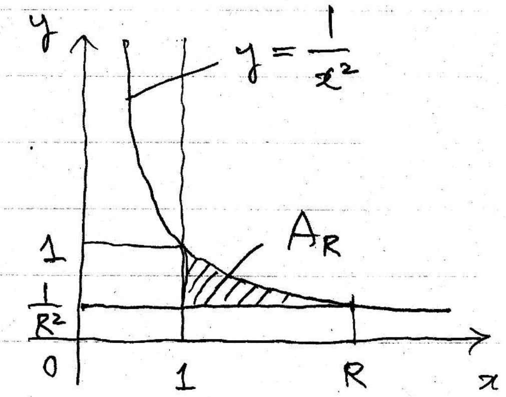
レポート



(1) $\displaystyle\iint_A \frac{dxdy}{\sqrt{xy}}$, $A: x \geqq 1,\ 0 \leqq y \leqq \frac{1}{x^2}$

図の領域を A_R とすると

$$\iint_{A_R} \frac{dxdy}{\sqrt{xy}} = \int_1^R \left(\int_{\frac{1}{R^2}}^{\frac{1}{x^2}} \frac{dy}{\sqrt{xy}} \right) dx$$

$$= \int_1^R \frac{1}{\sqrt{x}} \left(\int_{\frac{1}{R^2}}^{\frac{1}{x^2}} \frac{dy}{\sqrt{y}} \right) dx$$

$$= \int_1^R \frac{1}{\sqrt{x}} \Big[2\sqrt{y} \Big]_{y=\frac{1}{R^2}}^{\frac{1}{x^2}} dx$$

$$= \int_1^R \frac{2}{\sqrt{x}} \left(\frac{1}{x} - \frac{1}{R} \right) dx = 2\int_1^R \left(\frac{1}{x\sqrt{x}} - \frac{1}{R}\frac{1}{\sqrt{x}} \right) dx$$

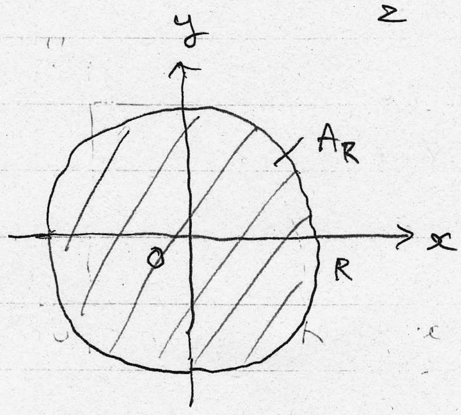
$$= 2\left[-\frac{2}{\sqrt{x}} - \frac{2}{R}\sqrt{x} \right]_{x=1}^{R}$$

$$= 4\left\{ \left(1 + \frac{1}{R}\right) - \left(\frac{1}{\sqrt{R}} + \frac{1}{\sqrt{R}} \right) \right\}$$

$$= 4\left(1 + \frac{1}{R} - \frac{2}{\sqrt{R}}\right) \longrightarrow 4 \quad (R \to \infty)$$

よって広義積分は収束して

$$\iint_A \frac{dxdy}{\sqrt{xy}} = 4$$



(2) $\displaystyle\iint_{\mathbb{R}^2} \frac{dxdy}{(1+x^2+y^2)^{\frac{\alpha}{2}}} \quad (\alpha > 0)$

図の領域 A_R で考える.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

により

$$A_R \longleftrightarrow A_R' \quad \begin{cases} 0 \leq r \leq R \\ 0 \leq \theta \leq 2\pi \end{cases}$$

また $dxdy = r\,dr\,d\theta$, よって

$$I_R := \iint_{A_R} \frac{dxdy}{(1+x^2+y^2)^{\frac{\alpha}{2}}} = \iint_{A_R'} \frac{r\,dr\,d\theta}{(1+r^2)^{\frac{\alpha}{2}}}$$

$$= \int_0^{2\pi} \left(\int_0^R \frac{r\,dr}{(1+r^2)^{\frac{\alpha}{2}}} \right) d\theta = \int_0^R \frac{r\,dr}{(1+r^2)^{\frac{\alpha}{2}}} \cdot \int_0^{2\pi} d\theta$$

$$= 2\pi \int_0^R \frac{r\,dr}{(1+r^2)^{\frac{\alpha}{2}}}$$

<u>case 1</u> $\alpha > 2$ のとき

$$I_R = 2\pi \left[-\frac{(1+r^2)^{-(\frac{\alpha}{2}-1)}}{2(\frac{\alpha}{2}-1)} \right]_{r=0}^{R}$$

$$= \frac{2\pi}{\alpha-2} \left\{ 1 - (1+R^2)^{-(\frac{\alpha}{2}-1)} \right\} \longrightarrow \frac{2\pi}{\alpha-2} \quad (R \to \infty)$$

<u>case 2</u>　$0 < \alpha < 2$ のとき

$$I_R = 2\pi \left[\frac{(1+r^2)^{1-\frac{\alpha}{2}}}{2\left(1-\frac{\alpha}{2}\right)} \right]_{r=0}^{R}$$

$$= \frac{2\pi}{2-\alpha} \left\{ (1+R)^{1-\frac{\alpha}{2}} - 1 \right\} \longrightarrow \infty \quad (R \to \infty)$$

<u>case 3</u>　$\alpha = 2$ のとき

$$I_R = 2\pi \int_0^R \frac{r\,dr}{1+r^2} = 2\pi \left[\frac{1}{2} \log(1+r^2) \right]_{r=0}^{R}$$

$$= \pi \log(1+R^2) \longrightarrow \infty \quad (R \to \infty)$$

以上より

$\alpha > 2$ のとき 広義積分は収束して

$$\iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+x^2+y^2)^{\frac{\alpha}{2}}} = \frac{2\pi}{\alpha-2}$$

$0 < \alpha \leqq 2$ のとき 広義積分は $+\infty$ に発散する.