Relational T-algebra and the category of topological spaces

Yoshihiro Mizoguchi
Institute of Mathematics for Industry
Kyushu University, JAPAN

Workshop on Logic Algebra and Category Theory : LAC2018
La Trobe University, Melbourne
February 14, 2018

Table of Contents

First step of an algebra of logic

- Boolean Algebra (1847) : an algebra of logic!

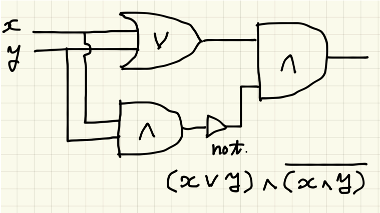
$$\mathcal{B} = (B, \bot, \top, \wedge, \vee, -)$$

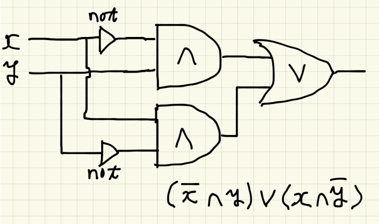
- De Morgan's Law (1864) : a formula of logic!

$\overline{(x \vee y)} = \overline{x} \wedge \overline{y}, \qquad \overline{(x \wedge y)} = \overline{x} \vee \overline{y}.$

- Symbolic Computing : A merit of algebraic formalization!

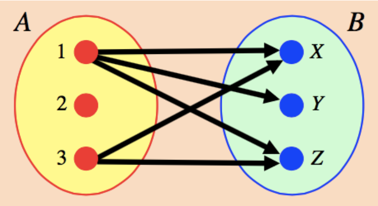
$$\begin{aligned}
(x \vee y) \wedge \overline{(x \wedge y)} &= (x \vee y) \wedge (\overline{x} \vee \overline{y}) \\
&= (x \wedge \overline{x}) \vee (x \wedge \overline{y}) \vee (y \wedge \overline{x}) \vee (y \wedge \overline{y}) \\
&= \bot \vee (x \wedge \overline{y}) \vee (y \wedge \overline{x}) \vee \bot = (x \wedge \overline{y}) \vee (\overline{x} \wedge y)
\end{aligned}$$





Theory of Relational Calculus (1)

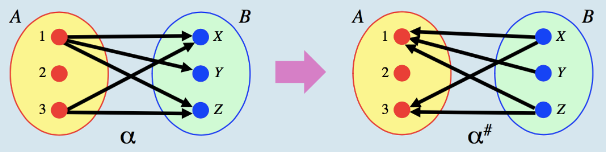
(1) A **relation** α from a set A into another set B is a subset of the Cartesian product $A \times B$ and denoted by $\alpha : A \rightharpoondown B$.



$$
\begin{aligned}
A &= \{1, 2, 3\} \\
B &= \{X, Y, Z\} \\
\alpha &= \{(1, X), (1, Y), (1, Z), (3, X), (3, Z)\} \\
\alpha &\subseteq A \times B
\end{aligned}
$$

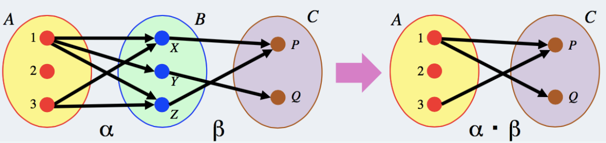
Theory of Relational Calculus (2)

(2) The **inverse relation** $\alpha^\sharp : B \rightharpoondown A$ of α is a relation such that $(b, a) \in \alpha^\sharp$ if and only if $(a, b) \in \alpha$.



We note $\alpha \subseteq A \times B$ and $\alpha^\sharp \subseteq B \times A$.

(3) The **composite** $\alpha \cdot \beta : A \rightharpoondown C$ of $\alpha : A \rightharpoondown B$ followed by $\beta : B \rightharpoondown C$ is a relation such that $(a, c) \in \alpha \cdot \beta$ if and only if there exists $b \in B$ with $(a, b) \in \alpha$ and $(b, c) \in \beta$.

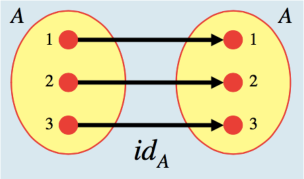


We note $\alpha \cdot \beta \subseteq A \times C$.

Theory of Relational Calculus (3)

(4) As a relation of a set A into a set B is a subset of $A \times B$, the inclusion relation, union, intersection and difference of them are available as usual and denoted by $\sqsubseteq$, $\sqcup$, $\sqcap$ and $-$, respectively.

(5) The **identity relation** $\mathrm{id}_A : A \rightharpoondown A$ is a relation with $\mathrm{id}_A = \{(a, a) \in A \times A \,|\, a \in A\}$.



(6) The **empty relation** $\phi \subseteq A \times B$ is denoted by $\mathbf{0}_{AB}$. The entire set $A \times B$ is called the **universal relation** and denoted by ∇_{AB}.

(7) The one point set $\{*\}$ is denoted by $\mathbf{I}$. We note that $\nabla_{II} = \mathrm{id}_I$.

Theory of Relational Calculus (4)

- Axiom
  - $\alpha \cdot \mathrm{id} = \alpha$
  - $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$
  - $(\alpha \cdot \beta)^\sharp = \beta^\sharp \cdot \alpha^\sharp$
  - $(\alpha^\sharp)^\sharp = \alpha$
  - If $\alpha \sqsubseteq \alpha'$ then $\alpha^\sharp \sqsubseteq \alpha'^\sharp$.
  - $(\alpha \cdot \beta) \sqcup \gamma \sqsubseteq \alpha \cdot (\beta \sqcap (\alpha^\sharp \cdot \gamma))$
  - ...
- Lemma
  - $\alpha \cdot (\beta \sqcup \gamma) = (\alpha \cdot \beta) \sqcup (\alpha \cdot \gamma)$
  - $\alpha \cdot (\beta \sqcap \gamma) \sqsubseteq (\alpha \cdot \beta) \sqcap (\alpha \cdot \gamma) \sqsubseteq \alpha \cdot (\beta \sqcap (\alpha^\sharp \cdot \alpha \cdot \gamma))$
  - $\alpha \sqsubseteq \alpha \cdot \alpha^\sharp \cdot \alpha$
  - If $\beta \sqsubseteq \beta'$ then $\alpha \cdot \beta \cdot \gamma \sqsubseteq \alpha \cdot \beta' \cdot \gamma$.
  - ...

composition of an injection and an injection is an injection (relational formula)

Proposition

Let $f : X \to Y$, $g : Y \to Z$ be injections. Then $f \cdot g : X \to Z$ is an injection.

$$(f \cdot f^\sharp \sqsubseteq id_X) \wedge (g \cdot g^\sharp \sqsubseteq id_Y) \Rightarrow ((f \cdot g) \cdot (f \cdot g)^\sharp \sqsubseteq id_X)$$

$$\begin{array}{lll}
 & (f \cdot g) \cdot (f \cdot g)^\sharp & \\
= & (f \cdot g) \cdot (g^\sharp \cdot f^\sharp) & (\because (\alpha \cdot \beta)^\sharp = \beta^\sharp \cdot \alpha^\sharp) \\
= & f \cdot (g \cdot g^\sharp) \cdot f^\sharp & (\because \text{associative law}) \\
\sqsubseteq & f \cdot \mathrm{id}_Y \cdot f^\sharp & (\because g \cdot g^\sharp \sqsubseteq \mathrm{id}_Y) \\
= & f \cdot f^\sharp & (\because \mathrm{id}_Y \text{is unit}) \\
\sqsubseteq & \mathrm{id}_X & (\because f \cdot f^\sharp \sqsubseteq \mathrm{id}_X)
\end{array}$$

Proof can be done using symbolic transformations.

composition of an injection and an injection is an injection (relational formula)

```
 Theorem injection_composite_rel_tactic
  {X Y Z : eqType} {f : Rel X Y} {g : Rel Y Z}:
  (f  ·  (f #)) ⊆  Id X /\ (g  ·  (g #)) ⊆  Id Y ->
  ((f  ·  g)  ·   ((f  ·  g) #)) ⊆  Id X.
Proof.
  Rel_simpl2.
Qed.
```

※ We can implement an automatic prover (Tactic).

The long and winding load to the relational T-algebra

- Algebra (Group,Ring,Field) $\to$ (Ω, E)-algebra (Universal Algebra) $\to$ T-algebra ($\mathbf{Set}^T$) $\to$ relational T-algebra ($Rel(T)$)
- Monad (Triple) (T, η, μ)
- Kleisli Category $(T, \eta, \circ)$
- Haskell's monad (T, `return`, >>=)
- ultrafilter monad (U, η_U, μ_U) $\to$ $\mathbf{Set}^U \cong CH \to \mathrm{Rel}(U) \cong Top$

Our goal is to refine those theories and introduce a formal proof using relational calculus and Coq a proof assistant system.

Category

Definition

A **category** $\mathcal{C}$ is defined by the following data and axioms.

- Datum1 $Obj(\mathcal{C})$: a class of objects in $\mathcal{C}$.
- Datum2 $\mathcal{C}(A, B)$: a class of $\mathcal{C}$-morphisms for objects A and B.
- Datum3 $id_A \in \mathcal{C}(A, A)$: the identity morphism id_A for any object A.
- Datum4 $g \cdot f \in \mathcal{C}(A, C)$ is defined by $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$.
- Axiom1 For any $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$ and $h \in \mathcal{C}(C, D)$, $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.
- Axiom2 For any $f \in \mathcal{C}(A, B)$, $f \cdot id_A = f = id_B \cdot f$.
- Axiom3 If $A \neq A'$ and $B \neq B'$ then $\mathcal{C}(A, B) \cap \mathcal{C}(A', B') = \phi$.

Functor

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **functor** $H : \mathcal{C} \to \mathcal{D}$ is defined by the following data and axioms.

- Datum1 $HA \in Obj(\mathcal{D})$ is defined by $A \in Obj(\mathcal{C})$.
- Datum2 $Hf \in \mathcal{D}(HA, HB)$ is defined by $f \in \mathcal{C}(A, B)$.
- Axiom1 $Hid_A = id_{HA}$.
- Axiom2 For any $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$, $H(g \cdot f) = Hg \cdot Hf$.

Natural transformation

Definition

$$\begin{array}{ccc} HA & \xrightarrow{Hf} & HB \\ {\scriptstyle \alpha A}\downarrow & & \downarrow{\scriptstyle \alpha B} \\ H'A & \xrightarrow[H'f]{} & H'B \end{array}$$

Let $H, H' : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\alpha : H \to H'$ is defined by the following datum and axiom.

Datum $\alpha A \in \mathcal{D}(HA, H'A)$ is defined by $A \in Obj(\mathcal{C})$.

Axiom For any $f \in \mathcal{C}(A, B)$, the following diagram commutes.

Example

Set(sets and functions), **Lin**(linear spaces and linear maps), **Grp**(groups and homomorphisms).

Ω-algebra

Definition

Let Ω_n be a label set of n-ary operators for $n = 0, 1, \ldots$ and $\Omega = \{\Omega_n | n = 0, 1, \ldots\}$. For a given set X, $\delta = \{\delta_\omega | \omega \in \Omega_n, n = 0, 1, \ldots\}$ is a set of n-ary functions $\delta_\omega : X^n \to X$. A pair (X, δ) is called a Ω-**algebra**.

Let (X, δ) and (Y, γ) are Ω-algebras. A function $f : X \to Y$ is an Ω-**morphism** if

$$f \cdot \delta_\omega(x_1, x_2, \cdots, x_n) = \gamma_\omega(f(x_1), f(x_2), \cdots, f(x_n))$$

for any $\omega \in \Omega_n$.

Ω-term

Definition

Let A be a set. The set ΩA of all ω-**terms** over A is defined as follows.

1. $a \in A \Rightarrow a \in \Omega A$
2. $\omega \in \Omega_n,\ p_1, \ldots, p_n \in \Omega A \Rightarrow \omega(p_1, \ldots, p_n) \in \Omega A.$

Definition

Let $V = \{v_1, v_2, \ldots, v_n, \ldots\}$ be a set of variables. For two elements $e_1, e_2 \in \Omega V$, a set $\{e_1, e_2\}$ is called Ω-**equation**. A pair (Ω, E) of Ω and a set E of Ω-equation is called an **equational presentation**.

Example

Example

Let m(multiple), i(inverse) and e(unit) be labels of operators. Let $\Omega_0 = \{e\}$, $\Omega_1 = \{i\}$, $\Omega_2 = \{m\}$ and $E = \{\{m(v_1, m(v_2, v_3)), m(m(v_1, v_2), v_3)\}, \{m(v_1, e), v_1\}, \{m(e, v_1), v_1\}, \{m(v_1, i(v_1), e\}, \{m(i(v_1), v_1), e\}\}$. Then a group X can be considered as an Ω-algebra.

Example

For a division function $d(x, y) = m(x, i(y))$ in a group, we define $\Omega_2 = \{d\}$ and

$$E = \{\{d(x, d(d(d(d(x, x), y), z), d(d(d(x, x), x), z))), y\}\}.$$

Then an (Ω,E)-algebra can be considered as a group[4].

Ω-algebra

Example (The Total Description Map)

Let (X, δ) be an Ω-algebra. δ can be naturally extended to $\delta^{@} : \Omega X \to X$.

- $\delta^{@}(x) = x (x \in X)$,
- $\delta^{@}(\omega(p_1, \cdots, p_n)) = \delta_\omega(\delta^{@}(p_1), \cdots, \delta^{@}(p_n)) (\omega \in \Omega_n)$.

Definition (Ω-algebra)

For a given Ω-algebra (X, δ) and an assignment $r : V \to X$, an extension map $r^\sharp : \Omega V \to X$ is defined by $\delta^{@} \cdot \Omega r$. Let $\{e_1, e_2\}$ be a Ω-equation. If $r^\sharp(e_1) = r^\sharp(e_2)$ for any $r : V \to X$ then we say (X, δ) satisfies $\{e_1, e_2\}$. If an Ω-algebra satisfies all equations in E, then it is called as an (Ω,E)-algebra.

Ω-morphism

Definition

For a given set A, we define an equivalence relation E_A over ΩA by

$$E_A = \{(p, q) | \forall (X, \delta) : \Omega - algebra, \forall f : A \to X, f^{\sharp}(p) = f^{\sharp}(q)\}.$$

We denote a quotient set of ΩA by an equivalence relation E_A as $TA = \Omega A / E_A$. We denote an equivalence class including $p \in \Omega A$ as $\rho A(p) = [p]$. Then $\rho A : \Omega A \to TA$.

Proposition

Let $\omega_n \in \Omega_n$, $\omega_n([p_1], \cdots, [p_n]) = [\omega_n(p_1, \cdots, p_n)]$ and $\omega = \{\omega_n | n = 0, 1, \ldots\}$. An Ω-algebra (TA, ω) is an (Ω, E)-algebra and $\rho A : \Omega A \to TA$ is an Ω-morphism.

(Ω, E)-algebra (1)

Definition

We denote (TA, ω) as TA and it is called a **free** (Ω, E)**-algebra** over A.

Proposition (The Universal Property of TA)

A function $\eta A : A \to TA$ is defined by $\eta A(a) = [a]$. For any (Ω, E)-algebra (X, δ) and a function $f : A \to X$, there exists a unique Ω-morphism $f^{\sharp\sharp} : TA \to (X, \delta)$ of f such that $f^{\sharp\sharp} \cdot \eta A = f$.

Monad (Triple)

Definition (Algebraic theory and monad)

A monad type **algebraic theory** over a category $\mathcal{C}$ is a triple $T = (T, \eta, \mu)$ satisfies followings.
$T : \mathcal{C} \to \mathcal{C}$ is a functor.
$\eta : I \to T$, $\mu : TT \to T$ is a natural transformation.
$\mu_A \cdot \eta_{TA} = id_{TA}$, $\mu_A \cdot T\eta_A = id_{TA}$ and $\mu_A \cdot T\mu_A = \mu_A \cdot \mu_{TA}$ hold for any $A \in Obj(\mathcal{C})$.
A category $\mathcal{C}^T$ of T-algebra and T-homomorphisms is defined by

$$Obj(\mathcal{C}^T) = \{(X, x) \,|\, X \in Obj(\mathcal{C}), x : TX \to X, x \cdot \eta_X = id_X, x \cdot Tx = x \cdot \mu_X\}$$

and

$$\mathcal{C}^T((X, x), (X', x')) = \{f \in \mathcal{C}(X, X') \,|\, x' \cdot Tf = f \cdot x\}.$$

We call an object in $\mathcal{C}^T$ as T-algebra and a morphism as T-homomorphism.

Kleisli Category

Definition (Algebraic theory and Kleisli category)

A clone type **algebraic theory** over a category $\mathcal{C}$ is a triple $T = (T, \eta, \circ)$ satisfies followings.
T is a map $T : Obj(\mathcal{C}) \to Obj(\mathcal{C})$.
A morphism $\eta A : A \to TA$ is defined for any object $A \in Obj(\mathcal{C})$.
$\circ$ is a map $\circ : \mathcal{C}(A, TB) \times \mathcal{C}(B, TC) \to \mathcal{C}(A, TC)$.
For any $f \in \mathcal{C}(A, B)$, $f^{\Delta} : A \to TB$ is defined by $f^{\Delta} = A \xrightarrow{f} B \xrightarrow{\eta B} TB$.
For any morphisms $\alpha \in \mathcal{C}(A, TB)$, $\beta \in \mathcal{C}(B, TC)$ and $\gamma \in \mathcal{C}(C, TD)$, the followings hold.

- $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$
- $\alpha \circ \eta B = \alpha$
- $\beta \circ \alpha^{\Delta} = (\beta\alpha)^{\Delta}$

A **Kleisli category** $\mathcal{C}_T$ is a category defined by $Obj(\mathcal{C}_T) = Obj(\mathcal{C})$, $\mathcal{C}_T(A, B) = \mathcal{C}(A, TB)$ and a composition of morphisms are defined by $\circ$.
We note the identity $id_A \in \mathcal{C}_T(A, A)$ is $\eta A : A \to TA$.

Monad and Kleisli category

Theorem ([7])

There exists a bijective correspondence between a clone type algebraic theory $T = (T, \eta, \circ)$ *and a monad type algebraic theory* $T = (T, \eta, \mu)$.

Theorem

A category of (Ω, E)*-algebra is isomorphic to a category of* $\mathbf{Set}^T$ *defined by its algebraic theory* $T = (T, \eta, \circ)$.

Example

Example

Let $\Omega_0 = \{zero\}$, $\Omega_1 = \{succ\}$, $\Omega_2 = \{plus\}$, $E = \{\{plus(zero, x), x\},$ $\{plus(succ(x), y), succ(plus(x, y))\}\}$. For (Ω, E), we have

$$(\Omega, E) \vdash plus(succ(zero), zero) = succ(zero)$$

(Ω, E)-algebra (2)

Definition

A **clone category** $\mathbf{Set}(\Omega, E)$ of (Ω, E) is defined as follows.

Object $Obj(\mathbf{Set}(\Omega, E)) = Obj(\mathbf{Set})$

Morphism $\mathbf{Set}(\Omega, E)(A, B) = \mathbf{Set}(A, TB)$. For any $\alpha : A \to TB$ and $\beta : B \to TC$, we define

$$(A \xrightarrow{\alpha} B) \circ (B \xrightarrow{\beta} C) = A \xrightarrow{\alpha} TB \xrightarrow{\beta^\sharp} TC.$$

Identity $id_A \in \mathbf{Set}(\Omega, E)(A, A)$ is $id_A = \eta A : A \to TA$.

Proposition

$\mathbf{Set}(\Omega, E)$ *is a category and* $T\phi$ *is the initial object in* $\mathbf{Set}(\Omega, E)$.

Haskell's monad (1)

- Let **Set** be the category of sets and functions.
- Let **List** be the category of free monoids and homomorphisms.
- Let $A = Integer$ (the set of all integers).
- Let $F : Set \to List$ be a functor creating a free monoid.
  We note $1, 2, 3 \in A$ and $[1, 2, 3] \in FA$.
  For $f : A \to B$ we define $Ff : FA \to FB$ as $Ff = (map\ f)$.
  Ff[1,2,3] = (map f [1,2,3]) = [f(1),f(2),f(3)]
- $concat : FFA \to FA$ is a natural transfromation.
  $concat[[1, 2], [3], [4, 5, 6]] = [1, 2, 3, 4, 5, 6]$
- $return : A \to FA$ is a natural transformation.
  $return\ x = [x]$

Haskell's monad (2)

Haskell's monad is constructed by a triple $(F, return, >>=)$.

- $F : Obj(Set) \to Obj(Set)$.
- $return : A \to FA$ (for $A \in Obj(Set)$)
- $>>=: Set(A, FA) \to Set(FA, FB)$
  We denote $>>= (f)(l_a)$ as $l >>= f$.
  $l_a >>= f$ is defined as $(concat\ (map\ f\ l_a))$.
-

$$\begin{aligned} & [1,2,3] >>= (\lambda\ x.[x, 2x]) \\ = \quad & concat\ (map\ (\lambda\ x.[x, 2x])\ [1,2,3]) \\ = \quad & concat\ [[1,2],[2,4],[3,6]] \\ = \quad & [1,2,2,4,3,6] \end{aligned}$$

Haskell's monad (3)

- Haskell's monad is a triple $(F, \mathit{return}, >>=)$.
  - $\mathit{return} : A \to FA$
  - $>>=: (A \to FB) \to (FA \to FB)$
- Kleisli category is a triple $(F, \eta, \circ)$.
  - $\eta : A \to FA$
  - $\circ : (A \to FB) \times (B \to FC) \to (A \to FC)$
- Correspondence between $(F, \mathit{return}, >>=)$ and $(F, \eta, \circ)$.
  - $\eta = \mathit{return}$
  - $\alpha \circ \beta = \lambda\, x.((\alpha\ x) >>= \beta)$.

Ultra filter monad

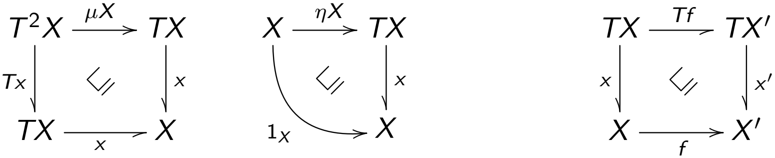
Let $U : \mathrm{Set} \to \mathrm{Set}$ be a functer, and $\eta U : 1_{\mathrm{Set}} \to U$ and $\mu U : U^2 \to U$ natural transformations. For a set X, UX is a set of ultra filters over X. For a set Y and a function $\Psi : X \to Y$, we define $U\Psi : UX \to UY$, $\eta UX : X \to UX$, and $\mu UX : U^2X \to UX$ by

$$\begin{array}{rcl} U\Psi(\mathcal{U}) & := & \{B \sqsubseteq Y \mid \Psi^\sharp \cdot B \in \mathcal{U}\} \\ \eta UX(a) & := & \{A \sqsubseteq X \mid a \in A\} \\ \mu UX(\mathscr{U}) & := & \{A \sqsubseteq X \mid \pi UX(A) \in \mathscr{U}\}. \end{array}$$

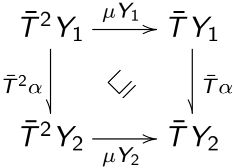
where $\pi UX(A) := \{\mathcal{U} \sqsubseteq 2^X \mid A \in \mathcal{U}\}$.
We note $\mathbf{U} = (U, \eta, \mu)$ is a monad over Set and called ultra filter monad.

Relational **T**-algebra

Let $\mathbf{T} = (T, \eta, \mu)$ a monad on Set. If $x \cdot Tx \sqsubseteq x \cdot \mu X$, and $1_X \sqsubseteq x \cdot \eta X$ holds for a pair (X, x) of a set X and a function $x : TX \rightharpoondown X$, then (X, x) is called a relational **T**-algebra. For two relational **T**-algebra (X, x), and (X', x'), A function $f : (X, x) \to (X', x')$is called a relational **T**-morphism if $f \cdot x \sqsubseteq x' \cdot Tf$. We denote the category of relational **T**-algebra and relational **T**-relations as Rel(**T**).



A proof using relational calculus



i.e.

$$\mu Y_2 \cdot \bar{T}^2\alpha \sqsubseteq \bar{T}\alpha \cdot \mu Y_1$$

can be proved as follows:

$$\begin{array}{ll} & \mu Y_2 \cdot \bar{T}^2\alpha \\ = & \mu Y_2 \cdot T^2 g_\alpha \cdot (T^2 f_\alpha)^\sharp \\ \sqsubseteq & \mu Y_2 \cdot T^2 g_\alpha \cdot (\mu R_\alpha)^\sharp \cdot (T f_\alpha)^\sharp \cdot \mu Y_1 \\ \sqsubseteq & \mu Y_2 \cdot (\mu Y_2)^\sharp \cdot T g_\alpha \cdot (T f_\alpha)^\sharp \cdot \mu Y_1 \\ \sqsubseteq & T g_\alpha \cdot (T f_\alpha)^\sharp \cdot \mu Y_1 \\ \sqsubseteq & \bar{T}\alpha \cdot \mu Y_1 \end{array}$$

Our motivation of formalization of mathematics using relational calculus

Closure Space

Let X be a set, $\alpha, \alpha' : I \rightharpoondown X$ subsets in X. Then $\Gamma : 2^X \to 2^X$ holds: (a_1), (a_2), and (a_3) then Γ is called a closure of X, further if it satisfies (b) then its called a closure system.

- (a_1) $\alpha \sqsubseteq \Gamma\alpha$
- (a_2) $\alpha \sqsubseteq \alpha' \to \Gamma\alpha \sqsubseteq \Gamma\alpha'$
- (a_3) $\Gamma^2\alpha = \Gamma\alpha$
- (b) $\Gamma(\alpha \sqcup \alpha') = \Gamma\alpha \sqcup \Gamma\alpha'$

For closure systems (X, Γ), and (X', Γ'), a function $f : (X, \Gamma) \to (X', \Gamma')$ is continuous if for all $\alpha : I \rightharpoondown X$ such that $f \cdot \Gamma\alpha \sqsubseteq \Delta(f \cdot \alpha)$.
We denote the closure space as Clos. and the category of topological spaces and continuous functions as $\mathrm{To}p$.

Conclusion

Definition

We define $C : \mathsf{Rel}(\mathbf{T}) \to \mathsf{Clos}$ as follows: For a relational $\mathbf{T}$-algebra (X, x), we define $C(X, x) = (X, \Gamma_x)$, where $\Gamma_x : 2^X \to 2^X$ is defined for $\alpha : I \rightharpoonup X$ by $\Gamma_x \alpha = x \cdot \bar{T}\alpha \cdot \eta I$. We define $Cf := f$ for a $\mathbf{T}$-morphism $f : (X, x) \to (X', x')$.

Definition

We define $J : \mathrm{Top} \to \mathrm{Rel}(\mathbf{U})$ as follows: for a topological space (X, Γ) $J(X, \Gamma) := (X, r_\Gamma)$. where,

$$(\mathcal{U}, a) \in r_\Gamma \quad \leftrightarrow \quad a \in \lim \mathcal{U} \ (= \sqcap_{F \in \mathcal{U}} \Gamma F)$$

For a topological space (X', Γ') and a continuous function $\Psi : (X, \Gamma) \to (X', \Gamma')$, we define $J\Psi := \Psi$.

Theorem (Barr(1970))

- $C \cdot J = 1_{\mathrm{Top}}$, $J \cdot C = 1_{\mathsf{Rel}(\mathbf{U})}$, $\mathsf{Rel}(\mathbf{U}) \cong \mathrm{Top}$

References I

- M. Barr, Relational algebra, Lecture Notes in Math., 137(1970), 39–55.
- A. Day, Filter monads, continuous lattices and closure systems, Can. J. Math., 27(1975), 50–59.
- H. Furusawa, Y. Kawahara, Point axioms and related conditions in Dedekind categories, J. of Logical and Algebraic Methods in Programming, Vol.84(2015), 359–376.
- G.Higman, B.H.Neumann, Groups as groupoids with one law, Publ. Math. Debrecen 2, 215–221,1952.
- Y.Kawahara, Y.Mizoguchi, Categorical assertion semantics in toposes, Advances in Software Science and Technology, Vol.4(1992), 137-150. https://catalog.lib.kyushu-u.ac.jp/opac_download_md/25296/cas-jssst.pdf

References II

- E.G.Manes, a triple-theoretic construction of compact algebras, Lecture Notes in Mathematics, Vol.80(1969), Springer-Verlag, 91–118.
- E.G.Manes, Algebraic Theories, Springer-Verlag, 1976.
- S.Mac Lane, Categories for the working mathematician, Springer-Verlag, 1978.
- Y.Mizoguchi, Powerset monad, filter monad and primfilter monad in the category of set with monoid actions, Bull. of Informatics and Cybernetics, lVol.21(1985), 83-95. https://catalog.lib.kyushu-u.ac.jp/opac_download_md/13370/p083.pdf
- Y.Mizoguchi, H.Tanaka, S.Inokuchi, Formalization of proofs using relational calculus, Proc. International Symposium on Information Theory and Its Applications (ISITA2016), pp. 532-536, November, 2016.