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DECAY PROPERTY FOR A PLATE EQUATION WITH MEMORY-TYPE DISSIPATION

YONGQIN LIU

Department of Mathematics, North China Electric Power University
Beijing 102208, China
Faculty of Mathematics, Kyushu University
Fukuoka 819-0395, Japan

SHUICHI KAWASHIMA

Faculty of Mathematics, Kyushu University
Fukuoka 819-0395, Japan

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Abstract. In this paper we focus on the initial value problem of the semi-linear plate equation with memory in multi-dimensions ($n \geq 1$), the decay structure of which is of regularity-loss property. By using Fourier transform and Laplace transform, we obtain the fundamental solutions and thus the solution to the corresponding linear problem. Appealing to the point-wise estimate in the Fourier space of solutions to the linear problem, we get estimates and properties of solution operators, by exploiting which decay estimates of solutions to the linear problem are obtained. Also by introducing a set of time-weighted Sobolev spaces and using the contraction mapping theorem, we obtain the global in-time existence and the optimal decay estimates of solutions to the semi-linear problem under smallness assumption on the initial data.

1. **Introduction.** In this paper we consider the initial value problem of the following semi-linear plate equation with memory term in multi-dimensional space $\mathbb{R}^n$ with $n \geq 1$:

$$u_{tt} + \Delta^2 u + u + g * \Delta u = f(u), \tag{1.1}$$

with the initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{1.2}$$

Here $u = u(x,t)$ is the unknown function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$, and represents the transversal displacement of the plate at the point x and the time t. The term $g * \Delta u := \int_0^t g(t-\tau)\Delta u(\tau)d\tau$ corresponds to the memory term, and g satisfies:

Assumption [A]:

a) $g \in C^2(\mathbb{R}^+) \cap W^{2,1}(\mathbb{R}^+)$,
b) $g(s) > 0$, $-C_0 g(s) \leq g'(s) \leq -C_1 g(s)$, $|g''(s)| \leq C_2 g(s)$, $\forall s \in \mathbb{R}^+$,
c) $1 - \int_0^t g(s)ds \geq C_3$, $\forall t \in \mathbb{R}^+$,

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where $C_i (i = 0, 1, 2, 3)$ are positive constants.
Assumption [B]: Assume that $f \in C^\infty(\mathbb{R} \setminus \{0\})$, and $f(u) = O(|u|^\alpha)$ as $|u| \to 0$, here $\alpha > \alpha_n$ and $\alpha_n := 1 + \frac{2}{n}, n \geq 1$, and α is assumed to be an integer for $n \geq 3$.

In [10], we studied the inertial model of quasilinear dissipative plate equation, whose linear part in a simpler case is given by:

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + u_t = 0, \tag{1.3}$$

here $-\Delta u_{tt}$ corresponds to the rotational inertia, and u_t is the linear dissipative term. In that paper, we obtained the global existence and asymptotic behavior of solutions by employing the time-weighted energy method combined with a semi-group argument. In this paper, one point worth noticing is that , the dissipation given by the memory term $g * \Delta u$ is relatively weaker compared with the linear term u_t. This weak dissipative mechanism could be reflected from the decay structure of solutions. Same as the inertial model of dissipative plate equation (1.3) in [10, 17], the plate equation with memory (1.4) is also of regularity-loss property. The decay structure of the regularity-loss type is characterized by the property

$$\rho(\xi) = \frac{|\xi|^2}{1 + |\xi|^4},$$

where $\rho(\xi)$ is introduced in the point-wise estimate in the Fourier space (3.1) of solutions to the linear problem. It is not difficult to see that the decay structure is very weak in high frequency region since $\rho(\xi)$ may tend to zero as $|\xi| \to \infty$. A similar decay structure of the regularity-loss type was also observed for the dissipative Timoshenko system ([9, 15]) and a hyperbolic-elliptic system related to a radiating gas ([8]). For more studies on various aspects of dissipation of plate equations, we refer to [1, 2, 3, 4, 6, 12, 14, 16, 18]. Also, as for the study of decay properties for wave equations and hyperbolic systems of memory-type dissipation, we refer to [5, 7, 11, 13].

The main purpose of this paper is to study decay estimates of solutions to the initial value problems (1.4), (1.2) and (1.1), (1.2). For our problem, it is difficult to obtain explicitly the solution operator or its Fourier transform due to the presence of memory term. However, by using Fourier transform and Laplace transform, we obtain the solution $\bar{u}$ to the linear problem (1.4), (1.2) given by (2.5) and the solution operators $G(t)*$ and $H(t)*$. Moreover by employing the energy method in the Fourier space, we obtain the point-wise estimate in the Fourier space of solutions to the corresponding linear problem (1.2) and

$$u_{tt} + \Delta^2 u + u + g * \Delta u = 0. \tag{1.4}$$

Appealing to this point-wise estimate, the corresponding point-wise estimate of solution operators and their properties are obtained. Consequently, the decay estimates of solutions to (1.4) (1.2), and the global existence and optimal decay estimates of solutions to (1.1), (1.2) are achieved. As for the semi-linear problem, one point worthy to be mentioned is that we obtain the results for $\alpha > \alpha_n$ in the case $n = 1, 2$, while $\alpha_n = 1 + \frac{2}{n}$ is the well-known critical Fujita exponent in dealing with the global existence and blow up of solutions to some semi-linear parabolic differential equations.

The contents of the paper are as follows. Solution formula are obtained in section 2. In section 3, we obtain the estimates and properties of solutions operators, which is based on the point-wise estimate in the Fourier space of solutions to the corresponding linear problem. In section 4, we prove the decay estimates of solutions

to the linear problem by virtue of the properties of solution operators. In the last section, the global existence and the optimal decay estimates of solutions to the initial value problem (1.1), (1.2) are obtained.

Before closing this section, we give some notations to be used below. Let $\mathcal{F}[f]$ denote the Fourier transform of f defined by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx,$$

and we denote its inverse transform as $\mathcal{F}^{-1}$.

Let $\mathcal{L}[f]$ denote the Laplace transform of f defined by

$$\mathcal{L}[f](\lambda) := \int_0^{+\infty} e^{-\lambda t} f(t)dt,$$

and we denote its inverse transform as $\mathcal{L}^{-1}$.

$L^p = L^p(\mathbb{R}^n)(1 \le p \le \infty)$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. $W^{m,\,p}(\mathbb{R}^n), m \in \mathbf{Z}_+, p \in [1,\infty]$ denote the usual Sobolev space with its norm

$$\|f\|_{W^{m,p}} := \Big(\sum_{k=0}^{m} \|\partial_x^k f\|_{L^p}^p\Big)^{\frac{1}{p}}.$$

In particular, we use $W^{m,\,2} = H^m$. Here, for a nonnegative integer k, ∂_x^k denotes the totality or each of all the k-th order derivatives with respect to $x \in \mathbb{R}^n$. Also, $C^k(I; H^m(\mathbb{R}^n))$ denotes the space of k-times continuously differentiable functions on the interval I with values in the Sobolev space $H^m = H^m(\mathbb{R}^n)$.

Finally, in this paper, we denote every positive constant by the same symbol C or c without confusion. $[\cdot]$ is Gauss' symbol.

2. **Solution formula.**

In this section we try to obtain the solution formula for the problems (1.4) (1.2) and (1.1) (1.2). Assume that $G(x,t)$ and $H(x,t)$ are solutions to the following problem,

$$\begin{cases} G_{tt} + (1+\Delta^2)G + g * \Delta G = 0, \\ G(x,0) = \delta(x), \\ G_t(x,0) = 0, \end{cases} \tag{2.1}$$

$$\begin{cases} H_{tt} + (1+\Delta^2)H + g * \Delta H = 0, \\ H(x,0) = 0, \\ H_t(x,0) = \delta(x). \end{cases} \tag{2.2}$$

Apply Fourier transform and Laplace transform to (2.1) and (2.2), then we have formally that

$$\hat{G}(\xi,t) = C\mathcal{L}^{-1}\Big[\frac{\lambda}{\lambda^2 + 1 + |\xi|^4 - |\xi|^2\mathcal{L}[g](\lambda)}\Big](\xi,t),$$

$$\hat{H}(\xi,t) = C\mathcal{L}^{-1}\Big[\frac{1}{\lambda^2 + 1 + |\xi|^4 - |\xi|^2\mathcal{L}[g](\lambda)}\Big](\xi,t),$$

here C is a constant determined by the initial data in (2.1) and (2.2). The following lemma guarantees that $\hat{G}(\xi,t)$ and $\hat{H}(\xi,t)$ are well defined.

Lemma 2.1. *$\hat{G}(\xi,t)$ and $\hat{H}(\xi,t)$ exist.*

Proof. We only prove $\hat{G}(\xi,t)$ exists; similarly we could prove $\hat{H}(\xi,t)$ exists. Denote $F(\lambda):=\lambda^2+1+|\xi|^4-|\xi|^2\mathcal{L}[g](\lambda)$. To prove $\mathcal{L}^{-1}[\frac{\lambda}{F(\lambda)}]$ exists, we need to consider the zero points of $F(\lambda)$. Denote $\lambda=\sigma+i\nu$, $\sigma>-C_1$, C_1 is same as that in Assumption [A] b), then $\mathcal{L}[g](\lambda)$ exists. Assume that $\lambda_1=\sigma_1+i\nu_1$ is a zero point of $F(\lambda)$ and $\sigma_1>-C_1$, then σ_1 and ν_1 satisfy

$$\begin{cases} \mathrm{Re}F(\lambda_1)=\sigma_1^2-\nu_1^2+1+|\xi|^4-|\xi|^2\int_0^\infty\cos(\nu_1 t)e^{-\sigma_1 t}g(t)dt=0,\\ \mathrm{Im}F(\lambda_1)=2\sigma_1\nu_1+|\xi|^2\int_0^\infty\sin(\nu_1 t)e^{-\sigma_1 t}g(t)dt=0.\end{cases}\tag{2.3}$$

Case 1: $\xi=0$.

From (2.3), we know that $\sigma_1=0$, $\nu_1=\pm1$.

Case 2: $\xi\neq0$.

We claim that $\sigma_1<0$. Now we prove the claim by contradiction.

Assume that $\sigma_1\geq0$. If $\nu_1=0$, then in view of $\int_0^\infty g(t)dt<1$, we obtain that

$$\mathrm{Re}F(\lambda_1)=\sigma_1^2+1+|\xi|^4-|\xi|^2\int_0^\infty e^{-\sigma_1 t}g(t)dt\geq\frac{1}{2},$$

it yields contradiction with $(2.3)_1$.

If $\nu_1\neq0$, then we have that

$$\mathrm{Im}F(\lambda_1)=\nu_1\Big(2\sigma_1+|\xi|^2\int_0^\infty\frac{\sin(\nu_1 t)}{\nu_1}e^{-\sigma_1 t}g(t)dt\Big).$$

Next we prove that $\int_0^\infty\frac{\sin(\nu_1 t)}{\nu_1}e^{-\sigma_1 t}g(t)dt>0$. Denote $a_m=\int_0^{\frac{2m\pi}{|\nu_1|}}\frac{\sin|\nu_1 t|}{|\nu_1|}e^{-\sigma_1 t}g(t)dt$, and we will prove $\{a_m\}_{m=1}^\infty$ is a convergent sequence. By direct computation, we have that

$$a_1=\int_0^{\frac{\pi}{|\nu_1|}}\frac{\sin|\nu_1 t|}{|\nu_1|}\big(e^{-\sigma_1 t}g(t)-e^{-\sigma_1(t+\frac{\pi}{|\nu_1|})}g(t+\frac{\pi}{|\nu_1|})\big)dt.$$

Since $\partial_t\big(e^{-\sigma_1 t}g(t)\big)<0$, we have that $0<a_1<\int_0^{\frac{\pi}{|\nu_1|}}te^{-\sigma_1 t}g(t)dt$. Similarly,

$$a_{m+1}-a_m=\int_{\frac{2m\pi}{|\nu_1|}}^{\frac{2m\pi+\pi}{|\nu_1|}}\frac{\sin|\nu_1 t|}{|\nu_1|}\big(e^{-\sigma_1 t}g(t)-e^{-\sigma_1(t+\frac{\pi}{|\nu_1|})}g(t+\frac{\pi}{|\nu_1|})\big)dt,$$

so we have that $0<a_{m+1}-a_m<\int_{\frac{2m\pi}{|\nu_1|}}^{\frac{2m\pi+\pi}{|\nu_1|}}te^{-\sigma_1 t}g(t)dt$. It yields that

$$0<a_m<\int_0^{\frac{2m\pi}{|\nu_1|}}te^{-\sigma_1 t}g(t)dt\leq\frac{g(0)}{(\sigma_1+C_1)^2},$$

so $\{a_m\}_{m=1}^\infty$ is a bounded and monotonic increasing sequence. Since $a_1>0$, $a(\lambda_1):=\lim\limits_{m\to\infty}a_m>0$. Thus we proved that $\int_0^\infty\frac{\sin(\nu_1 t)}{\nu_1}e^{-\sigma_1 t}g(t)dt>0$. Also, because $\xi\neq0$, $\sigma_1\geq0$ and $\nu_1\neq0$, It results that $\mathrm{Im}F(\lambda_1)\neq0$. This contradicts with $(2.3)_2$. Thus by contradiction we proved the claim $\sigma_1<0$.

Combining the two cases, we know that $\frac{\lambda}{F(\lambda)}$ is analytic in $\{\lambda\in\mathbb{C};\mathrm{Re}(\lambda)>0\}$ if $\xi=0$ and in $\{\lambda\in\mathbb{C};\mathrm{Re}(\lambda)\geq0\}$ if $\xi\neq0$. Take $\lambda=\sigma+i\nu$, $\sigma>\max\{\mathrm{Re}\lambda_s\}$, here $\{\lambda_s\}$ is the set of all the singular points of $F(\lambda)$, then we have that

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{\lambda}{F(\lambda)}\right](t)&=\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\lambda e^{\lambda t}}{F(\lambda)}d\lambda=\int_{-\infty}^{+\infty}\frac{i(\sigma+i\nu)e^{(\sigma+i\nu)t}}{F(\sigma+i\nu)}d\nu\\&=\int_{\{\nu;|\nu|\leq R\}}+\int_{\{\nu;|\nu|>R\}}=:J_1+J_2.\end{aligned}$$

J_1 converges, so we only need to consider J_2. Notice that $\frac{\lambda}{F(\lambda)}=\frac{1}{\lambda}-\frac{1+|\xi|^4-|\xi|^2\mathcal{L}[g](\lambda)}{\lambda F(\lambda)}$ and $|\mathcal{L}[g](\lambda)|\leq C$, then it is not difficult to prove that J_2 converges. The constant

C in the expression of $\hat{G}(\xi,t)$ is determined by the initial data of $G(x,t)$. So far we complete the proof. □

In view of Lemma 2.1 and Duhamel principle, the solution to the problem (1.1)(1.2) could be expressed as following:

$$u(t) = G(t) * u_0 + H(t) * u_1 + \int_0^t H(t-\tau) * f(u)(\tau) d\tau. \tag{2.4}$$

We denote

$$\bar{u}(t) := G(t) * u_0 + H(t) * u_1, \tag{2.5}$$

then $\bar{u}(x,t)$ is the solution to the linear problem (1.4)(1.2).

3. **Decay properties of solution operators.** In this section our aim is to obtain the following decay estimates of the solution operators $G(t)*$ and $H(t)*$ appearing in the solution formula (2.4).

Proposition 1. *Let k, l be integers, $\varphi \in H^{s+1}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $\psi \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \le p \le 2$, then the following estimates hold:*
1) $\|\partial_x^k G(t) * \varphi\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} \|\varphi\|_{L^p} + C(1+t)^{-\frac{l}{2}} \|\partial_x^{k+l}\varphi\|_{L^2}$,
for $k \ge 0,\ l \ge 0,\ k+l \le s+1$.
2) $\|\partial_x^k G_t(t) * \varphi\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} \|\varphi\|_{L^p} + C(1+t)^{-\frac{l}{2}} \|\partial_x^{k+l+2}\varphi\|_{L^2}$,
for $k \ge 0,\ l \ge 0,\ k+l \le s-1$.
3) $\|\partial_x^k H(t) * \psi\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} \|\psi\|_{L^p} + C(1+t)^{-\frac{l+2}{2}} \|\partial_x^{k+l}\psi\|_{L^2}$,
for $k \ge 0,\ l+2 \ge 0,\ 0 \le k+l \le s$.
4) $\|\partial_x^k H_t(t) * \psi\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} \|\psi\|_{L^p} + C(1+t)^{-\frac{l}{2}} \|\partial_x^{k+l}\psi\|_{L^2}$,
for $k \ge 0,\ l \ge 0,\ k+l \le s$.

To prove the proposition, the key point is to obtain the point-wise estimates of the fundamental solutions in the Fourier space. In fact this could be achieved by using the following point-wise estimate of solutions to the linear problem (1.4) (1.2).

Lemma 3.1 (point-wise estimate)**.** *Assume u is the solution of (1.4)(1.2), then it satisfies the following point-wise estimate in the Fourier space:*

$$\begin{aligned} &|\hat{u}_t(\xi,t)|^2 + (1+|\xi|^4)|\hat{u}|^2 + |\xi|^2 g\Box\hat{u} \\ &\le Ce^{-C\rho(\xi)t}\Big(|\hat{u}_1(\xi)|^2 + (1+|\xi|^4)|\hat{u}_0(\xi)|^2\Big), \end{aligned} \tag{3.1}$$

here $\rho(\xi) = \frac{|\xi|^2}{1+|\xi|^4}$.

To prove Lemma 3.1 we need some notations. For any real or complex-valued function $f(t)$, we define

$$(g * f)(t) := \int_0^t g(t-\tau) f(\tau) d\tau,$$

$$(g \diamond f)(t) := \int_0^t g(t-\tau)\big(f(\tau) - f(t)\big) d\tau,$$

$$(g\Box f)(t) := \int_0^t g(t-\tau)|f(t) - f(\tau)|^2 d\tau,$$

By direct calculation we have the following lemma, which is useful in obtaining our point-wise estimate of solutions in the Fourier space.

Lemma 3.2. *For any function $k \in C(\mathbb{R})$, and any $\phi \in W^{1,2}(0,T)$, it holds that*

1) $(k*\phi)(t) = (k\diamond\phi)(t) + \Big(\int_0^t k(\tau)d\tau\Big)\phi(t)$,

2) $$\begin{aligned}\mathrm{Re}\{(k*\phi)(t)\bar{\phi}_t(t)\} = & -\tfrac{1}{2}k(t)|\phi(t)|^2 + \tfrac{1}{2}(k'\Box\phi)(t)\\ & -\tfrac{1}{2}\tfrac{d}{dt}\Big\{(k\Box\phi)(t) - \big(\int_0^t k(\tau)d\tau\big)|\phi(t)|^2\Big\},\end{aligned}$$

3) $|(k\diamond\phi)(t)|^2 \le \Big(\int_0^t |k(\tau)|d\tau\Big)(|k|\Box\phi)(t)$.

Now we will come to get the point-wise estimates in the Fourier space.

Proof of Lemma 3.1. Step 1: Apply Fourier transform to (1.4) we have that

$$\hat{u}_{tt} + (1+|\xi|^4)\hat{u} - |\xi|^2 g*\hat{u} = 0. \tag{3.2}$$

By multiplying (3.2) by $\bar{\hat{u}}_t$ and taking the real part, we have that

$$\Big\{\frac{1}{2}|\hat{u}_t|^2 + \frac{1}{2}(1+|\xi|^4)|\hat{u}|^2\Big\}_t - |\xi|^2\mathrm{Re}\{g*\hat{u}\bar{\hat{u}}_t\} = 0. \tag{3.3}$$

Apply Lemma 3.2 2) to the term $\mathrm{Re}\{g*\hat{u}\bar{\hat{u}}_t\}$ in (3.3), and denote

$$E_1(\xi,t) := |\hat{u}_t|^2 + (1+|\xi|^4)|\hat{u}|^2 + |\xi|^2 g\Box\hat{u} - |\xi|^2\Big(\int_0^t g(s)ds\Big)|\hat{u}|^2,$$

$$F_1(\xi,t) := |\xi|^2\Big(g|\hat{u}|^2 - g'\Box\hat{u}\Big),$$

then we obtain that

$$\frac{\partial}{\partial t}E_1(\xi,t) + F_1(\xi,t) = 0. \tag{3.4}$$

Step 2: By multiplying (3.2) by $\{-(g*\bar{\hat{u}})_t\}$ and taking the real part, we have that

$$\Big\{\frac{1}{2}|\xi|^2|g*\hat{u}|^2\Big\}_t - \mathrm{Re}\{\hat{u}_{tt}(g*\bar{\hat{u}})_t\} - \mathrm{Re}\{(1+|\xi|^4)\hat{u}(g*\bar{\hat{u}})_t\} = 0. \tag{3.5}$$

Since $(g*\bar{\hat{u}})_t = g(0)\bar{\hat{u}} + g'*\bar{\hat{u}}$, the second term in (3.5) yields that,

$$\begin{aligned}-\mathrm{Re}\{\hat{u}_{tt}(g*\bar{\hat{u}})_t\} &= -\mathrm{Re}\{\hat{u}_t(g*\bar{\hat{u}})_t\}_t + \mathrm{Re}\{\hat{u}_t(g*\bar{\hat{u}})_{tt}\}\\ &= -\mathrm{Re}\{\hat{u}_t(g*\bar{\hat{u}})_t\}_t + \mathrm{Re}\{g(0)|\hat{u}_t|^2 + \hat{u}_t(g'*\bar{\hat{u}})_t\}.\end{aligned}$$

Denote

$$E_2(\xi,t) := \frac{1}{2}|\xi|^2|g*\hat{u}|^2 - \mathrm{Re}\{\hat{u}_t(g*\bar{\hat{u}})_t\},$$

$$F_2(\xi,t) := g(0)|\hat{u}_t|^2,$$

$$R_2(\xi,t) := \mathrm{Re}\big\{-\hat{u}_t(g'*\bar{\hat{u}})_t + (1+|\xi|^4)\hat{u}(g*\bar{\hat{u}})_t\big\},$$

then (3.5) yields that

$$\frac{\partial}{\partial t}E_2(\xi,t) + F_2(\xi,t) = R_2(\xi,t). \tag{3.6}$$

Step 3: By multiplying (3.2) by $\bar{\hat{u}}$ and taking the real part, we have that

$$\Big\{\mathrm{Re}\{\hat{u}_t\bar{\hat{u}}\}\Big\}_t - |\hat{u}_t|^2 + (1+|\xi|^4)|\hat{u}|^2 - |\xi|^2\mathrm{Re}\{g*\hat{u}\bar{\hat{u}}\} = 0. \tag{3.7}$$

In view of Lemma 3.2 1), we have that

$$\mathrm{Re}\{g*\hat{u}\bar{\hat{u}}\} = \Big(\int_0^t g(s)ds\Big)|\hat{u}|^2 + \mathrm{Re}\{g\diamond\hat{u}\bar{\hat{u}}\}.$$

Denote

$$E_3(\xi,t) := \mathrm{Re}\{\hat{u}_t\bar{\hat{u}}\},$$

$$F_3(\xi,t) := (1+|\xi|^4)|\hat{u}|^2 - |\xi|^2\Big(\int_0^t g(s)ds\Big)|\hat{u}|^2,$$
$$R_3(\xi,t) := |\hat{u}_t|^2 + |\xi|^2\mathrm{Re}\{g\diamond\hat{u}\bar{\hat{u}}\},$$

then (3.7) yields that

$$\frac{\partial}{\partial t}E_3(\xi,t) + F_3(\xi,t) = R_3(\xi,t). \tag{3.8}$$

Define $\rho(\xi) = \frac{|\xi|^2}{1+|\xi|^4}$, and denote

$$E(\xi,t) := E_1(\xi,t) + \rho(\xi)\big(\alpha E_2(\xi,t) + \beta E_3(\xi,t)\big),$$
$$F(\xi,t) := F_1(\xi,t) + \rho(\xi)\big(\alpha F_2(\xi,t) + \beta F_3(\xi,t)\big),$$
$$R(\xi,t) := \rho(\xi)\big(\alpha R_2(\xi,t) + \beta R_3(\xi,t)\big),$$

where α,β are positive constants. then (3.4)(3.6)(3.8) yield that

$$\frac{\partial}{\partial t}E(\xi,t) + F(\xi,t) = R(\xi,t). \tag{3.9}$$

We introduce Lyapunov functionals:

$$E_0(\xi,t) := |\hat{u}_t|^2 + (1+|\xi|^4)|\hat{u}|^2 + |\xi|^2 g\Box\hat{u},$$
$$F_0(\xi,t) := g\Box\hat{u} + g|\hat{u}|^2.$$

From the definitions of $E_1(\xi,t)$ and $F_1(\xi,t)$, we know that there exist positive constants $c_i (i=1,2,3)$ such that the following relations hold:

$$c_1E_0(\xi,t) \le E_1(\xi,t) \le c_2E_0(\xi,t),\ \ F_1(\xi,t) \ge c_3|\xi|^2F_0(\xi,t). \tag{3.10}$$

On the other hand, since

$$|E_2(\xi,t)| \le C|\hat{u}_t|^2 + C(1+|\xi|^2)(|\hat{u}|^2 + g\Box\hat{u}),$$

and

$$|E_3(\xi,t)| \le C(|\hat{u}_t|^2 + |\hat{u}|^2),$$
$$\begin{aligned}|\rho(\xi)\big(\alpha E_2(\xi,t) + \beta E_3(\xi,t)\big)| &\le C(\alpha+\beta)\Big\{|\hat{u}_t|^2 + |\xi|^2(|\hat{u}|^2 + g\Box\hat{u})\Big\}\\ &\le c_4(\alpha+\beta)E_0(\xi,t).\end{aligned}$$

Choose α,β suitably small such that $c_4(\alpha+\beta) \le \min\{\frac{c_1}{2},\frac{c_2}{2}\}$, by virtue of (3.10) we have that

$$\frac{c_1}{2}E_0(\xi,t) \le E(\xi,t) \le \frac{3c_2}{2}E_0(\xi,t). \tag{3.11}$$

In view of (3.10) and the fact that $0 \le \int_0^t g(s)ds < 1$, it is easy to verify that

$$F(\xi,t) \ge c_3|\xi|^2F_0(\xi,t) + \rho(\xi)\Big(\alpha g(0)|\hat{u}_t|^2 + \frac{\beta}{2}(1+|\xi|^4)|\hat{u}|^2\Big). \tag{3.12}$$

Since

$$|R_2(\xi,t)| \le \varepsilon|\hat{u}_t|^2 + \delta(1+|\xi|^4)|\hat{u}|^2 + C_{\varepsilon,\delta}(1+|\xi|^4)F_0(\xi,t),$$

and

$$|R_3(\xi,t)| \le |\hat{u}_t|^2 + \gamma|\xi|^2|\hat{u}|^2 + C_\gamma|\xi|^2 g\Box\hat{u},$$

where $\varepsilon,\delta,\gamma$ are any positive constants, we have

$$\begin{aligned}|R(\xi,t)| &\le \rho(\xi)\Big\{(\alpha\varepsilon+\beta)|\hat{u}_t|^2 + (\alpha\delta+\gamma\beta)(1+|\xi|^4)|\hat{u}|^2\\ &\quad + \alpha C_{\varepsilon,\delta}(1+|\xi|^4)F_0(\xi,t) + \beta C_\gamma|\xi|^2 g\Box\hat{u}\Big\}\\ &\le (\alpha\varepsilon+\beta)\rho(\xi)|\hat{u}_t|^2 + (\alpha\delta+\gamma\beta)|\xi|^2|\hat{u}|^2 + (\alpha+\beta)C_{\varepsilon,\delta,\gamma}|\xi|^2F_0(\xi,t).\end{aligned}$$

We claim that there exist $\gamma, \varepsilon, \delta, \alpha, \beta$ such that

$$|R(\xi,t)| \le \frac{1}{2}F(\xi,t). \tag{3.13}$$

First choose $\gamma = \frac{1}{8}, \varepsilon = \frac{1}{4}g(0), \delta = \frac{1}{32}g(0), \beta = \frac{1}{4}\alpha g(0)$, then the following two inequalities hold:

$$\alpha\varepsilon + \beta \le \frac{1}{2}\alpha g(0),\ \ \alpha\delta + \gamma\beta \le \frac{1}{4}\beta.$$

In order to prove the claim (3.13) (here (3.11) is also considered), it suffices to choose α suitably small such that

$$\Big(1 + \frac{1}{4}g(0)\Big)\alpha \le \min\Big\{\frac{c_3}{2C_{\varepsilon,\delta,\gamma}}, \frac{c_1}{2c_4}, \frac{c_2}{2c_4}\Big\}.$$

In view of (3.13), (3.9) yields that

$$\frac{\partial}{\partial t}E(\xi,t) + \frac{1}{2}F(\xi,t) \le 0. \tag{3.14}$$

On the other hand, (3.11) and (3.12) yield that

$$F(\xi,t) \ge C\rho(\xi)E(\xi,t). \tag{3.15}$$

(3.14) and (3.15) yield that

$$E(\xi,t) \le e^{-C\rho(\xi)t}E(\xi,0). \tag{3.16}$$

By virtue of (3.11) and (3.16), we obtain the point-wise estimate of solutions to (1.4) (1.2) in the Fourier space. □

As a simple corollary of Lemma 3.1, we have the following point-wise estimates of the fundamental solutions $G(x,t)$ and $H(x,t)$ in the Fourier space.

Lemma 3.3. *$G(x,t)$ and $H(x,t)$ satisfy*
1). $|\hat{G}(\xi,t)| \le Ce^{-C\rho(\xi)t}$;
2). $|\hat{G}_t(\xi,t)| \le Ce^{-C\rho(\xi)t}(1+|\xi|^4)^{\frac{1}{2}}$;
3). $|\hat{H}(\xi,t)| \le Ce^{-C\rho(\xi)t}(1+|\xi|^4)^{-\frac{1}{2}}$;
4). $|\hat{H}_t(\xi,t)| \le Ce^{-C\rho(\xi)t}$,
where $\rho(\xi) = \frac{|\xi|^2}{1+|\xi|^4}$.

Proof. Putting (2.5) with $u_1 = 0$ in (3.1), it results that

$$|\hat{G}_t(\xi,t)|^2 + (1+|\xi|^4)|\hat{G}(\xi,t)|^2 \le Ce^{-C\rho(\xi)t}(1+|\xi|^4).$$

It yields 1) and 2).

Putting (2.5) with $u_0 = 0$ in (3.1), it results that

$$|\hat{H}_t(\xi,t)|^2 + (1+|\xi|^4)|\hat{H}(\xi,t)|^2 \le Ce^{-C\rho(\xi)t}.$$

It yields 3) and 4). □

Now we use Lemma 3.3 to prove Proposition 1.

Proof of Proposition 1. In view of Lemma 3.3 1), we have that

$$\begin{aligned}
&\|\partial_x^k G(t) * \varphi\|_{L^2}^2 \le C\int_{\mathbb{R}^n} |\xi|^{2k}e^{-C\rho(\xi)t}|\hat{\varphi}(\xi)|^2 d\xi \\
&\le C\int_{\{\xi;|\xi|\le 1\}} |\xi|^{2k}e^{-\frac{C}{2}|\xi|^2 t}|\hat{\varphi}(\xi)|^2 d\xi + C\int_{\{\xi;|\xi|\ge 1\}} |\xi|^{2k}e^{-\frac{Ct}{2|\xi|^2}}|\hat{\varphi}(\xi)|^2 d\xi \\
&=: K_1 + K_2.
\end{aligned}$$

Assume that p' satisfies $\frac{1}{p}+\frac{1}{p'}=1$, then

$$K_1 \le C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-k}\|\hat{\varphi}\|^2_{L^{p'}} \le C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-k}\|\varphi\|^2_{L^p}.$$

On the other hand, $K_2 \le C(1+t)^{-l}\|\partial_x^{k+l}\varphi\|^2_{L^2}$, here $k\ge 0,\ l\ge 0,\ k+l\le s+1$. Thus 1) is proved.

Due to Lemma 3.3 2), it results that

$$\begin{aligned}
&\|\partial_x^k G_t(t)*\varphi\|^2_{L^2} \le C\int_{\mathbb{R}^n}|\xi|^{2k}(1+|\xi|^4)e^{-C\rho(\xi)t}|\hat{\varphi}(\xi)|^2d\xi\\
&\le C\int_{\{\xi;|\xi|\le 1\}}|\xi|^{2k}e^{-\frac{C}{2}|\xi|^2t}|\hat{\varphi}(\xi)|^2d\xi + C\int_{\{\xi;|\xi|\ge 1\}}|\xi|^{2k+4}e^{-\frac{Ct}{2|\xi|^2}}|\hat{\varphi}(\xi)|^2d\xi\\
&\le C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-k}\|\varphi\|^2_{L^p} + C(1+t)^{-l}\|\partial_x^{k+l+2}\varphi\|^2_{L^2},
\end{aligned}$$

here $k\ge 0,\ l\ge 0,\ k+l+2\le s+1$. Thus 2) is proved.

Next we prove 3) and 4). It follows from Lemma 3.3 3) that

$$\begin{aligned}
&\|\partial_x^k H(t)*\psi\|^2_{L^2} \le C\int_{\mathbb{R}^n}|\xi|^{2k}(1+|\xi|^4)^{-1}e^{-C\rho(\xi)t}|\hat{\psi}(\xi)|^2d\xi\\
&\le C\int_{\{\xi;|\xi|\le 1\}}|\xi|^{2k}e^{-\frac{C}{2}|\xi|^2t}|\hat{\psi}(\xi)|^2d\xi + C\int_{\{\xi;|\xi|\ge 1\}}|\xi|^{2k}|\xi|^{-4}e^{-\frac{Ct}{2|\xi|^2}}|\hat{\psi}(\xi)|^2d\xi\\
&\le C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-k}\|\psi\|^2_{L^p} + C(1+t)^{-l-2}\|\partial_x^{k+l}\psi\|^2_{L^2},
\end{aligned}$$

here $k\ge 0,\ l\ge -2,\ 0\le k+l\le s$.

Lemma 3.3 4) yields that

$$\begin{aligned}
\|\partial_x^k H_t(t)*\psi\|^2_{L^2} &\le C\int_{\mathbb{R}^n}|\xi|^{2k}e^{-C\rho(\xi)t}|\hat{\psi}(\xi)|^2d\xi\\
&\le C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-k}\|\psi\|^2_{L^p} + C(1+t)^{-l}\|\partial_x^{k+l}\psi\|^2_{L^2},
\end{aligned}$$

here $k\ge 0,\ l\ge 0,\ k+l\le s$. Thus 3) and 4) are proved. □

4. **Decay estimates for linear problem.** In this section we study the decay estimates of solutions to the linear problem (1.4) (1.2). By using the point-wise estimate of solutions in the Fourier space (3.1) we obtain the energy estimate of solutions to the linear problem (1.4) (1.2).

Theorem 4.1 (energy estimate for linear problem)**.** *Let $s\ge 1$ be an integer. Assume that $u_0\in H^{s+1}(\mathbb{R}^n)$ and $u_1\in H^s(\mathbb{R}^n)$, and put*

$$I_0 := \|u_0\|_{H^{s+1}} + \|u_1\|_{H^s}.$$

Then the solution $\bar{u}$ to the problem (1.4), (1.2) given by (2.5) satisfies

$$\bar{u}\in C^0([0,\infty);H^{s+1}(\mathbb{R}^n))\cap C^1([0,\infty);H^{s-1}(\mathbb{R}^n)),$$

and the following energy estimate:

$$\|\bar{u}(t)\|^2_{H^{s+1}} + \|\bar{u}_t(t)\|^2_{H^{s-1}} + \int_0^t\big(\|\partial_x\bar{u}(\tau)\|^2_{H^{s-1}} + \|\partial_x\bar{u}_t(\tau)\|^2_{H^{s-3}}\big)d\tau \le CI_0^2.$$

Proof. We have obtained the solution $\bar{u}$ of (1.4)(1.2) given by (2.5) and proved that it satisfies the point-wise estimate in the Fourier space (3.1). From (3.14) and (3.15) we have that

$$\frac{\partial}{\partial t}E(\xi,t) + C\rho(\xi)E(\xi,t)\le 0.$$

Integrate the previous inequality with respect to t and appeal to (3.11), then we obtain

$$E_0(\xi,t) + \int_0^t \rho(\xi,t)E_0(\xi,\tau)d\tau \le CE_0(\xi,0). \tag{4.1}$$

Multiply (4.1) by $(1+|\xi|^2)^{s-1}$ and integrate the resulting inequality with respect to $\xi \in \mathbb{R}^n$, then we have that

$$\|\bar{u}(t)\|^2_{H^{s+1}} + \|\bar{u}_t(t)\|^2_{H^{s-1}} + \int_0^t \big(\|\partial_x \bar{u}(\tau)\|^2_{H^{s-1}} + \|\partial_x \bar{u}_t(\tau)\|^2_{H^{s-3}}\big) d\tau \le CI_0^2. \quad (4.2)$$

(4.2) guarantees the regularity of the solution (2.5). So far we complete the proof of Theorem 4.1. □

By using Proposition 1 with $p=2$, we obtain the following decay estimates of $\bar{u}$ given by (2.5), if initial data $u_0 \in H^{s+1}(\mathbb{R}^n)$ and $u_1 \in H^s(\mathbb{R}^n)$.

Theorem 4.2 (decay estimates for linear problem)**.** *Under the same assumptions as in Theorem 4.1, then $\bar{u}$ given by (2.5) satisfies the decay estimate:*

$$\|\partial_x^k \bar{u}(t)\|_{H^{s+1-2k}} \le CI_0(1+t)^{-\frac{k}{2}}, \quad (4.3)$$

for $0 \le k \le [\frac{s+1}{2}]$.

Remark 1. Under the same assumptions as in Theorem 4.1, $\bar{u}$ given by (2.5) also satisfies the following decay estimate, which we think is not optimal, for $0 \le k \le [\frac{s-1}{2}]$,

$$\|\partial_x^k \bar{u}_t(t)\|_{H^{s-1-2k}} \le CI_0(1+t)^{-\frac{k}{2}}. \quad (4.4)$$

Also, if initial data $u_0 \in H^{s+1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $u_1 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then by using Proposition 1 with $p=1$ we have the sharp decay estimates of the solution $\bar{u}$ to (1.4), (1.2).

Denote

$$\sigma(k,n) = 2k + \left[\frac{n+1}{2}\right], \ n \ge 1, \quad (4.5)$$

then the theorem can be stated as follows.

Theorem 4.3 (sharp decay estimates for linear problem)**.** *Let $s \ge 1$ be an integer. Assume that $u_0 \in H^{s+1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $u_1 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and put*

$$I_1 := \|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} + \|(u_0,u_1)\|_{L^1}.$$

Then the solution $\bar{u}$ to (1.4)(1.2) given by (2.5) satisfies the following decay estimates:

$$\|\partial_x^k \bar{u}(t)\|_{H^{s+1-2k-j}} \le CI_1(1+t)^{-\frac{k+j}{2}}, \quad (4.6)$$

with $0 \le j \le [\frac{n}{2}]$, $k \ge 0$ and $2k+j \le s+1$;

$$\|\partial_x^k \bar{u}(t)\|_{H^{s+1-\sigma(k,n)}} \le CI_1(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \quad (4.7)$$

for $k \ge 0$ and $\sigma(k,n) \le s+1$, $s \ge [\frac{n+1}{2}] - 1$.

Remark 2. In addition to the above decay estimates, by similar computation we also have the following estimates, which we think are not optimal,

$$\|\partial_x^k \bar{u}_t(t)\|_{H^{s-1-2k-j}} \le CI_1(1+t)^{-\frac{k+j}{2}}, \quad (4.8)$$

with $0 \le j \le [\frac{n}{2}]$, $k \ge 0$ and $2k+j \le s-1$;

$$\|\partial_x^k \bar{u}_t(t)\|_{H^{s-1-\sigma(k,n)}} \le CI_1(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \quad (4.9)$$

for $k \ge 0$ and $\sigma(k,n) \le s-1$, $s \ge [\frac{n+1}{2}] + 1$, here $\sigma(k,n)$ is defined in (4.5).

Since proof of Theorem 4.2 and 4.3 are similar, here we only prove Theorem 4.3.

Proof of Theorem 4.3. First we prove (4.6). Let $k \geq 0,\ m \geq 0$ be integers. In view of (2.5), by using Proposition 1 1) and 3) with $p = 1$, we have that

$$\begin{aligned} \|\partial_x^{k+m}\bar{u}(t)\|_{L^2} &\leq \|\partial_x^{k+m}G(t) * u_0\|_{L^2} + \|\partial_x^{k+m}H(t) * u_1\|_{L^2} \\ &\leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|(u_0,u_1)\|_{L^1} + C(1+t)^{-\frac{l_1}{2}}\|\partial_x^{k+m+l_1}u_0\|_{L^2} \\ &\quad + C(1+t)^{-\frac{l_2+2}{2}}\|\partial_x^{k+m+l_2}u_1\|_{L^2}, \end{aligned} \tag{4.10}$$

here $l_1 \geq 0,\ l_2 \geq -2,\ k+m+l_1 \leq s+1,\ k+m+l_2 \leq s$.

Choose the smallest integers $l_1,\ l_2$ satisfying

$$\frac{l_1}{2} \geq \frac{k+j}{2}, \frac{l_2+2}{2} \geq \frac{k+j}{2},$$

i.e. $l_1 = k+j, l_2 = l_1 - 2$, then we obtain that

$$\|\partial_x^{k+m}\bar{u}(t)\|_{L^2} \leq CI_1(1+t)^{-\frac{k+j}{2}}$$

with $0 \leq m \leq s+1-2k-j$. Take sum with $0 \leq m \leq s+1-2k-j$, then we get (4.6).

To prove (4.7), we choose l_1 and l_2 in (4.10) as the smallest integer satisfying

$$\frac{l_1}{2} \geq \frac{n}{4} + \frac{k}{2},\ \frac{l_2+2}{2} \geq \frac{n}{4} + \frac{k}{2}.$$

It yields that $l_1 = k + [\frac{n+1}{2}] = \sigma(k,n) - k,\ l_2 = k + [\frac{n+1}{2}] - 2 = l_1 - 2$, here $\sigma(k,n)$ is defined in (4.5). Thus m satisfies $0 \leq m \leq s+1-\sigma(k,n)$.

Take sum of (4.10) with $0 \leq m \leq s+1-\sigma(k,n)$ we obtain that

$$\|\partial_x^k\bar{u}(t)\|_{H^{s+1-\sigma(k,n)}} \leq CI_1(1+t)^{-\frac{n}{4}-\frac{k}{2}}.$$

Thus (4.7) is proved. □

Remark 3. The estimates in Theorem 4.2 and 4.3 indicate that the decay structure of solutions to the linear problem (1.4) (1.2) is of regularity-loss type.

5. Global existence and decay for semi-linear problem.

In this section we will first introduce a set of time-weighted Sobolev spaces and employ the contraction mapping theorem to prove the global existence and optimal decay estimates of solutions to the semi-linear problem, then obtain the decay estimate of u_t by using the decay estimate for u and the semi-group method.

First we give some useful lemmas.

Lemma 5.1. *Assume that p, q, r and k are integers, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $k \geq 0$, then*

$$\|\partial_x^k(uv)\|_{L^p} \leq C\big(\|u\|_{L^q}\|\partial_x^k v\|_{L^r} + \|v\|_{L^q}\|\partial_x^k u\|_{L^r}\big).$$

Proof of Lemma 5.1 can be seen in [10]. By using Lemma 5.1, we have

Lemma 5.2. *Assume that p, q, r, k, α and β are integers, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, $k \geq 0$, $\alpha \geq 1$ and $\beta \geq 1$, then*

$$\|\partial_x^k(u^\alpha v^\beta)\|_{L^p} \leq C\|u\|_{L^\infty}^{\alpha-1}\|v\|_{L^\infty}^{\beta-1}\big(\|u\|_{L^q}\|\partial_x^k v\|_{L^r} + \|v\|_{L^q}\|\partial_x^k u\|_{L^r}\big).$$

Recall Assumption [B], we know that $f \in C^\infty(\mathbb{R}\setminus\{0\})$, and $f(u) = O(|u|^\alpha)$ as $|u| \to 0$, here $\alpha > \alpha_n$ and $\alpha_n := 1 + \frac{2}{n}, n \geq 1$, and α is assumed to be an integer for $n \geq 3$. By using the decay properties of solution operators and the above lemmas, we obtain the following result about the global existence and optimal decay estimates of solutions to the semi-linear problems (1.1)(1.2).

Theorem 5.3 (existence and decay estimates for semi-linear problem)**.** *Let s be an integer, $s \geq 2$ for $n = 1$ and $s \geq [\frac{n}{2}]$ for $n \geq 2$. Also assume that $s + 1 \leq \alpha$ for $n = 1, 2$. Let $u_0 \in H^{s+1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $u_1 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and put*

$$I_1 := \|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} + \|(u_0, u_1)\|_{L^1}.$$

If I_1 is suitably small, then there exists a unique solution $u(x,t)$ of (1.1)(1.2) in

$$u \in C^0([0,\infty); H^{s+1}(\mathbb{R}^n))$$

satisfying the following decay estimates:

$$\|\partial_x^k u(t)\|_{H^{s+1-2k-j}} \leq CI_1(1+t)^{-\frac{k+j}{2}}, \tag{5.1}$$

for $k \geq 0$, $0 \leq j \leq [\frac{n}{2}]$ and $2k + j \leq s + 1$;

$$\|\partial_x^k u(t)\|_{H^{s+1-\sigma(k,n)}} \leq CI_1(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \tag{5.2}$$

for $k \geq 0$ and $\sigma(k,n) \leq s + 1$, here $\sigma(k,n)$ is defined in (4.5).

Proof. (1). Define

$$X := \Big\{u \in C\big([0,\infty), H^{s+1}(\mathbb{R}^n)\big); \|u\|_X \leq \infty\Big\},$$

here

$$\begin{aligned}\|u\|_X :\ &= \sum_{j=0}^{[\frac{n}{2}]} \sum_{2k+j\leq s+1} \sup_{t\geq 0}(1+t)^{\frac{k+j}{2}}\|\partial_x^k u(t)\|_{H^{s+1-2k-j}} \\ &\quad + \sum_{\sigma(k,n)\leq s+1} \sup_{t\geq 0}(1+t)^{\frac{n}{4}+\frac{k}{2}}\|\partial_x^k u(t)\|_{H^{s+1-\sigma(k,n)}},\end{aligned}$$

here the second sum is taken over all integers $k \geq 0$ satisfying $2k + j \leq s + 1$ with $0 \leq j \leq [\frac{n}{2}]$, the third sum is taken over all integers $k \geq 0$ satisfying $\sigma(k,n) \leq s+1$. Denote

$$\begin{aligned}&S_R := \{u \in X; \|u\|_X \leq R\}, \forall R > 0,\\ &\Phi[u](t) := G(t) * u_0 + H(t) * u_1 + \textstyle\int_0^t H(t-\tau) * f(u)(\tau)d\tau,\\ &\Phi_0(t) := G(t) * u_0 + H(t) * u_1.\end{aligned}$$

$\forall v, w \in X$, $\Phi[v](t) - \Phi[w](t) = \int_0^t H(t-\tau) * \big(f(v) - f(w)\big)(\tau)d\tau$.

Noticing that $f(v) = O(|v|^\alpha)$ and using Lemma 5.2, we have the following inequalities for $k \geq 0$:

$$\begin{aligned}&\|\partial_x^k\big(f(v) - f(w)\big)(\tau)\|_{L^1}\\ &\leq C\|(v,w)(\tau)\|_{L^\infty}^{\alpha-2}\\ &\quad\Big(\|(v,w)(\tau)\|_{L^2}\|\partial_x^k(v-w)(\tau)\|_{L^2} + \|\partial_x^k(v,w)(\tau)\|_{L^2}\|(v-w)(\tau)\|_{L^2}\Big),\end{aligned} \tag{5.3}$$

$$\begin{aligned}&\|\partial_x^k\big(f(v) - f(w)\big)(\tau)\|_{L^2}\\ &\leq C\|(v,w)(\tau)\|_{L^\infty}^{\alpha-2}\\ &\quad\Big(\|(v,w)(\tau)\|_{L^\infty}\|\partial_x^k(v-w)(\tau)\|_{L^2} + \|\partial_x^k(v,w)(\tau)\|_{L^2}\|(v-w)(\tau)\|_{L^\infty}\Big).\end{aligned} \tag{5.4}$$

Also, if $v \in X$, then the following estimate holds:

$$\|v(\tau)\|_{L^\infty} \le C\|v\|_X(1+\tau)^{-d}, \tag{5.5}$$

here $d \ge d_n$, and d_n satisfies

$$d_n \begin{cases} = \frac{1}{2}, & n=1, \\ > 0, & n \ge 2. \end{cases}$$

In fact, by using Gagliardo-Nirenberg inequality, we have

$$\|v(\tau)\|_{L^\infty} \le C\|v(\tau)\|_{L^2}^{1-\theta}\|\partial_x^{s_0}v(\tau)\|_{L^2}^{\theta},$$

here $s_0 = [\frac{n}{2}]+1, \theta = \frac{n}{2s_0}$. In the case $n=1$, $\|v(\tau)\|_{L^2} \le (1+\tau)^{-\frac{1}{4}}\|v\|_X$ with $s+1-\sigma(0,1) \ge 0$, i.e. $s \ge 0$. $\|\partial_x^{s_0}v(\tau)\|_{L^2} \le (1+\tau)^{-\frac{1}{4}-\frac{1}{2}}\|v\|_X$ with $s+1-\sigma(1,1) \ge 0$, i.e. $s \ge 2$. Then $d_n = (1-\theta)\frac{1}{4}+\theta(\frac{1}{4}+\frac{1}{2}) = \frac{1}{2}$, it yields (5.5) with $n=1$. In the case of $n \ge 2$, $\|v(\tau)\|_{L^2} \le \|v\|_X(1+\tau)^{-\frac{1}{2}}$ with $s \ge 0$. $\|\partial_x^{s_0}v(\tau)\|_{L^2} \le \|v\|_X$ with $s+1 \ge s_0$, i.e. $s \ge [\frac{n}{2}]$. Then $d_n = \frac{1-\theta}{2} > 0$, it yields (5.5) with $n \ge 2$.

First we prove the estimate:

$$\|\partial_x^k(\Phi[v]-\Phi[w])(t)\|_{H^{s+1-2k-j}} \le C(1+t)^{-\frac{k+j}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X, \tag{5.6}$$

with $0 \le j \le [\frac{n}{2}]$, $k \ge 0$ and $2k+j \le s+1$.

Assume that k, j and m are non-negative integers, $j \le [\frac{n}{2}]$ and $2k+j \le s+1$. By applying ∂_x^{k+m} to $\Phi[v]-\Phi[w]$, we have that

$$\begin{aligned} &\|\partial_x^{k+m}(\Phi[v]-\Phi[w])(t)\|_{L^2} \\ &\le \Big(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\Big)\|\partial_x^{k+m}H(t-\tau) * \big(f(v)-f(w)\big)(\tau)\|_{L^2}d\tau \\ &=: I_1 + I_2. \end{aligned} \tag{5.7}$$

By virtue of Proposition 1 3) with $p=1$, we have

$$\begin{aligned} I_1 \quad &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{k+m}{2}}\|\big(f(v)-f(w)\big)(\tau)\|_{L^1}d\tau \\ &\quad + C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{l+2}{2}}\|\partial_x^{k+m+l}\big(f(v)-f(w)\big)(\tau)\|_{L^2}d\tau \\ &=: I_{11} + I_{12}. \end{aligned} \tag{5.8}$$

By using (5.3) with $k=0$ and (5.5), we have that

$$\|\big(f(v)-f(w)\big)(\tau)\|_{L^1} \le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X(1+\tau)^{-d(\alpha-2)-\frac{n}{2}}. \tag{5.9}$$

Since $\alpha > \alpha_n = 1+\frac{2}{n}$ for $n=1,2$ and $\alpha \ge 2$ for $n \ge 3$, appealing to (5.9), we have

$$I_{11} \le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \tag{5.10}$$

If $k+m+l \le s+1$, by virtue of (5.4) with k replaced by $k+m+l$ and (5.5), it yields that

$$\|\partial_x^{k+m+l}\big(f(v)-f(w)\big)(\tau)\|_{L^2} \le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d(\alpha-1)}. \tag{5.11}$$

Take $l = k+j$, appeal to (5.11) and notice that $d > 0, \alpha \ge 2$, then we have that

$$I_{12} \le C(1+t)^{-\frac{k+j}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X,$$

with $0 \le m \le s+1-2k-j$. Put the estimates for I_{11} and I_{12} in (5.8), then we obtain

$$I_1 \le C(1+t)^{-\frac{k+j}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X, \tag{5.12}$$

with $0 \le m \le s+1-2k-j$.

Also by employing Proposition 1 3) with $l=0$ and $p=1$ to the term I_2, we have

$$\begin{aligned} I_2 &\le C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{n}{4}-\frac{m}{2}}\|\partial_x^k\big(f(v)-f(w)\big)(\tau)\|_{L^1}d\tau \\ &\quad +C\int_{\frac{t}{2}}^t (1+t-\tau)^{-1}\|\partial_x^{k+m}\big(f(v)-f(w)\big)(\tau)\|_{L^2}d\tau \\ &=: I_{21}+I_{22}. \end{aligned} \tag{5.13}$$

In view of (5.3) and (5.5), we have that

$$\|\partial_x^k\big(f(v)-f(w)\big)(\tau)\|_{L^1} \le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d(\alpha-2)-\frac{n}{4}-\frac{k+j}{2}}, \tag{5.14}$$

for k,j satisfying $0\le j\le [\frac{n}{2}]$ and $2k+j\le s+1$.

Since $\alpha>\alpha_n=1+\frac{2}{n}$ for $n=1,2$ and $\alpha\ge 2$ for $n\ge 3$, appealing to (5.14), we have

$$I_{21}\le C(1+t)^{-\frac{k+j}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X,$$

with $0\le j\le [\frac{n}{2}]$ and $2k+j\le s+1$.

If $m\le s+1-2k-j$, by applying (5.4) with k replaced by $k+m$, we have that

$$\|\partial_x^{k+m}\big(f(v)-f(w)\big)(\tau)\|_{L^2} \le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d(\alpha-1)-\frac{k+j}{2}}. \tag{5.15}$$

It yields that

$$I_{22}\le C(1+t)^{-\frac{k+j}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X,$$

with $0\le m\le s+1-2k-j$. Put the estimates for I_{21} and I_{22} in (5.13), we have that

$$I_2\le C(1+t)^{-\frac{k+j}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X, \tag{5.16}$$

with $0\le m\le s+1-2k-j$. Put the estimates (5.12) and (5.16) into (5.7) and take sum with $0\le m\le s+1-2k-j$, then we have (5.6).

Next we prove the estimate:

$$\|\partial_x^k(\Phi[v]-\Phi[w])(t)\|_{H^{s+1-\sigma(k,n)}} \le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X, \tag{5.17}$$

with $\sigma(k,n)\le s+1$.

Assume that k and m are non-negative integers and $\sigma(k,n)\le s+1$. Apply ∂_x^{k+m} to $\Phi[v]-\Phi[w]$, then we have (5.7), (5.8) and (5.10). Now we come to estimate the term I_{12}. If $k+m+l\le s+1$, we still have (5.11). Take $l=\sigma(k,n)-k=k+[\frac{n+1}{2}]$, then we obtain

$$I_{12}\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X, \tag{5.18}$$

with $0\le m\le s+1-\sigma(k,n)$. Put the estimates (5.10) (5.18) for I_{11} and I_{12} into (5.8), then we obtain

$$I_1\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X, \tag{5.19}$$

with $0\le m\le s+1-\sigma(k,n)$.

Similarly we have (5.13). Since $\sigma(k,n)\le s+1$, in view of(5.3) and (5.5), we have that

$$\|\partial_x^k\big(f(v)-f(w)\big)(\tau)\|_{L^1} \le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d(\alpha-2)-\frac{n}{2}-\frac{k}{2}}.$$

It yields that

$$I_{21}\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X.$$

Since $\|\partial_x^{k+m}\big(f(v)-f(w)\big)(\tau)\|_{L^2}\le\|\partial_x^k\big(f(v)-f(w)\big)(\tau)\|_{H^m}$, if $m\le s+1-\sigma(k,n)$, by using (5.4) and (5.5) and noting that $\sigma(k,n)\le s+1$, we have

$$\|\partial_x^{k+m}\big(f(v)-f(w)\big)(\tau)\|_{L^2} \le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X(1+\tau)^{-d(\alpha-1)-\frac{n}{4}-\frac{k}{2}}.$$

It yields that

$$I_{22} \le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X,$$

with $0 \le m \le s+1-\sigma(k,n)$. Put the estimates for I_{21} and I_{22} into (5.13), we have that

$$I_2 \le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|(v,w)\|_X^{\alpha-1}\|v-w\|_X, \tag{5.20}$$

with $0 \le m \le s+1-\sigma(k,n)$. Put (5.19) and (5.20) into (5.7) and take sum with $0 \le m \le s+1-\sigma(k,n)$, then we obtain (5.17).

Combining the estimates (5.6) and (5.17), we obtain that

$$\|\Phi[v]-\Phi[w]\|_X \le C\|(v,w)\|_X^{\alpha-1}\|v-w\|_X. \tag{5.21}$$

So far we proved that $\|\Phi[v]-\Phi[w]\|_X \le C_1R^{\alpha-1}\|v-w\|_X$ if $v,w \in S_R$. On the other hand, $\Phi[0](t)=\Phi_0(t)=\bar{u}(t)$, and from Theorem 4.3 we know that $\|\Phi_0\|_X \le C_2I_1$ if I_1 is suitably small. Take $R=2C_2I_1$. If I_1 is suitably small such that $R<1$ and $C_1R \le \frac{1}{2}$, then we have that

$$\|\Phi[v]-\Phi[w]\|_X \le \frac{1}{2}\|v-w\|_X.$$

It yields that, for $v \in S_R$,

$$\|\Phi[v]\|_X \le \|\Phi_0\|_X + \frac{1}{2}\|v\|_X \le C_2I_1 + \frac{1}{2}R = R,$$

i.e. $\Phi[v] \in S_R$. Thus $v \to \Phi[v]$ is a contraction mapping on S_R, so there exists a unique $u \in S_R$ satisfying $\Phi[u]=u$, and it is the solution to the semi-linear problem (1.1)(1.2) satisfying the decay estimate (5.1) and (5.2). So we complete the proof of Theorem 5.3. □

As a corollary of Theorem 5.3, we have the following result, which maybe is not optimal.

Corollary 1. *Let s be an integer, $s \ge [\frac{n+1}{2}]+1$ for $n \ge 1$. Assume that $u_0 \in H^{s+1}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ and $u_1 \in H^s(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$, and put*

$$I_1 := \|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} + \|(u_0,u_1)\|_{L^1},$$

then the solution $u(x,t)$ of (1.1)(1.2) obtained in Theorem 5.3 satisfies

$$u \in C^1([0,\infty); H^{s-1}(\mathbb{R}^n))$$

and the following estimates:

$$\|\partial_x^k u_t(t)\|_{H^{s-1-2k-j}} \le CI_1(1+t)^{-\frac{k+j}{2}}, \tag{5.22}$$

for $k \ge 0$, $0 \le j \le [\frac{n}{2}]$ and $2k+j \le s-1$;

$$\|\partial_x^k u_t(t)\|_{H^{s-1-\sigma(k,n)}} \le CI_1(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \tag{5.23}$$

for $k \ge 0$ and $\sigma(k,n) \le s-1$, here $\sigma(k,n)$ is defined in (4.5).

Proof. By using (2.4), we have that

$$\begin{aligned} u_t(t) &= G_t(t)*u_0 + H_t(t)*u_1 + \textstyle\int_0^t H_t(t-\tau)*f(u)(\tau)d\tau \\ &=: \Psi_0(t) + \Psi[u](t), \end{aligned} \tag{5.24}$$

here $\Psi_0(t) = G_t(t)*u_0 + H_t(t)*u_1 = \bar{u}_t(t)$, $\Psi[u](t) = \int_0^t H_t(t-\tau)*f(u)(\tau)d\tau$.

By applying Proposition 1 4) with $p=1$, and in the similar way to (5.6), we obtain that

$$\|\partial_x^k \Psi[u](t)\|_{H^{s-1-2k-j}} \le CI_1^{\alpha}(1+t)^{-\frac{k+j}{2}}, \tag{5.25}$$

for $k \geq 0$, $0 \leq j \leq [\frac{n}{2}]$ and $2k + j \leq s - 1$. In view of (4.8) in Remark 2, we know that if I_1 is suitably small,

$$\|\partial_x^k \Psi_0(t)\|_{H^{s-1-2k-j}} \leq CI_1(1+t)^{-\frac{k+j}{2}},$$

which combined with (5.25) and (5.24) yields the decay estimate (5.22). Also in view of Proposition 1 4) with $p = 1$ and by the similar proof as for (5.17), we obtain that

$$\|\partial_x^k \Psi[u](t)\|_{H^{s-1-\sigma(k,n)}} \leq C{I_1}^{\alpha}(1+t)^{-\frac{n}{4}-\frac{k}{2}}. \tag{5.26}$$

In view of (4.9) in Remark 2, we know that if $s \geq [\frac{n+1}{2}] + 1$ and I_1 is suitably small, then

$$\|\partial_x^k \Psi_0(t)\|_{H^{s-1-\sigma(k,n)}} \leq CI_1(1+t)^{-\frac{n}{4}-\frac{k}{2}},$$

which combined with (5.26) and (5.24) yields the decay estimate (5.23). So far we complete the proof of Corollary 1.

□

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E-mail address: yqliu2@yahoo.com.cn
E-mail address: kawashim@math.kyushu-u.ac.jp