}$:
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Theorem For any modal algebra $\mathbb{A}$:
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Suggestion Develop the modal theory of $Q^*$
Overview

- Introduction
- Modal Dualities
- Subdirectly irreducible algebras and rooted structures
- Vietoris via modal logic
- Final remarks
The Vietoris construction

Let $X = \langle X, \tau \rangle$ be a topological space. $K(X)$ denotes the collection of compact sets. With $U \subseteq \omega \tau$, define $\nabla U := \{ F \in K(X) | (F, U) \in P(\in) \}$, where $(F, U) \in P(\in)$ if $F$ is 'properly covered' by $U$: $\forall s \in F \exists U \in U. s \in U$ and $\forall U \in U \exists s \in F. s \in U$. These sets $\nabla U$ together provide a basis for a topology on $K(X)$, the Vietoris topology $\nu \tau$. $V(X) := \langle K(X), \nu \tau \rangle$ is the Vietoris space of $X$.
The Vietoris construction

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The Vietoris construction

- Let $\mathbb{X} = \langle X, \tau \rangle$ be a topological space.
- $K(\mathbb{X})$ denotes the collection of compact sets
- With $\mathcal{U} \subseteq \wp(\tau)$, define
  \[
  \nabla \mathcal{U} := \{ F \in K(\mathbb{X}) \mid (F, \mathcal{U}) \in \mathcal{P}(\mathcal{E}) \},
  \]
- $\nabla \mathcal{U}$ together provide a basis for a topology on $K(\mathbb{X})$, the Vietoris topology $\nu(\mathbb{X}) := \langle K(\mathbb{X}), \nu(\tau) \rangle$ is the Vietoris space of $\mathbb{X}$. 
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With $\mathcal{U} \subseteq \omega \tau$, define

$$\nabla \mathcal{U} := \{ F \in K(\mathcal{X}) \mid (F, \mathcal{U}) \in \mathcal{P}(\epsilon) \},$$

where $(F, \mathcal{U}) \in \mathcal{P}(\epsilon)$ if $F$ is ‘properly covered’ by $\mathcal{U}$.
The Vietoris construction

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These sets $\nabla \mathcal{U}$ together provide a basis for a topology on $K(\mathbb{X})$. 

The Vietoris construction

Let \( X = \langle X, \tau \rangle \) be a topological space.
\( K(X) \) denotes the collection of compact sets
With \( U \subseteq \omega \tau \), define
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\]
where \((F, U) \in \overline{P}(\in)\) if \( F \) is ‘properly covered’ by \( U \):
\[\begin{align*}
\forall s \in F & \exists U \in U. \ s \in U \quad \text{and} \\
\forall U \in U & \exists s \in F \ . \ s \in U
\end{align*}\]
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The Vietoris construction

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These sets $\nabla \mathcal{U}$ together provide a basis for a topology on $K(X)$, the Vietoris topology $\nu_\tau$.

$V(X) := \langle K(X), \nu_\tau \rangle$ is the Vietoris space of $X$. 
Different presentation:

- **For** $a \in \tau$, **define**

  - $\Diamond a := \{ F \in K(\mathbb{R}) | F \cap a \neq \emptyset \}$
  - $\Box a := \{ F \in K(\mathbb{R}) | F \subseteq a \}$
Different presentation:

- For \( a \in \tau \), define

\[
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\]

- Generate \( \nu_\tau \) from \( \{ \langle \exists \rangle a, [\exists] \mid a \in \tau \} \) as a subbasis.
Different presentation:

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  \]

- Generate \( \nu_\tau \) from \( \{ \langle \exists \rangle a, [\exists] \mid a \in \tau \} \) as a subbasis.

**Fact** The Vietoris construction preserves various properties, including:

- compactness
- compact Hausdorffness
- zero-dimensionality
The Vietoris functor

From now on we restrict to the category KHaus of

- objects: compact Hausdorff spaces
- arrows: continuous maps

**Fact** Given \( f : X \to Y \),
From now on we restrict to the category KHaus of
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**Fact** Given $f : X \to Y$, let $Vf : K(X) \to P(Y)$ be given by

$$Vf(F) := f[F] = \{fx \mid x \in F\}$$
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Then \( Vf \) maps compact sets to compact sets.
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**Fact**

$V$ is a functor on the categories $\text{KHaus}$ and $\text{Stone}$. 
Two observations

**Observation** Stone duality and the Vietoris functor:

- **BA** \(\xrightarrow{S} \text{Stone}\)
- **P** \(\xleftarrow{S} \text{BA}\)

In a TKS \((S, R, \tau)\),

\[
R : S \to P(S)
\]

Theorem

Topological Kripke frames are Vietoris coalgebras over Stone.
Two observations

**Observation** Stone duality and the Vietoris functor:

\[ S : \text{BA} \rightarrow \text{P} \]

\[ R : \text{(S, } \tau) \rightarrow \text{V(S, } \tau) \]

Theorem Topological Kripke frames are Vietoris coalgebras over Stone
Observation Stone duality and the Vietoris functor:
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\[ \text{BA} \xrightarrow{?} \text{S} \xrightarrow{\text{V}} \text{Stone} \xrightarrow{P} \text{P}(S) \]

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[Diagram]

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Universal Coalgebra

Universal Coalgebra (Rutten, 2000) is a general mathematical theory for evolving systems.
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Universal Coalgebra (Rutten, 2000) is a general mathematical theory for evolving systems. It provides a natural framework for notions like behavior, bisimulation/behavioral equivalence, and invariants. Sufficiently general to model notions like input, output, non-determinism, interaction, probability, . . .
Let $T : C \to C$ be an endofunctor on the category $C$. 

Examples:

- Kripke structures are $P$-coalgebras over $\text{Set}$
- Deterministic finite automata are coalgebras over $\text{Set}$
Coalgebras and their morphisms

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\[
\begin{array}{ccc}
   c' & \xrightarrow{f} & c \\
   \downarrow{\gamma'} & & \downarrow{\gamma} \\
   Tc' & \xrightarrow{Tf} & Tc
\end{array}
\]
Coalgebras and their morphisms

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\begin{array}{c}
c' \\
\gamma' \\
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Tc
\end{array}
\]

Examples:

- Kripke structures are \( P \)-coalgebras over \( \text{Set} \)
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Theorem \( \text{TKS} \cong \text{Coalg}_V(\text{Stone}) \)
Vietoris coalgebras

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Manifestations:
- The final \( V \)-coalgebra \( \sim \) the canonical general frame \( (C, R, \tau) \),
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**Theorem** $\text{TKS} \cong \text{Coalg}_V(\text{Stone})$

Manifestations:
- The final $V$-coalgebra $\sim$ the canonical general frame $(C, R, \tau)$,
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Duality:
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Duality:
Modal Logic Dualizes the Vietoris Functor

Johnstone: describe $M$ via generators and relations
- Given a BA $B$, $M_B$ is the Boolean algebra generated by the set $\{3b : b \in B\}$ modulo the relations $3(a \lor b) = 3a \lor 3b$ and $3\top = \top$.

Theorem (Kupke, Kurz & Venema) ModAlg $\cong \text{ALg BA}(M)$.

The topological modal duality is an algebra | coalgebra duality.
Modal Logic Dualizes the Vietoris Functor

Johnstone: describe M via generators and relations
Modal Logic Dualizes the Vietoris Functor

- Johnstone: describe $M$ via generators and relations
- Given a BA $\mathbb{B}$, $M_{\mathbb{B}}$ is the Boolean algebra
  - generated by the set $\{\Diamond b : b \in B\}$
  - modulo the relations $\Diamond (a \lor b) = \Diamond a \lor \Diamond b$ and $\Diamond \top = \top$
Modal Logic Dualizes the Vietoris Functor

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**Theorem** (Kupke, Kurz & Venema) $\text{ModAlg} \cong \text{ALg}_{BA}(M)$.
Modal Logic Dualizes the Vietoris Functor

Johnstone: describe $M$ via generators and relations

Given a BA $B$, $M_B$ is the Boolean algebra

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**Theorem** (Kupke, Kurz & Venema) $\text{ModAlg} \cong \text{ALg}_{BA}(M)$.

The topological modal duality is an algebra|coalgebra duality
Variation: Pointfree Topology

Frames/Locales provide pointfree versions of topologies.
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\[
\text{KRFr} \leftrightarrow \text{K Haus}
\]

Variation: Pointfree Topology
Frames/Locales provide pointfree versions of topologies.

Variation: Pointfree Topology

Geometric modal logic dualizes/axiomatizes the Vietoris functor (Johnstone)
Variation: Pointfree Topology

Frames/Locales provide pointfree versions of topologies.

\[ \text{M} \rightarrow \text{KRFr} \leftarrow \text{P} \rightarrow \text{Khaus} \rightarrow \text{V} \]
Variation: Pointfree Topology

Frames/Locales provide pointfree versions of topologies.

Geometric modal logic dualizes/axiomatizes the Vietoris functor (Johnstone)
Vietoris pointfree (Johnstone Functor)

Given a frame $\mathbb{L}$, define $L\Box := \{\Box a \mid a \in L\}$ and $L\Diamond := \{\Diamond a \mid a \in L\}$.

$$M\mathbb{L} := \text{Fr}\langle L\Box \cup L\Diamond \mid \Box(\bigwedge A) = \bigwedge_{a \in A} \Box a \quad (A \in P_\omega L)
\Diamond(\bigvee A) = \bigvee_{a \in A} \Diamond a \quad (A \in P_\omega L)
\Box a \land \Diamond b \leq \Diamond(a \land b)
\Box(a \lor b) \leq \Box a \lor \Diamond b
\Box(\bigcup A) = \bigcup_{a \in A} \Box a \quad (A \in PL \text{ directed})
\Diamond(\bigcup A) = \bigcup_{a \in A} \Diamond a \quad (A \in PL \text{ directed}) \rangle$$
Vietoris and the Cover Modality $\nabla$

- Vietoris used the $\nabla$-constructor on $P_{\omega^\tau}$
Vietoris and the Cover Modality $\nabla$

- Vietoris used the $\nabla$-constructor on $P_{\omega^\tau}$
- Now think of $\nabla$ as a primitive modality
Vietoris and the Cover Modality $\nabla$

- Vietoris used the $\nabla$-constructor on $P_{\omega^\tau}$
- Now think of $\nabla$ as a primitive modality
- This modality has many manifestations in modal logic
  - normal forms (Fine)
  - coalgebraic modal logic (Moss)
  - automata theory (Walukiewicz)
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Vietoris and the Cover Modality $\nabla$

- Vietoris used the $\nabla$-constructor on $P_{\omega^\tau}$
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- This modality has many manifestations in modal logic
  - normal forms (Fine)
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- May develop $\nabla$-logic . . .
- . . . and formulate the functor $M$ accordingly, in terms of $\nabla$
New directions

Fix a standard set functor $T$ that preserves weak pullbacks.

Define the $T$-powerlocale of a frame $L$ as $M^T_L := \text{Fr} \langle T \omega L \mid (\nabla_1), (\nabla_2), (\nabla_3) \rangle$, where the relations are as follows:

$(\nabla_1) \nabla \alpha \leq \nabla \beta (\alpha T \leq \beta)$

$(\nabla_2) \wedge \gamma \in \Gamma \nabla \gamma \leq \bigvee \{\nabla (T \wedge) \Psi \mid \Psi \in \text{SRD}(\Gamma) \}$ ($\Gamma \in \text{P} \omega T \omega L$)

$(\nabla_3) \nabla (T \bigvee) \Phi \leq \bigvee \{\nabla \beta \mid \beta T \in \Phi \}$ ($\Phi \in T \omega \text{P} L$)
New directions

Fix a standard set functor $T$ that preserves weak pullbacks.
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2. $(\nabla 2)$ $\bigwedge_{\gamma \in \Gamma} \nabla \gamma \leq \bigvee \{ \nabla (T \land) \psi \mid \psi \in SRD(\Gamma) \}$ $\quad (\Gamma \in \mathcal{P}_\omega T_\omega L)$
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(\nabla 2) & \quad \bigwedge_{\gamma \in \Gamma} \nabla \gamma \leq \bigvee \{ \nabla (T \wedge) \Psi \mid \Psi \in \text{SRD}(\Gamma) \} & (\Gamma \in P_\omega T_\omega \mathbb{L}) \\
(\nabla 3) & \quad \nabla (T \vee) \Phi \leq \bigvee \{ \nabla \beta \mid \beta \bar{T} \in \Phi \} & (\Phi \in T_\omega \mathbb{P} \mathbb{L})
\end{align*}
Some results

Theorem (V., Vickers & Vosmaer)

Given a set functor $T$ that preserves weak pullbacks:

- $M_T$ provides a functor on the category Fr of frames.
- $M_T$ generalizes Johnstone's $M$: $M_T \sim M_P$.
- $M_T$ preserves regularity, zero-dimensionality, and Stone-ness.
- $M_T$ restricts to a functor on $\text{KRFr}$ (compact regular frames) provided $T$ preserves finiteness.
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Question

\[
\begin{array}{c}
\text{KRFr} \\
\overset{S}{\rightarrow}
\end{array}
\begin{array}{c}
\text{KHaus} \\
\overset{P}{\rightarrow}
\end{array}
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**Question**

Describe the dual of $M_T$ for an arbitrary set functor $T$!
Overview

- Introduction
- Modal Dualities
- Subdirectly irreducible algebras and rooted structures
- Vietoris via modal logic
- Final remarks
Final Remarks

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  - isolate interesting concepts
Final Remarks

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Dualities can be used ‘on the other side’ to:
- solve problems
- isolate interesting concepts
- trigger interesting questions
References


http://staff.fnwi.uva.nl/y.venema