応用数学 B　レポート問題

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Consider the initial value problem

$$
\begin{aligned}
&(1-\Delta)u_{tt}+\Delta^2 u-\mu\Delta u_t-a^2\Delta u=\Delta f(u),\\
&u(0)=u_1, \qquad u_t(0)=u_1:=\Lambda U_1,
\end{aligned}
\tag{1}
$$

where μ and a are positive constants, $f(u)=u^2$, and $\Lambda u=\mathcal{F}^{-1}[i|\xi|\hat{u}]$.

Problem 1. Verify that the problem (1) is transformed to the integral equation

$$
u(t)=\Lambda G(t)*U_1+H(t)*u_0+\int_0^t \Lambda G(t-\tau)*(1-\Delta)^{-1}\Lambda f(u)(\tau)d\tau, \tag{2}
$$

where $G(t)$ and $H(t)$ are the fundamental solutions corresponding to u_1 and u_0, respectively.

Problem 2. Prove the following decay estimates:

$$
\begin{aligned}
&\|\partial_x^k \Lambda G(t)*\phi\|_{L^2}+\|\partial_x^k H(t)*\phi\|_{L^2}\\
&\quad\le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k-j}{2}}\|\partial_x^j\phi\|_{L^q}+Ce^{-ct}\|\partial_x^{k+l}\phi\|_{L^2},
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
&\|\partial_x^k \Lambda G(t)*(1-\Delta)^{-1}\Lambda f\|_{L^2}\\
&\quad\le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1-j}{2}}\|\partial_x^j f\|_{L^q}+Ce^{-ct}\|\partial_x^{k+l-1}\phi\|_{L^2},
\end{aligned}
\tag{4}
$$

where $1\le q\le 2$, $0\le j\le k$, and $l\ge 0$ in (3), and $1\le q\le 2$, $0\le j\le k+1$, $k+l-1\ge 0$, and $k,\ l\ge 0$ in (4).

Problem 3. Let $n\ge 1$ and $s\ge[\frac{n}{2}]$. Suppose that $(u_0,U_1)\in H^{s+1}\cap L^1$ and put $E_1=\|(u_0,U_1)\|_{H^{s+1}\cap L^1}$. Let u be the global solution to the problem (2). When E_1 is suitably small, prove that the solution u satisfies the decay estimate:

$$
\|\partial_x^k u(t)\|_{L^2}\le CE_1(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \tag{5}
$$

where $0\le k\le s+1$.