# ASSOCIAHEDRA, MULTIPLIHEDRA AND UNITS IN $A_\infty$ FORM

NORIO IWASE

Abstract. Jim Stasheff gave two apparently distinct definitions of an $A_m$ form, $m \leq \infty$ in [17, 18]. It is also claimed that the two definitions are equivalent in [17, 18], while it is not apparently clear for us. That is why we are trying to clarify related things and to show that the claim is actually true under a 'loop-like' hypothesis in this paper. Along with these two definitions, we must construct Associahedra and Multiplihedra as convex polytopes with piecewise-linearly decomposed faces to manipulate units in $A_\infty$ form. This is done in Iwase [9, 10], Iwase-Mimura [11] or by Haiman [8] especially on Associahedra, followed recently by Forcey [7] and Mau-Woodward [14], while the origin of Associahedra goes back to Tamari [19]. In this paper, we follow [11] on the geometric constructions of Associahedra and Multiplihedra. In Appendix, we also explain how we can construct Associahedra or Multiplihedra as polytopes on the (half) lattice by taking a shadow or collecting words from trivalent or bearded trees.

## Contents



*Date*: January 24, 2013.

2000 *Mathematics Subject Classification.* Primary 55P48, Secondary 18D50,18D10,18D20,18D35,55R05,55R35.

*Key words and phrases.* $A_\infty$ structure, $A_\infty$ form, Associahedra, Multiplihedra, $A_\infty$ space, $A_\infty$ map, internal category, internal pre-category, $A_\infty$ category, $A_\infty$ functor, $A_\infty$ action, $A_\infty$ equivariant, trivalent tree, bearded tree, integral lattice.

The author is supported by the Grant-in-Aid for Scientific Research (B) #22340014 and Grant-in-Aid for Exploratory Research #24654013 from Japan Society for the Promotion of Science.

## 0. Introduction

The notion of an $A_m$ structure for a space is first introduced in Stasheff [17], whose idea is spread into a wide area of mathematics and becomes a basic idea in both mathematics and physics.

On the other hand from the beginning, two apparently distinct definitions of an '$A_m$ form' for a space are given to exist: the original definition described in [17] requires an existence of *strict-unit* which is used to construct a projective or a classifying space, while the second definition described in Stasheff [18] was claimed to be equivalent in [17] and used extensively in [18] which requires just a *hopf-unit* (a strict unit as an h-space) or even an *h-unit* (a homotopy unit as an h-space) a slightly weaker condition. To follow the arguments in [17] on the equivalence of two definitions, we must use Lemma 7 in [17] recursively. However, at the stage to obtain a new 3-form, the proof must use Lemma 7 which is depending on retractile arguments due to James [12], and thus, the relative homotopy types of old and new 3-forms are possibly different (see [12], [17] or Zabrodsky [20]). That means, at least, we can not guarantee that there exist higher forms for new 3-form. In particular, it is shown in Iwase and Mimura [11] that some $A_\infty$ space has exotic $n$-form which does not admit $n{+}1$-form for any $n \geq 2$. Alternatively, if one tries to adopt the approach, we actually discussed here, one should encounter problems to show a version of Proposition 24 of [17] which gives an crucial step to show Theorem 5 of [17]. In fact, Proposition 24 of [17] may not hold without a *strict unit*. These problems puzzled the author for many years after [9].

In this paper, we work in $\underline{\underline{\mathcal{T}}}$ the category of spaces and continuous maps, unless otherwise stated. Let us remark that the exponential law does not always hold in $\underline{\underline{\mathcal{T}}}$. We follow the definitions and notations mostly in Mimura [15] or [11]. So we say that a space $X$ admits an $A_m$ *structure*, if there is a sequence $\{q^X_n, n{\leq}m\}$ of maps of pairs $q^X_n : (D^n, E^n) \to (P^n, P^{n-1})$ such that $p^X_n = q^X_n|_{E^n} : E^n \to P^{n-1}$ is a quasi-fibration and $E^n$ is contractible in $D^n$. A set of higher multiplications are called $A_m$ form if it satisfies boundary and unit conditions.

For a space $X$ with an element $e \in X$, which is always assumed to be non-degenerate. For a based multiplication $\mu : X{\times}X \to X$, we say $(X, \mu, e)$ is an h-space with *h-unit*, if the restriction of $\mu$ to $X{\vee}X \subset X{\times}X$ is (based) homotopic to the folding map $\nabla_X : X{\vee}X \to X$ given by $\nabla_X(x, e) = \nabla_X(e, x) = x$. We also say $(X, \mu, e)$ is an h-space with *hopf-unit* if $e$ is a two-sided strict unit of $\mu$. Similarly, we say $f : X \to X'$ is a map between h-spaces regarding *h-units*, if $(X, \mu, e)$ and $(X', \mu', e')$ are h-spaces with *h-units* and $f(e)$ lies in the same connected component of $e' \in X'$. We also say $f : X \to X'$ is a map between h-spaces regarding *hopf-units*, if $(X, \mu, e)$ and $(X', \mu', e')$ are h-spaces with *hopf-units* and $f(e) = e'$.

We say an h-space with *h-unit* (or *hopf-unit*) is loop-like, if both right and left translations are homotopy-equivalences. So a CW complex h-space with *h-unit*, whose $\pi_0$ is a loop (an algebraic generalization of a group), is a loop-like h-space with *hopf-unit*. In Page 34 of Adams [1], it is pointed out that Theorem 5 of [17] is implicitly shown to be true for a CW complex h-space whose $\pi_0$ is a group when $m{=}\infty$. A corresponding generalization including the case when $m = 2$ would be for a CW complex h-space whose $\pi_0$ is a loop, or equivalently, a CW complex loop-like space. In fact, in the proof of Theorem 5 of [17], this latter assumption is necessary to apply the arguments given in Dold and Thom [6] or Mimura-Toda [16] to deduce that the projections obtained using $A_m$ form are all quasi-fibrations by using induction arguments.

**Lemma 0.1.** *If a CW complex $X$ admits an $A_m$ structure, then there exists a homotopy-equivalence inclusion map $j : X \hookrightarrow \tilde{X}$ which is an $A_m$ map in the sense of* [11], *where $\tilde{X}$ has an $A_m$ form with strict-unit.*

Using this lemma, we show the following theorem giving a deformation of an $A_m$ form with *h-unit* into an $A_m$ form with *strict-unit.*

**Theorem 0.2.** [1] *A connected CW complex $X$ has an $A_m$ form with h-unit if and only if $X$ has an $A_m$ form with strict-unit. More generally, a CW complex h-space $X$ whose $\pi_0$ is a loop, has an $A_m$ form with h-unit if and only if $X$ has an $A_m$ form with strict-unit.*

When $m{=}\infty$, we obtain the following corollary.

**Corollary 0.3.** *A connected CW complex $X$ has an $A_\infty$ form with h-unit if and only if $X$ has an $A_\infty$ form with strict-unit.*

We can also show a slightly stronger result of Theorem 11.5 in [18] using the idea of $A_\infty$ form *without unit* as follows:

**Theorem 0.4.** *Let $\mu : X{\times}X \to X$ be a multiplication on a space $X$ without unit. If $X$ has an $A_\infty$ form $\{a(n), n{\geq}2\}$ with $a(2) = \mu$, then $X$ is a deformation retract of a space $M$ with associative multiplication such that the inclusion $X \subset M$ has an $A_\infty$ form in the sense of Stasheff* [18]. *If the $A_\infty$ form has further a strict-unit, then we can choose $M$ as a monoid, and the inclusion becomes an $A_\infty$ map regarding strict-units.*

Stasheff cells denoted by $K_n$, now known as Associahedra, were introduced in [17] in 1963 to characterize an $A_\infty$ structure for a space, while the origin goes back to Tamari [19] in 1951 (see also Müller-Hoissen, Pallo and Stasheff [13]). To deal with *strict-units*, both Associahedra and Multiplihedra were designed to be convex polytopes with faces decomposed into piecewise-linear subfaces by Iwase [9, 10] in 1983 or by Haiman [8] in 1984, which are in 1989 redesigned and denoted by $K(n)$ and $J(n)$ in [11]. In [9, 10] or [11], we need to design Multiplihedra to fit in with the design of Associahedra to manipulate $A_m$ structures for maps between $A_m$ spaces. The author is also aware of appearance of new constructions of Multiplihedra by Forcey [7] in 2008 and Mau

[1]The author has come to know that, in algebraic context, Jacob Lurie has obtained a similar result using higher algebra and higher topos theories.

and Woodward [14] in 2010, and is used extensively in the world of combinatorial mathematics.

In this paper, we follow the definitions and notations of Associahedra and Multiplihedra in [11], which form sequences $\{K(n) \subset \mathbb{R}^n_+\,;\, n{\geq}1\}$ and $\{J(n) \subset \mathbb{R}^n_+\,;\, n{\geq}1\}$ of convex polytopes, where we denote $\mathbb{R}_+ = [0,\infty)$. We remark that $K(1) = \{0\} \subset \mathbb{R}_+$, $K(2) = \{(0,1)\} \subset \mathbb{R}^2_+$ and $J(1) = \{\frac{1}{2}\} \subset \mathbb{R}_+$, all of which are one-point sets. In Appendix, we show how trivalent or bearded trees relate to $K(n)$ or $J(n)$, saying that $K(n)$ and $J(n)$ are convex hulls of a (half) integral lattice points, say $K_L(n) \subset L = \mathbb{Z}^n$ and $J_L(n) \subset \frac{1}{2}L$. We can consider $K_L(n)$ as a down-to-left shadow of all trivalent trees on the lattice with one root and $n$ top-branches, and both $K_L(n)$ and $J_L(n)$ can be obtained in terms of languages of trivalent and bearded trees, respectively.

## 1. $A_\infty$ operad for objects

**1.1. Topological $A_\infty$ operad for objects.** Let us recall the definition of Associahedra designed as in [11]:

**Definition 1.1** (An $A_\infty$ operad for objects in $\underline{\underline{\mathcal{T}}}$)**.**

$$K(n) = \left\{ (u_1,\cdots,u_n) \in \mathbb{R}^n_+ \,\middle|\, u_j \leq \sum_{i=1}^{j-1}(1{-}u_i),\ u_n = \sum_{i=1}^{n-1}(1{-}u_i) \right\}$$

*In this definition, we assume that $n \geq 1$, i.e., we have $K(1)$ as well.*

Using trivalent trees, Boardman and Vogt gave an alternative description of Associahedra in [5]. See Appendix E.1 for the relation between two descriptions of Associahedra.

**Remark 1.2.** (1) *For any $(u_1,\cdots,u_n) \in K(n)$, we always have $u_1 = 0$ and $u_n \geq 1$ if $n \geq 2$.*
(2) *$K(1) = \{(0)\}$, $K(2) = \{(0,1)\}$, $K(3) = \{(0,t,2{-}t)\,|\,0{\leq}t{\leq}1\} \approx [0,1]$ (homeomorphic). The elements $(0) \in K(1)$ and $(0,1) \in K(2)$ are often denoted as $\alpha_1$ and $\alpha_2$, respectively.*

**Definition 1.3.** *Boundary operators of $K(n)$'s are defined as follows:*

$$\partial_j : K(r){\times}K(t) \to K(n), \quad 1 \leq j \leq r,\ 2 \leq r,t,\ r{+}t = n{+}1,$$
$$\partial_j(v_1,\cdots,v_r;u_1,\cdots,u_t) = (v_1,\cdots,v_{j-1},u_1,\cdots,u_{t-1},u_t{+}v_j,\cdots,v_r).$$

*Then by definition, $\partial_k(\rho,\tau)$ is affine in $\rho$ and $\tau$. Let us denote*

$$K_j(r,t) = \partial_j(K(r){\times}K(t)), \quad 1 \leq j \leq r,\ 2 \leq t \leq n,\ r{+}t = n{+}1$$

*so that we have $\partial K(n) = \bigcup K_k(r,s)$.*

We often denote the adjoint map of $\partial_j$ by the same symbol

$$\partial_j : K(t) \longrightarrow Map(K(r),K(n)), \quad \partial_j(\tau)(\rho) = \partial_j(\rho,\tau). \tag{1.1}$$

Then we obtain the following proposition.

**Proposition 1.4.** *Let $\tau \in K(t), \sigma \in K(s)$. Then we have*

$$\partial_k(\sigma)\circ\partial_j(\tau) = \begin{cases} \partial_{j+s-1}(\tau)\circ\partial_k(\sigma), & k < j, \\ \partial_j(\partial_{k-j+1}(\tau,\sigma)), & j \le k < j+t, \\ \partial_j(\tau)\circ\partial_{k-t+1}(\sigma), & k \ge j+t. \end{cases}$$

**Remark 1.5.** *We can dualize the definition of boundary operators as $\partial'_j = \partial_{r-j+1} : K(r)\times K(t) \to K(n),\ 1 \le j \le r,\ 2 \le t \le n,\ r+t = n+1$. Then we have the following equality:*

$$\begin{aligned} \partial'_k(\sigma)\circ\partial'_j(\tau) &= \partial_{n+t-k}(\sigma)\circ\partial_{n-j+1}(\tau) \\ &= \begin{cases} \partial_{n-j+1}(\tau)\circ\partial_{n-k+1}(\sigma), & n+t-k \ge n+t-j+1. \\ \partial_{n-j+1}(\partial_{t-k+j}(\tau,\sigma)), & n-j+1 \le n+t-k < n+t-j+1, \\ \partial_{n+s-j}(\tau)\circ\partial_{n+t-k}(\sigma), & n+t-k < n-j+1, \end{cases} \\ &= \begin{cases} \partial'_{j+s-1}(\tau)\circ\partial'_k(\sigma), & k < j, \\ \partial'_j(\partial'_{k-j+1}(\tau,\sigma)), & j \le k < j+t, \\ \partial'_j(\tau)\circ\partial'_{k-t+1}(\sigma), & k \ge j+t. \end{cases} \end{aligned}$$

*The relations in Proposition 1.4 tells us that each face $K_j(r,t)$ of $K(n)$ meets on its face with just one another, and hence they give a piecewise-linear decomposition of $n{-}3$ sphere $\partial K(n)$, $n{\geq}3$, and we have*

$$\partial K(n) = \bigcup_{\substack{1\le j\le r,\ 2\le r,t,\\ r+t=n+1}} K_j(r,t).$$

We define special faces of $K(n)$ for $n \ge 2$ as follows.

$$K_j(n) = \bigcup_{\substack{2\le r,t,\\ r+t=n+1}} K_j(r,t) = \{(u_1,\cdots,u_n) \in K(n) \mid u_j = 0\},\ 1{<}j{<}n,$$

$$K_1(n) = \bigcup_{\substack{2\le r,t,\\ r+t=n+1}} K_1(r,t) = \bigcup_{1<t<n} \{(u_1,\cdots,u_n) \in K(n) \mid u_t = \sum_{i=1}^{t-1}(1{-}u_i)\}.$$

## 2. $A_\infty$ operad for morphisms

2.1. **Topological $A_\infty$ operad for morphisms.** Let us recall the definition of Multiplihedra designed as in [11]:

**Definition 2.1** (An $A_\infty$ operad for morphisms in $\underline{\underline{\mathcal{T}}}$)**.**

$$J^a(n) = \left\{ (v_1,\cdots,v_n) \in \mathbb{R}^n_+ \,\middle|\, v_j \le \sum_{i=1}^{j-1}(1{-}v_i){+}a,\ v_n = \sum_{i=1}^{n-1}(1{-}v_i){+}a \right\},$$

$$J(n) = J^{\frac{1}{2}}(n).$$

*While we usually assume that $0{<}a{<}1$ in this definition, it is clear by definition that $J^0(n) = K(n)$ and $K(n{+}1) = \{0\}\times J^1(n)$.*

**Remark 2.2.** (1) *By definition,* $J^0(n) = K(n)$*. Moreover for any* $a \in \mathbb{R}$ *with* $0 \le a \le 1$*,* $J^a(n)$ *can be embedded in* $K(n+1)$ *by identifying* $(u_1, \cdots, u_n) \in J^a(n)$ *with* $(0, u_1, \cdots, u_{n-1}, u_n+1-a) \in K(n+1)$ *so that* $J^a(n) \subseteq K(n+1)$ *and* $K(n+1) = J^1(n)$*.*

(2) *For any* $(v_1, \cdots, v_n) \in J^a(n)$*, we have* $v_n \ge 1$ *if* $n \ge 2$*.*

(3) $J^a(1) = \{(a)\}$*,* $J^a(2) = \{(t, 1+a-t) \,|\, 0 \le t \le a\} \approx [0, a]$*. The special element* $(a)$ *is often denoted by* $\beta_1 = \beta_1^a \in J^a(1)$*.*

**Definition 2.3.** (1) *The boundary operators,* $\delta_j^a : J^a(r) \times K(s) \to J^a(n)$*,* $1 \le j \le r$*,* $2 \le s \le n$*,* $r+s=n+1$*, are defined by*

$$\delta_j^a(v_1, \cdots, v_r; u_1, \cdots, u_t) = (v_1, \cdots, v_{j-1}, u_1, \cdots, u_t+v_j, \cdots, v_r),$$

*where we often denote the adjoint map of* $\delta_j^a$ *by the same sumbol* $\delta_j^a : K(s) \to \mathrm{Map}(J^a(r), J^a(n))$*.*

(2) *The boundary operators,* $\delta^a : K(t) \times J^a(n_1) \times \cdots \times J^a(n_t) \to J^a(n)$*,* $1 \le n_i$ $(1 \le i \le t)$*,* $\sum_{i=1}^{t} n_i = n$*, are defined by*

$$\delta^a(u_1, \cdots, u_t; v_1^{(1)}, \cdots, v_{n_1}^{(1)}; \cdots; v_1^{(t)}, \cdots, v_{n_t}^{(t)}) = (v_1^{(1)}, \cdots, v_{n_1}^{(1)}+(1-a)u_1; \cdots; v_1^{(t)}, \cdots, v_{n_t}^{(t)}+(1-a)u_t).$$

*where we often denote the adjoint of* $\delta^a$ *by the same symbol* $\delta^a : J^a(t_1) \times \cdots \times J^a(t_n) \to \mathrm{Map}(K(t), J^a(n))$*.*

(3) *When* $a = \frac{1}{2}$*, we often abbreviate '*$\frac{1}{2}$*' as*

$$\delta_j : J(r) \times K(s) \to J(n), \quad 1 \le j \le r,\ 2 \le s \le n,\ r+s=n+1,$$

$$\delta : K(t) \times J(n_1) \times \cdots \times J(n_t) \to J(n) \quad 1 \le n_i,\ \sum_{i=1}^{t} n_i = n,$$

*where we denote their adjoints by* $\delta_j : K(s) \to \mathrm{Map}(J(r), J(n))$ *and* $\delta : J(n_1 \times \cdots \times J(n_t)) \to \mathrm{Map}(K(t), J(n))$*.*

**Remark 2.4.** *The map* $\delta_k^a(\rho, \tau) = \delta_k^a(\sigma)(\rho)$ *is affine in* $\rho$ *and* $\tau$*, and* $\delta^a(\tau; \rho_1, \cdots, \rho_t) = \delta^a(\tau)(\rho_1, \cdots, \rho_t)$ *is affine in* $\tau$ *and all* $\rho_i$*s,* $1 \le i \le t$*.*

The faces $J_j^a(r,t) = \delta_j^a(K(s))(J(r))$ $(1 \le j \le r,\ 2 \le t,\ r+t = n+1)$ and $J^a(t; n_1, \cdots, n_t) = \delta^a(J^a(n_1) \times \cdots \times J^a(n_t))(K(t))$ $(2 \le t,\ 1 \le n_i$ $(1 \le i \le t),\ \sum_{i=1}^{t} n_i = n)$ of $J^a(n)$ satisfy the following equation.

$$\partial J^a(n) = \bigcup_{\substack{1 \le j \le r,\ 2 \le t, \\ r+t=n+1}} J_j^a(r,t) \ \cup \bigcup_{\substack{2 \le t,\ 1 \le n_i\ (1 \le i \le t) \\ \sum_{i=1}^{t} n_i = n}} J^a(t; n_1, \cdots, n_t).$$

Hence for taking $a = \frac{1}{2}$, we have

$$\partial J(n) = \bigcup_{\substack{1 \le j \le r,\ 2 \le t, \\ r+t=n+1}} J_j(r,t) \ \cup \bigcup_{\substack{2 \le t,\ 1 \le n_i\ (1 \le i \le t) \\ \sum_{i=1}^{t} n_i = n}} J(t; n_1, \cdots, n_t),$$

**Proposition 2.5.** *Let $\tau \in K(t), \sigma \in K(s)$. Then the following holds:*

$$\delta^a_k(\sigma)\circ\delta^a_j(\tau) = \begin{cases} \delta^a_{j+s}(\tau)\circ\delta^a_k(\sigma)), & k < j, \\ \delta^a_j(\partial_{k-j+1}(\sigma)(\tau)), & j \le k < j+t, \\ \delta^a_j(\tau)\circ\delta^a_{k-t+1}(\sigma)), & k \ge j+t. \end{cases}$$

$$\delta^a_k(\sigma)\circ\delta^a(\rho_1,\cdots,\rho_t) = \delta^a(\rho_1,\cdots,\rho_{j-1},\delta^a_{k'}(\sigma)(\rho_j),\rho_{j+1},\cdots,\rho_t),$$
$$k = r_1+\cdots+r_{j-1}+k', 1 \le k' \le r_j.$$

$$\delta^a(\rho_1,\cdots,\rho_t)\circ\partial_j(\sigma) = \delta^a(\rho_1,\cdots,\rho_{j-1},\delta^a(\rho_j,\cdots,\rho_{j+s-1})(\sigma),\cdots,\rho_t).$$

We define special faces of $J^a(n)$ and $J(n)$ as follows.

**Definition 2.6.** *For $a$, $0 < a < 1$, $n \ge 1$ and $0 \le j < n$, we define*

$$J^a_j(n) = \bigcup_{\substack{2\le r,t,\\ r+t=n+1}} J^a_j(r,t) = \{(u_1,\cdots,u_n) \in J^a(n) \mid u_j = 0\},\ j \ne 0,$$

$$J^a_0(n) = \bigcup_{2\le t,\ 1\le n_i\ (1\le i\le t),\ \sum_{i=1}^t n_i=n} J^a(t;n_1,\cdots,n_t)$$
$$= \{(v_1,\cdots,v_n) \in J^a(n) \mid \exists_{i\,(1\le i<n)} \sum_{j=1}^i v_j = i-1+a\},\ j = 0,$$

$$J^a_k(r,s)_0 = \delta^a_k(r,s)(J^a_0(r)\times K(s)),$$

*and, when $a = \frac{1}{2}$, we define*

$$J_j(n) = J^{\frac{1}{2}}_j(n) = \bigcup_{2\le r,t,\ r+t=n+1} J_j(r,t),\ j \ne 0,$$

$$J_0(n) = J^{\frac{1}{2}}_0(n) = \bigcup_{2\le t,\ 1\le n_i\ (1\le i\le t),\ \sum_{i=1}^t n_i=n} J_0(t;n_1,\cdots,n_t),\ j = 0.$$

$$J_k(r,s)_0 = \delta_k(r,s)(J_0(r)\times K(s)),$$

As for $\delta^a$, we can slightly extend its definition as follows.

**Definition 2.7.** *For $0 \le a \le b \le 1$, $2 \le t$, $r = \frac{b-a}{1-a}$, $1 \le n_i$ $(1\le i\le t)$ and $\sum_{i=1}^t n_i = n$, we define $\delta^{b/a} : J^r(t) \times J^a(n_1)\times\cdots\times J^a(n_t) \to J^b(n)$ by*

$$\delta^{b/a}(u_1,\cdots,u_t;v^{(1)}_1,\cdots,v^{(1)}_{n_1};\cdots;v^{(t)}_1,\cdots,v^{(t)}_{n_t})$$
$$= (v^{(1)}_1,\cdots,v^{(1)}_{n_1}+(1-a)u_1;\cdots;v^{(t)}_1,\cdots,v^{(t)}_{n_t}+(1-a)u_t).$$

Then we can easily see the following equation.

**Proposition 2.8.** *If $0 \le a \le b \le 1$, then for any $n \ge 2$, we have*

$$J^b(n) = J^a(n) \cup \bigcup_{2\le t,\ 1\le n_i\ (1\le i\le t),\ \sum_{i=1}^t n_i=n} J^{\frac{b-a}{1-a}}(t)\times J^a(n_1)\times\cdots\times J^a(n_t).$$

*Hence we also have the following equation.*

$$K(n{+}1) = J^a(n) \cup \bigcup_{2\le t,\ 1\le n_i\ (1\le i\le t),\ \sum_{i=1}^{t} n_i = n} K(t{+}1)\times J^a(n_1)\times\cdots\times J^a(n_t).$$

This immediately implies that a composition of two $A_m$ maps is an $A_m$ map for any $m \le \infty$.

## 3. Degeneracy Operations

3.1. **Shift 1 map.** In this section, we introduce a shift 1 map $\xi : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $n \ge 1$, which give essentially the degeneracy operations.

**Definition 3.1.** *We define a map $\xi : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $n \ge 1$, by*

$$\xi(t_1,\cdots,t_n) = (t'_1,\cdots,t'_n),$$

*where $t'_i$'s are given inductively by the following formulas for $k \ge 1$.*

$$\begin{cases} t'_1 = \mathrm{Max}\,\{0, t_1{-}1\} \quad \textit{and} \\ t'_k = \mathrm{Min}\left\{t_k, \mathop{\mathrm{Max}}_{1\le j\le k}\left\{\sum_{i=1}^{j} t_i - j\right\} - \sum_{i=1}^{k-1} t'_i + (k{-}1)\right\} \end{cases} \tag{3.1}$$

From the above definition of $t'_k$, we can immediately show the following equation.

$$\sum_{i=1}^{k} t'_i = \mathrm{Min}\left\{\sum_{i=1}^{k-1} t'_i + t_k, \mathop{\mathrm{Max}}_{1\le j\le k}\left\{\sum_{i=1}^{j} t_i - j\right\} + (k{-}1)\right\} \tag{3.2}$$

This equation turns further into the following one.

$$\begin{aligned} \sum_{i=1}^{k}(t_i{-}t'_i) &= \sum_{i=1}^{k} t_i - \sum_{i=1}^{k} t'_i \\ &= \mathrm{Max}\left\{\sum_{i=1}^{k-1}(t_i{-}t'_i), \sum_{i=1}^{k} t_i - k - \mathop{\mathrm{Max}}_{1\le j\le k}\left\{\sum_{i=1}^{j} t_i - j\right\} + 1\right\}. \end{aligned} \tag{3.3}$$

Firstly, the following proposition implies that $\xi : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is well-defined for all $n \ge 1$.

**Proposition 3.2.** *For any $k \ge 1$, we have $0 \le t'_k \le t_k$.*

*Proof*: We show this by induction on $k$.
(Case: $k = 1$) We have $0 \le \mathrm{Max}\{0, t_1{-}1\}$ and $\mathrm{Max}\{0, t_1{-}1\} \le t_1$.
(Case: $k \ge 2$) By the induction hypothesis, we may assume that

$$0 \le t'_i \le t_i, \quad \text{for all } i < k.$$

Since $\mathrm{Min}\{t_k, x\} \le t_k$ for any $x \in \mathbb{R}$, we have $t'_k \le t_k$. On the other hand, we have $\underset{1\le j\le k}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\} \ge \sum_{i=1}^{k-1} t_i - (k-1)$, and hence we obtain

$$t'_k \ge \mathrm{Min}\left\{t_k, \sum_{i=1}^{k-1} t_i - (k-1) - \sum_{i=1}^{k-1} t'_i + (k-1)\right\} = \mathrm{Min}\left\{t_k, \sum_{i=1}^{k-1}(t_i - t'_i)\right\}.$$

Since $t_i - t'_i \ge 0$ for all $i < k$, we obtain $t'_k \ge \mathrm{Min}\{t_k, 0\} = 0$. It completes the proof of the proposition. □

Secondly, we show some other properties of $\xi$.

**Proposition 3.3.** *For any $k \ge 1$, we have $\sum_{i=1}^{k}(t_i - t'_i) \le 1$.*

*Proof*: We show this by induction on $k$.
(Case: $k = 1$) By the definition of $t'_1$, we have $t'_1 \ge t_1 - 1$ and hence

$$\sum_{i=1}^{1}(t_i - t'_i) = t_1 - t'_1 \le t_1 - (t_1 - 1) = 1.$$

(Case: $k \ge 2$) By the induction hypothesis, we may assume

$$\sum_{i=1}^{k-1}(t_i - t'_i) \le 1.$$

Again by induction hypothesis together with (3.3), we proceed as

$$\sum_{i=1}^{k}(t_i - t'_i) \le \mathrm{Max}\left\{1, \sum_{i=1}^{k} t_i - k - \underset{1\le j\le k}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\} + 1\right\}.$$

Since $\sum_{i=1}^{k} t_i - k \le \underset{1\le j\le k}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\}$, we have

$$\sum_{i=1}^{k} t_i - k - \underset{1\le j\le k}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\} + 1 \le 1.$$

Thus we obtain $\sum_{i=1}^{k}(t_i - t'_i) \le \mathrm{Max}\{1, 1\} = 1$. □

**Proposition 3.4.** *If $\sum_{i=1}^{k}(t_i - t'_i) = 1$ for some positive integer $k$, then we have $t'_{k'} = t_{k'}$ for all $k' > k$.*

*Proof*: We have

$$\sum_{i=1}^{k+1} t_i - (k+1) - \underset{1\le j\le k+1}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\} + 1$$
$$\le \sum_{i=1}^{k+1} t_i - (k+1) - \sum_{i=1}^{k+1} t_i + (k+1) + 1 = 1.$$

By the assumption, we have

$$\begin{aligned}(t'_{k+1}-t_{k+1})+1 &= (t'_{k+1}-t_{k+1})+\sum_{i=1}^{k}(t_i-t'_i) = \sum_{i=1}^{k+1}(t_i-t'_i)\\ &= \mathrm{Max}\left\{\sum_{i=1}^{k}(t_i-t'_i), \sum_{i=1}^{k+1}t_i-(k+1)-\underset{1\le j\le k+1}{\mathrm{Max}}\left\{\sum_{i=1}^{j}t_i-j\right\}+1\right\}\\ &\le \mathrm{Max}\{1,1\}=1,\end{aligned}$$

and hence $t'_{k+1}=t_{k+1}$. This immediately implies that $t'_{k'}=t_{k'}$ for all $k'>k$. □

**Proposition 3.5.** *If* $t_1\ge 1$, *then* $t_1-t'_1=1$.

*Proof*: If $t_1\ge 1$, then $t'_1=\mathrm{Max}\{0,t_1-1\}=t_1-1$, and hence we obtain $t_1-t'_1=1$. □

**Proposition 3.6.** *Let* $s\ge 2$ *and* $0\le a\le 1$. *If* $\sum_{i=1}^{j}t_i\le j-1+a$ *for all* $j\le s$, *then* $t'_1=0$ *and* $\sum_{i=2}^{k}t'_i\le k-2+a$ *for all* $k$, $2\le k\le s$.

*Proof*: Since $\sum_{i=1}^{j}t_i\le j-1+a$ for any $j\le s$, we have $t_1-1\le a-1\le 0$ and $\underset{1\le j\le k}{\mathrm{Max}}\left\{\sum_{i=1}^{j}t_i-j\right\}\le a-1$. Thus by (3.1) and (3.2), we have $t'_1=\mathrm{Max}\{0,t_1-a\}=0$ and $\sum_{i=1}^{k}t'_i=\mathrm{Min}\left\{\sum_{i=1}^{k-1}t'_i+t_k, a-1+k-1\right\}\le k-2+a$. Thus we obtain $\sum_{i=2}^{k}t'_i\le k-2+a$ for all $k$, $2\le k\le s$. □

**Proposition 3.7.** *Let* $s\ge 2$ *and* $0\le a\le 1$. *If* $\sum_{i=1}^{s}t_i\ge s-1+a$ *and* $\sum_{i=1}^{\ell}t_i\le \ell-1+a$ *for any* $\ell$, $1\le\ell<s$, *then* $\sum_{i=1}^{s}(t_i-t'_i)=1$ *and there exist real numbers* $\hat{t}_s,\bar{t}_s\ge 0$ *such that* $t_s=\hat{t}_s+\bar{t}_s$, $\sum_{i=1}^{s-1}t_i+\bar{t}_s=s-1+a$, $t'_\ell=t_\ell$, $\ell>s$, $t'_s=\hat{t}_s+\bar{t}'_s$ *and*

$$(t'_1,\cdots,t'_{s-1},\bar{t}'_s)=\xi(t_1,\cdots,t_{s-1},\bar{t}_s).$$

*Proof*: If $\sum_{i=1}^{s}t_i\ge s-1+a$ and $\sum_{i=1}^{\ell}t_i\le\ell-1+a$ for any $\ell<s$, then it follows

$$\underset{1\le j\le s}{\mathrm{Max}}\left\{\sum_{i=1}^{j}t_i-j\right\}=\sum_{i=1}^{s}t_i-s.$$

Hence by using (3.3) and Proposition 3.3, we obtain

$$\sum_{i=1}^{s}(t_i - t'_i) = \mathrm{Max}\left\{\sum_{i=1}^{s-1}(t_i - t'_i), 1\right\} = 1.$$

Hence $t'_\ell = t_\ell$, $\ell > s$ by Proosition 3.4. Let $\bar{t}_s = s{-}1{+}a - \sum_{i=1}^{s-1} t_i$ and $\hat{t}_s = t_s - \bar{t}_s$. We know $t_s - t'_s = 1 - \sum_{i=1}^{s-1}(t_i - t'_i)$. The same argument also implies $\bar{t}_s - \bar{t}'_s = 1 - \sum_{i=1}^{s-1}(t_i - t'_i)$ where $(t'_1, \cdots, t'_{s-1}, \bar{t}'_s) = \xi(t_1, \cdots, t_{s-1}, \bar{t}_s)$. Thus we obtain $t_s - t'_s = \bar{t}_s - \bar{t}'_s$ and hence $t'_s - \bar{t}'_s = t_s - \bar{t}_s = \hat{t}_s$. □

**Proposition 3.8.** *If* $\sum_{i=1}^{s} t_{k+i-1} \geq s{-}1$ *and* $\sum_{i=1}^{\ell} t_{k+i-1} \leq \ell{-}1$ *for any* $\ell$, $1 \leq \ell < s$, *then there exist real numbers* $\hat{t}_{k+s-1}, \bar{t}_{k+s-1} \geq 0$ *such that*

$$t_{k+s-1} = \hat{t}_{k+s-1} + \bar{t}_{k+s-1}, \quad \sum_{i=1}^{s-1} t_{k+i-1} + \bar{t}_{k+s-1} = s{-}1.$$
$$t'_{k+\ell-1} = t_{k+\ell-1},\ 1 \leq \ell < s, \quad \textit{and} \quad t'_{k+s-1} = \hat{t}'_{k+s-1} + \bar{t}_{k+s-1},$$
$$(t'_1, \cdots, t'_{k-1}, \hat{t}'_{k+s-1}, t'_{k+s}, \cdots, t'_n) = \xi(t_1, \cdots, t_{k-1}, \hat{t}_{k+s-1}, t_{k+s}, \cdots, t_n).$$

*Proof*: Firstly for $1 \leq \ell < s$, we obtain $\underset{1\leq j\leq k+\ell-1}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\} = \underset{1\leq j\leq k-1}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\}$ and $\underset{1\leq j\leq k-1}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\} - \sum_{i=1}^{k-1} t'_i + (k{-}1) \geq 0$, and hence we have

$$\begin{aligned}&\underset{0\leq j\leq k+\ell-1}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\} - \sum_{i=1}^{k+\ell-2} t'_i + (k{+}\ell{-}2)\\ &\qquad \geq \underset{1\leq j\leq k-1}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\} - \sum_{i=1}^{k-1} t'_i + (k{-}1) - \sum_{i=1}^{\ell-1} t_{k+i-1} + (\ell{-}1).\end{aligned}$$

Assuming that $t'_{k+i-1} = t_{k+i-1}$, $1 \leq i < \ell$, we obtain

$$\begin{aligned}&\underset{1\leq j\leq k+\ell-1}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\} - \sum_{i=1}^{k+\ell-2} t'_i + (k{+}\ell{-}2)\\ &\qquad \geq -\sum_{i=1}^{\ell-1} t_{k+i-1} + (\ell{-}1) = -\sum_{i=1}^{\ell} t_{k+i-1} + (\ell{-}1) + t_{k+\ell-1} \geq t_{k+\ell-1},\end{aligned}$$

which implies $t'_{k+\ell-1} = t_{k+\ell-1}$ by the definition of $\xi$. Thus we have $t'_{k+\ell-1} = t_{k+\ell-1}$, $1 \leq \ell < s$.

Secondly, the equations

$$\underset{1\le j\le k+s-1}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\} = \mathrm{Max}\left\{\sum_{i=1}^{j} t_i - j,\ \sum_{i=1}^{k+s-1} t_i - (k+s-1)\,;\ 1\le j\le k-1\right\}$$
$$= \mathrm{Max}\left\{\sum_{i=1}^{j} t_i - j,\ \sum_{i=1}^{k-1} t_i + \hat{t}_{k+s-1} - k\,;\ 1\le j\le k-1\right\}$$

and $\sum_{i=1}^{k+s-2} t'_i = \sum_{i=1}^{k-1} t'_i + \sum_{i=1}^{s-1} t_{k+i-1} = \sum_{i=1}^{k-1} t'_i + (s-1) - \bar{t}_{k+s-1}$, imply the following by putting $\hat{t}'_{k+s-1} = t'_{k+s-1} - \bar{t}_{k+s-1}$ and $\hat{t}_{k+s-1} = t_{k+s-1} - \bar{t}_{k+s-1}$.

$$\hat{t}'_{k+s-1} = t'_{k+s-1} - \bar{t}_{k+s-1} = \mathrm{Min}\left\{\hat{t}_{k+s-1}, \mathrm{Max}\left\{\sum_{i=1}^{j} t_i - j,\right.\right.$$
$$\left.\left.\sum_{i=1}^{k-1} t_i + \hat{t}_{k+s-1} - k\,;\ 1\le j\le k-1\right\} - \sum_{i=1}^{k-1} t'_i + (k-1)\right\}.$$

Thirdly, for any $\ell$, $1 \le \ell \le n-k-s+1$, we obtain

$$\underset{1\le j\le k+s+\ell-1}{\mathrm{Max}}\left\{\sum_{i=1}^{j} t_i - j\right\} = \mathrm{Max}\left\{\sum_{i=1}^{j} t_i - j,\right.$$
$$\left.\sum_{i=1}^{k-1} t_i + \hat{t}_{k+s-1} + \sum_{i=1}^{j'} t_{k+s+i-1} - (k+j')\,;\ 1\le j\le k-1, 0\le j'\le \ell\right\}$$

and

$$\sum_{i=1}^{k+s+\ell-2} t'_i = \sum_{i=1}^{k-1} t'_i + (s-1) - \bar{t}_{k+s-1} + t'_{k+s-1} + \sum_{i=1}^{\ell-1} t'_{k+s+i-1}$$
$$= \sum_{i=1}^{k-1} t'_i + (s-1) + \hat{t}'_{k+s-1} + \sum_{i=1}^{\ell-1} t'_{k+s+i-1}.$$

Hence we obtain the following equation.

$$t'_{k+s+\ell-1} = \mathrm{Min}\left\{t_{k+s+\ell-1}, \mathrm{Max}\left\{\sum_{i=1}^{j} t_i - j,\right.\right.$$
$$\left.\sum_{i=1}^{k-1} t_i + \hat{t}_{k+s-1} + \sum_{i=1}^{j'} t_{k+s+i-1} - (k+j')\,;\ 1\le j\le k-1, 0\le j'\le \ell\right\}$$
$$\left. - \sum_{i=1}^{k-1} t'_i - \hat{t}'_{k+s-1} - \sum_{i=1}^{\ell-1} t'_{k+s+i-1} + (k+\ell-1)\right\}.$$

Thus we have

$$(t'_1, \cdots, t'_{k-1}, \hat{t}'_{k+s-1}, t'_{k+s}, \cdots, t'_n) = \xi(t_1, \cdots, t_{k-1}, \hat{t}_{k+s-1}, t_{k+s}, \cdots, t_n).$$

This completes the proof of the proposition. □

Finally, the above properties of $\xi$ yield the following results.

**Lemma 3.9.** *$\xi(K(n)) \subseteq \{0\}\times K(n-1)$ and $\xi(J(n)) \subseteq \{0\}\times J(n-1)$.*

*Proof*: It is sufficient to show that $\xi(J^a(n)) \subseteq \{0\}\times J^a(n{-}1)$ for all $a \in I$, and is a direct consequence of Proposition 3.6. □

**Lemma 3.10.** *Let $1 \le k \le r$ and $2 \le r, t$ with $r{+}t = n{+}1$. For any $\rho \in K(r)$ and $\tau \in K(t)$, the following equation holds.*

$$\xi(\partial_k(\tau)(\rho)) = \begin{cases} \partial_{k-1}(\tau)(\xi(\rho)), & 1 < k \ \&\ r > 2,\\ \partial_1(\xi(\tau))(\rho), & k = 1 \ \&\ t > 2,\\ \tau, & k = 2 \ \&\ r = 2,\\ \rho, & k = 1 \ \&\ t = 2, \end{cases}$$

**Lemma 3.11.** *Let $1 \le k \le r$ and $2 \le t$ with $r{+}t = n{+}1$. For any $\rho \in J(r)$ and $\tau \in K(t)$, the following equation holds.*

$$\xi(\delta^{(a)}_k(\tau)(\rho)) = \begin{cases} \delta^{(a)}_{k-1}(\tau)(\xi(\rho)), & 1 < k,\\ \delta^{(a)}_1(\xi(\tau))(\rho), & k = 1 \ \&\ t > 2,\\ \rho, & k = 1 \ \&\ t = 2, \end{cases}$$

3.2. **Canonical degeneracy operations.** The map $\xi$ satisfies the conditions required for a degeneracy operation.

**Definition 3.12.** (1) *Let $d^K_k : K(n) \to K(n{-}1)$, $1 \le k \le n$ be the degeneracy operation given by the following formulas.*

$$d^K_1(t_1, \cdots, t_n) = (t'_2, \cdots, t'_n), \ \textit{where } (t'_1, \cdots, t'_n) = \xi(t_1, \cdots, t_n),$$
$$d^K_k(t_1, \cdots, t_n) = (t_1, \cdots, t_{k-2}, t_{k-1}{+}t'_k, t'_{k+1}, \cdots, t'_n), \ k \ge 2,$$
$$\textit{where } (t'_k, \cdots, t'_n) = \xi(t_k, \cdots, t_n).$$

(2) *Let $d^{J,a}_k : J^a(n) \to J^a(n{-}1)$, $1 \le k \le n$ be the degeneracy operation given by the following formulas.*

$$d^{J,a}_1(t_1, \cdots, t_n) = (t'_2, \cdots, t'_n), \ \textit{where } (t'_1, \cdots, t'_n) = \xi(t_1, \cdots, t_n),$$
$$d^{J,a}_k(t_1, \cdots, t_n) = (t_1, \cdots, t_{k-2}, t_{k-1}{+}t'_k, t'_{k+1}, \cdots, t'_n), \ k \ge 2,$$
$$\textit{where } (t'_k, \cdots, t'_n) = \xi(t_k, \cdots, t_n).$$

(3) $d^J_k = d^{J,\frac{1}{2}}_k : J(n) \to J(n{-}1)$, $1 \le k \le n$.

By Lemma 3.10, direct calculations imply the following theorems.

**Theorem 3.13.** *Let $j \le n$, $k \le r < n$ and $2 \le r, t$ with $r{+}t = n{+}1$. For any $\rho \in K(r)$ and $\tau \in K(t)$, the following equation holds.*

$$d^K_j(\partial_k(\tau)(\rho)) = \begin{cases} \partial_{k-1}(\tau)(d^K_j(\rho)), & 1 \le j < k \ \&\ r > 2,\\ \partial_k(d^K_{j-k+1}(\tau))(\rho), & k \le j < k{+}t \ \&\ t > 2,\\ \partial_k(\tau)(d^K_{j-t}(\rho)), & k{+}t \le j \le n \ \&\ r > 2,\\ \rho, & j = k \ \&\ t = 2,\\ \tau, & j = k{+}1 \ \&\ r = 2,\\ \tau, & j = n \ \&\ k = 1 \ r = 2, \end{cases}$$

**Theorem 3.14.** *Let $j \leq n$, $k \leq r < n$ and $2 \leq t \leq n$ with $r+t = n+1$. For any $\rho \in J(r)$ and $\tau \in K(t)$, the following equation holds.*

$$d^J_j(\delta_k(\tau)(\rho)) = \begin{cases} \delta_{k-1}(\tau)(d^J_j(\rho)), & 1 \leq j < k, \\ \delta_k(d^K_{j-k+1}(\tau))(\rho), & k \leq j < k+t \ \&\ t > 2, \\ \delta_k(\tau)(d^J_{j-t}(\rho)), & k+t \leq j \leq n, \\ \rho, & j = k \ \&\ t = 2. \end{cases}$$

**Theorem 3.15.** *Let $j \leq n$, $r_1, \cdots, r_t < n$, $2 \leq t \leq n$ with $\sum_{i=1}^{t} r_i = n$. For any $\rho_i \in J(r_i)$ $(1 \leq i \leq t)$ and $\tau \in K(t)$, the following equation holds.*

$$d^J_j(\delta(\rho_1, \cdots, \rho_t)(\tau)) =$$
$$\begin{cases} \delta(\rho_1, \cdots, d^J_{j-s_{k-1}}(\rho_k), \cdots, \rho_t)(\tau), & s_{k-1} < j \leq s_k \ \&\ r_k > 1, \\ \delta(\rho_1, \cdots, \rho_{k-1}, \rho_{k+1}, \cdots, \rho_t)(d^K_k(\tau)), & j = s_k \ \&\ r_k = 1 \ \&\ t > 2, \\ \rho_2, & j = 1 \ \&\ r_1 = 1 \ \&\ t = 2, \\ \rho_1, & j = n \ \&\ r_2 = 1, \ \&\ t = 2, \end{cases}$$

*where $s_k = r_1 + \cdots + r_k$.*

3.3. **Degeneracy operations.** We call a set $\{d_j : K(n) \to K(n-1)\,;\, 1 \leq j \leq n\}$ a *degeneracy operations on* $K(n)$ if they satisfies the condition given in Theorem 3.13 with $d^K_j$ replaced by $d_j$. Similarly, we call a set $\{D_j : J^a(n) \to J^a(n-1)\,;\, 1 \leq j \leq n\}$ a *degeneracy operations on* $J^a(n)$ if they satisfies the condition given in Theorem 3.13 with $d^J_j$ replaced by $D_j$. Then we show that a choice of the set of degeneracy operations are not essential:

**Theorem 3.16.** *The existence of an $A_m$ form ($m \leq \infty$) for a space $X$ for a set of degeneracies on $K(n)$ ($n \leq m$) implies the existence of that for another set of degeneracies on $K(n)$ ($n \leq m$).*

*Proof*: Let $\{d_j\}$ and $\{d'_j\}$ be two sets of degeneracy operations on $K(n)$. Since $K(n)$ and its faces $K_k(r, s)$ are convex subspaces of the Euclidean space $\mathbb{R}^n$, we can construct a family of sets of degeneracy operations $\tilde{d}_j : I \times K(n) \to K(n-1)$, $1 \leq j \leq n$ as follows:

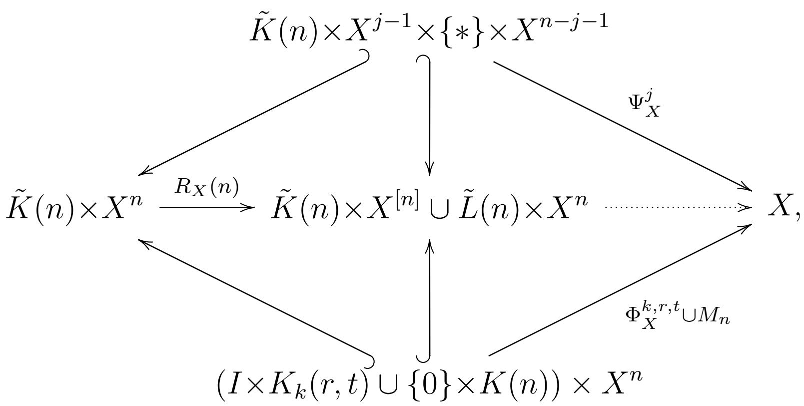
$$\tilde{d}_j(u, \sigma) = (1-u)d_j(\sigma) + ud'_j(\sigma).$$

Since $\partial_k(\tau)(\rho)$ is affine in $\rho$ and $\tau$, we can deduce that

$$\tilde{d}_j(u,\partial_k(\tau)(\rho)) = (1-u)d_j\partial_k(\tau)(\rho) + ud'_j\partial_k(\tau)(\rho)$$
$$= \begin{cases} \partial_{k-1}(\tau)(\tilde{d}_j(u,\rho)), & 1 \le j < k \ \&\ r > 2, \\ \partial_k(\tilde{d}_{j-k+1}(u,\tau))(\rho), & k \le j < k+t \ \&\ t > 2, \\ \partial_k(\tau)(\tilde{d}_{j-t}(u,\rho)), & k+t \le j \le n \ \&\ r > 2, \\ \rho, & j = k \ \&\ s = 2, \\ \tau, & j = k+1 \ \&\ r = 2, \\ \tau, & j = n \ \&\ k = 1\ r = 2. \end{cases}$$

Since the inclusion map $K_1(n) \hookrightarrow K(n)$ is a cofibration, the pair $(\tilde{K}(n), \tilde{L}(n)) = (I, \{0\}) \times (K(n), K_1(n))$ is a DR-pair in the sense of G. W. Whitehead (see [21]). It also follows that the pair $(\tilde{K}(n) \times X^n, \tilde{K}(n) \times X^{[n]} \cup \tilde{L}(n) \times X^n)$ is naturally a DR-pair, if the base point of $X$ is non-degenerate (see also [21]). Thus there exists a natural deformation retraction $R_X(n) : \tilde{K}(n) \times X^n \to \tilde{K}(n) \times X^{[n]} \cup \tilde{L}(n) \times X^n$.

Let $\{M_i\}$ be an $A_m$ form $(m \le \infty)$ of a space for a set of degeneracy operations $\{d_j\}$ on $K(n)$, and $\{d'_j\}$ be another set of degeneracies. Using the above $\{\tilde{d}_j\}$, we can construct a homotopy $\tilde{M}_n : I \times K(n) \times X^n \to X$ given inductively by $\tilde{M}_n = (\bigcup_j \Psi^j_X \cup \bigcup_{k,r,s} \Phi^{k,r,s}_X \cup M_n) \circ R_X(n)$;




where $\Psi^j_X$ and $\Phi^{k,r,t}_X$ are defined by

$$\Psi^j_X(u,\sigma; x_1, \cdots, x_{j-1}, *, x_{j+1}, \cdots, x_n)$$
$$= \tilde{M}_{n-1}(u, \tilde{d}_j(u,\sigma); x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n),$$
$$\Phi^{k,r,t}_X(u, \partial_k(\tau)(\rho); x_1, \cdots, x_n)$$
$$= \tilde{M}_r(u, \rho; x_1, \cdots, x_{k-1}, \tilde{M}_s(u, \tau; x_k, \cdots, x_{k+s-1}), x_{k+s}, \cdots, x_n).$$

Then we obtain a new $A_m$ form $\{M'_i\}$ given by the formula

$$M'_n(\sigma) = \tilde{M}_n(1, \sigma),$$

which satisfies the *strict-unit* condition with respect to the set of degeneracy operations $\{d'_j\}$. □

Let us fix $a \in (0,1]$. Similarly as above, we can show the following:

**Theorem 3.17.** *The existence of an $A_m$ form ($m \le \infty$) for a map $f$ of $A_\infty$ spaces for a set of degeneracies on $J^a(n)$ ($n \le m$) implies the existence of that for another set of degeneracies on $J^a(n)$ ($n \le m$).*

*Proof*: Let $\{D_j\}$ and $\{D'_j\}$ be two sets of degeneracy operations on $J^a(n)$ which are compatible with $\{d_j\}$ and $\{d'_j\}$ on $K(n)$, respectively. Since $J^a(n)$ and its faces $J^a_k(r,s)$ and $J^a(t,r_1,\cdots,r_t)$ are convex subspaces of the Euclidean space $\mathbb{R}^n$, we can construct a family of sets of degeneracy operations $\tilde{D}_j : I \times J^a(n) \to J^a(n{-}1)$, $1 \le j \le n$ as follows:

$$\tilde{D}_j(u,\sigma) = (1{-}u)D_j(\sigma) + uD'_j(\sigma).$$

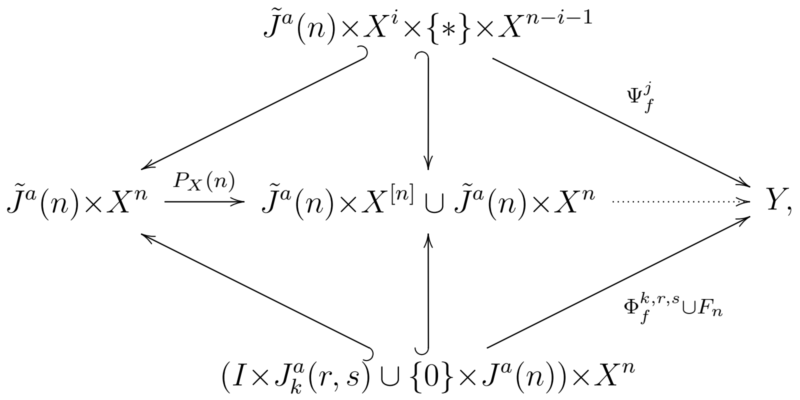
Because $\delta^a_k(\tau)(\rho)$ is affine in $\rho$ and $\tau$, and $\delta^a(t,r_1,\cdots,r_t)(\tau;\rho_1,\cdots,\rho_t)$ is affine in $\tau$ and all $\rho_i$s, the definition of $\tilde{D}_j$ yields the following.

$$\begin{aligned}\tilde{D}_j(u,\delta_k(\tau)(\rho)) &= (1{-}u)D_j\partial_k(\tau)(\rho) + uD'_j\partial_k(\tau)(\rho)\\ &= \begin{cases}\delta^a_{k-1}(\tau)(\tilde{d}_j(u,\rho)), & 1 \le j < k,\\ \delta^a_k(\tilde{d}_{j-k+1}(u,\tau))(\rho), & k \le j < k{+}t \ \&\ t > 2,\\ \delta^a_k(\tau)(\tilde{d}_{j-t}(u,\rho)), & k{+}t \le j \le n,\\ \rho, & j = k \ \&\ s = 2.\end{cases}\end{aligned}$$

$$\begin{aligned}&\tilde{D}_j(u,\delta^a(t,r_1,\cdots,r_t)(\tau;\rho_1,\cdots,\rho_t))\\ &\quad= (1{-}u)D_j\delta^a(t,r_1,\cdots,r_t)(\tau;\rho_1,\cdots,\rho_t)\\ &\qquad + uD'_j\delta^a(t,r_1,\cdots,r_t)(\tau;\rho_1,\cdots,\rho_t)\\ &\quad= \begin{cases}\delta(\rho_1,\cdots,\tilde{D}_{j-s_{k-1}}(\rho_k),\cdots,\rho_t)(\tau), & s_{k-1} < j \le s_k \ \&\ r_k > 1,\\ \delta(\rho_1,\cdots,\rho_{k-1},\rho_{k+1},\cdots,\rho_t)(\tilde{d}_k(\tau)), & j = s_k,\ r_k = 1 \ \&\ t > 2,\\ \rho_2, & j = 1,\ r_1 = 1 \ \&\ t = 2,\\ \rho_1, & j = n,\ r_2 = 1 \ \&\ t = 2.\end{cases}\end{aligned}$$

Since the inclusion map $J^a_0(n) \hookrightarrow J^a(n)$ is a cofibration, the pair $(\tilde{J}^a(n), \tilde{J}^a(n)) = (I,\{0\}) \times (J^a(n), J^a_0(n))$ is a DR-pair. It also follows that the pair $(\tilde{J}^a(n) \times X^n, \tilde{J}^a(n) \times X^{[n]} \cup \tilde{J}^a(n) \times X^n)$ is naturally a DR-pair, if the base point of $X$ is non-degenerate (see also [21]). Thus there exists a natural deformation retraction $P_X(n) : \tilde{J}^a(n) \times X^n \to \tilde{J}^a(n) \times X^{[n]} \cup \tilde{J}^a(n) \times X^n$.

Let $\{F_i\}$ be an $A_m$ form ($m \le \infty$) of a map $f : X \to Y$ of $A_m$ spaces $X$ and $Y$ for a set of degeneracy operations $\{D_j\}$ on $J^a(n)$ compatible with $\{d_j\}$ a set of degeneracy operations on $K(n)$, and $\{D'_j\}$ be another set of degeneracies compatible with $\{d'_j\}$. Using the above $\{\tilde{D}_j\}$ and $\{\tilde{d}_j\}$, we can construct a homotopy $\tilde{F}_n : I{\times}J^a(n) \times X^n \to Y$ given inductively by $\tilde{F}_n = (\bigcup_j \Psi_X^j \cup \bigcup_{k,r,s} \Phi_X^{k,r,s} \cup F_n){\circ}P_X(n)$;




where $\Psi_f^j$ and $\Phi_f^{k,r,t}$ are defined by

$$\begin{aligned}
&\Psi_f^j(u,\sigma;x_1,\cdots,x_{j-1},*,x_{j+1},\cdots,x_n)\\
&\qquad = \tilde{F}_{n-1}(u,\tilde{D}_j(u,\sigma);x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_n),\\
&\Phi_f^{k,r,t}(u,\delta_k^a(\tau)(\rho);x_1,\cdots,x_n)\\
&\qquad = \tilde{F}_r(u,\rho;x_1,\cdots,x_{k-1},\tilde{M}_s(u,\tau;x_k,\cdots,x_{k+s-1}),x_{k+s},\cdots,x_n),\\
&\Phi_f^{k,r,t}(u,\delta^a(\rho_1,\cdots,\rho_t)(\tau);x_1,\cdots,x_n)\\
&\qquad = \tilde{M}_r(u,\tau;\tilde{F}_{r_1}(u,\rho_1;x_1,\cdots,x_{r_1}),\cdots,\tilde{F}_{r_1}(u,\rho_t;x_{\sum_{i=1}^{t-1} r_i+1},\cdots,x_n)).
\end{aligned}$$

□

From now on, we do not specify explicitly the set of degeneracy operations actually in use, unless it requires the detailed expression.

3.4. **Homeomorphisms between $K(n)$ and $J_0^a(n)$.** We define by induction on $n$ a homeomorphism between $K(n)$ and $J_0^a(n) \subset J^a(n)$.

Firstly, we introduce special points in $K(n)$ and $J^a(n)$. Let $\beta_a(n) = (a,1,\cdots,1) \in I_a(n)$, $\beta^J(n) = \beta_{\frac{1}{2}}(n) \in J(n)$ and $\beta^K(n) = \beta_0(n) \in K(n)$. Some direct calculations yield the following.

**Theorem 3.18.** *For any $\rho \in J(n)$ and $\sigma \in K(n)$, we have*

(1) $d_j^K(t\beta^K(n) + (1{-}t)\sigma) = t\beta^K(n{-}1) + (1{-}t)d_j^K(\sigma)$.
(2) $d_j^J(t\beta^J(n){+}(1{-}t)\rho) = t\beta^J(n{-}1){+}(1{-}t)d_j^J(\rho)$, *except for $j = 1$.*

We also define $\alpha_a(n) = (0,1{-}\frac{a}{2},1,\cdots,1,1{+}\frac{a}{2}) \in K(n)$ for $n \geq 2$ and $\alpha(n) = \alpha_1(n)$, which lie in the interior of $K(n)$. We remark that $\beta_1(n)$ in $J^1(n)$ corresponds to $\alpha_0(n)$ in $K(n{+}1)$ by the natural homeomorphism $J^1(n) \approx K(n{+}1)$.

As is first introduced in Stasheff [17], we can define another set of degeneracy operations $d_j^S$ by induction using the following formulas:

$$d_j^S(t\alpha(n) + (1{-}t)\sigma) = t\alpha(n{-}1) + (1{-}t)d_j^S(\sigma).$$

These observations suggests us the following definition.

**Definition 3.19.** *Homeomorphisms* $\omega_n^a : K(n) \to J_0^a(n)$ *and* $\omega_n = \omega^{1/2}(n) : K(n) \to J_0(n)$ *are defined inductively by*

$$\omega_n^a(t\alpha_a(n) + (1{-}t)\partial_k(\tau)(\rho)) = t\beta_a(n) + (1{-}t)\delta_k^a(\tau)(\omega_r^a(\rho)),$$

*where* $\omega_n^0(\sigma) = \sigma \in J^0(n) = K(n)$*. Then we define* $\eta_n^a : [0,1] \times K(n) \to J^a(n)$*,* $\eta_n : [0,1] \times K(n) \to J(n)$ *and* $\eta_n^1 : [0,1] \times K(n) \to K(n{+}1)$ *by*

$$\eta_n^a(t,\sigma) = \omega_n^{at}(\sigma),\ \ \eta_n = \eta_n^{1/2},$$

*which implies* $\eta_n(0,\sigma) = \delta_1(1,n)(*,\sigma)$ *and* $\eta_n^1(0,\sigma) = \partial_2(2,n)(*,\sigma)$*.*

Since $\delta_k^a(\tau)(J_0^a(r)) \subset J_0^a(n)$, $\delta_k^a(\tau)(\eta_r^1(\rho)) = (u_1,\cdots,u_n)$ must satisfy $\sum_{i=1}^{\ell} u_i = \ell{-}1{+}a$ for some $\ell \le n$. Hence $t\beta_a(n) + (1{-}t)\delta_k^a(\tau)(\eta_r^1(\rho)) = (ta{+}(1{-}t)u_1) + \sum_{i=2}^{\ell}(t{+}(1{-}t)u_i) = t(a + \sum_{i=2}^{\ell} 1) + (1{-}t)(\sum_{i=1}^{\ell} u_i) = t(\ell{-}1{+}a) + (1{-}t)(\ell{-}1{+}a) = \ell{-}1{+}a$. This implies that $\omega_n^a(K(n)) \subset J_0^a(n)$ and hence $\eta_n^1$, $\eta_n$ and $\omega_n^a$ are well-defined.

By this definition, we easily see the following proposition.

**Proposition 3.20.** (1) $\omega_n^a(\partial_k(\tau)(\rho)) = \delta_k(\tau)(\omega_r^a(\rho))$*.*
(2) $d_j^J\omega_n^a(\sigma) = \omega_{n-1}^a(d_j^S(\sigma))$*,* $2 \le j \le n$*.*
(3) $d_{j+1}^K\omega_n^1(\sigma) = \omega_{n-1}^1(d_j^S(\sigma))$*,* $1 \le j \le n$*.*

Hence $\eta_n^1 : [0,1] \times K(n) \to K(n{+}1)$ and $\eta_n : [0,1] \times K(n) \to J(n)$ preserves both the face and degeneracy operations except $d_1$.

## 4. Internal pre-category

We introduce a notion of an internal pre-category using a notion of coalgebras and comodules in a regular monoidal category $\underline{\underline{\mathcal{C}}}$ with a tensor product $\otimes : \underline{\underline{\mathcal{C}}} \times \underline{\underline{\mathcal{C}}} \to \underline{\underline{\mathcal{C}}}$ with a unit object 1.

### 4.1. Coalgebras and comodules.

First we introduce the notion of a comonoid or a coalgebra in $\underline{\underline{\mathcal{C}}}$.

**Definition 4.1.** (1) *A triple* $(O,\nu,\epsilon)$*, or simply* $O$*, is called an 'coalgebra', if* $O$ *is an object in* $\underline{\underline{\mathcal{C}}}$*, morphisms* $\nu : O \to O{\otimes}O$ *and* $\epsilon : O \to 1$ *in* $\underline{\underline{\mathcal{C}}}$ *satisfies that* $(\nu{\otimes}1_O){\circ}\nu = (1_O{\otimes}\nu){\circ}\nu$ *and* $(1_O{\otimes}\epsilon){\circ}\nu = (\epsilon{\otimes}1_O){\circ}\nu = 1_O$ *the identity morphism. In that case, the morphisms* $\nu$ *and* $\epsilon$ *are often called a comultiplication and a counit, respectively, of a coalgebra* $O$*.*
(2) *Let* $\underline{\underline{\mathcal{C}}}$ *be symmetric regular monoidal. Then for a coalgebra* $O = (O,\nu,\epsilon)$*, we define a coalgebra* $O^* = (O,\nu^*,\epsilon)$ *by* $\nu^* = \gamma{\circ}\nu$ *where* $\gamma : O{\otimes}O \to O{\otimes}O$ *is the symmetry isomorphism of* $\underline{\underline{\mathcal{C}}}$*.*
(3) *A coalgebra* $O$ *in a symmetric regular monoidal category* $\underline{\underline{\mathcal{C}}}$ *is called cocommutative, if* $O = O^*$*.*

Then we define a 'bicomodules' under a coalgebra $O$:

**Definition 4.2.** *Let $O=(O,\nu,\epsilon)$ be a coalgebra in $\underline{\underline{\mathcal{C}}}$ and $M$ be an object in $\underline{\underline{\mathcal{C}}}$.*

(1) *$M=(M,\tau)$ is right comodule under $O$, if $\tau : M \to M{\otimes}O$ satisfies $(\tau{\otimes}1_O){\circ}\tau=(1_M{\otimes}\nu){\circ}\tau$ and $(1_M{\otimes}\epsilon){\circ}\tau=1_M$.*
(2) *$M=(M,\sigma)$ is left comodule under $O$, if $\sigma : M \to O{\otimes}M$ satisfies $(1_O{\otimes}\sigma){\circ}\sigma=(\nu{\otimes}1_M){\circ}\sigma$ and $(\epsilon{\otimes}1_M){\circ}\sigma=1_M$.*
(3) *$M=(M;\tau,\sigma)$ is bicomodule under $O$, if $(M,\tau)$ is a right comodule and $(M,\sigma)$ is a left comodule.*

For a morphism, we also introduce some more notions.

**Definition 4.3.** *Let $O=(O,\nu,\epsilon)$ and $O=(O',\nu',\epsilon')$ be coalgebras in $\underline{\underline{\mathcal{C}}}$ and $M=(M;\tau,\sigma)$ and $M'=(M';\tau',\sigma')$ be objects in $\underline{\underline{\mathcal{C}}}$.*

(1) *A morphism $\phi : O \to O'$ of coalgebras in $\underline{\underline{\mathcal{C}}}$ is called a 'homomorphism', if it satisfies $\nu'{\circ}\phi=(\phi{\otimes}\phi){\circ}\nu$ and $\epsilon'{\circ}\phi=\epsilon$.*
(2) *A pair $(f,\phi)$ of a morphism $f : M \to M'$ and a homomorphism $\phi : O \to O'$ where $M$ and $M'$ are right comodules by $\tau$ and $\tau'$ under $O$ and $O'$, respectively in $\underline{\underline{\mathcal{C}}}$ is called (right) 'equivariant', if it satisfies $(f{\otimes}\phi){\circ}\tau=\tau'{\circ}f$.*
(3) *A pair $(f,\phi)$ of a morphism $f : M \to M'$ and a homomorphism $\phi : O \to O'$ where $M$ and $M'$ are left comodules by $\sigma$ and $\sigma'$ under $O$ and $O'$, respectively in $\underline{\underline{\mathcal{C}}}$ is called (left) 'equivariant', if it satisfies $(\phi{\otimes}f){\circ}\sigma=\sigma'{\circ}f$.*
(4) *A pair $(f,\phi)$ of a morphism $f : M \to M'$ and a homomorphism $\phi : O \to O'$ where $M=(M;\tau,\sigma)$ and $M'=(M';\tau',\sigma')$ are bicomodules under $O$ and $O'$ respectively in $\underline{\underline{\mathcal{C}}}$ is called a 'biequivariant' morphism or an 'internal homomorphism', if it is both right and left equivariant.*

We remark that, in a slightly different context, Tamaki and Asashiba [3] have given a similar idea which is used to generalize quiver for a generalization of the Grothendiek construction.

4.2. **Internal pre-category and internal pre-functor.** We introduce a notion of internal pre-category in a regular monoidal category $\underline{\underline{\mathcal{C}}}$.

**Definition 4.4.** (1) *A pair $(M,O)$ of $O$ a coalgebra with a comultiplication $\nu$ and a counit $\epsilon$ and $M$ a bicomodule by $\tau : M \to M{\otimes}O$ and $\sigma : M \to O{\otimes}M$ under $O$ equipped with a morphism $\iota : O \to M$ is called an 'internal pre-category' in $\underline{\underline{\mathcal{C}}}$ and denoted by $(M,O;\sigma,\tau,\iota)$ if it satisfies the following two conditions:*

$$\sigma{\circ}\iota=(1_O{\otimes}\iota){\circ}\nu, \quad \tau{\circ}\iota=(\iota{\otimes}1_O){\circ}\nu.$$

(2) *Let $(M,O;\sigma,\tau,\iota)$ and $(M',O';\sigma',\tau',\iota')$ be internal pre-categories in $\underline{\underline{\mathcal{C}}}$. A pair of morphisms $(f : M \to M', \phi : O \to O')$ in $\underline{\underline{\mathcal{C}}}$ is*

*called an 'internal pre-functor' in $\underline{\underline{\mathcal{C}}}$ and denoted by $(f,\phi)$ if it is an internal homomorphism satisfying the following condition:*

$$f{\circ}\iota = \iota'{\circ}\phi.$$

From now on, we often abbreviate $(M,O;\sigma,\tau,\iota)$ by $(M,O)$ or simply by $M$, and $(f,\phi)$ by $f$. Let us denote by ${}^{ip}\underline{\underline{\mathcal{C}}}$ the category of internal pre-categories and internal pre-functors in $\underline{\underline{\mathcal{C}}}$.

**Example 4.5.** (1) *$O = (O,O;1_O,\nu,\nu)$ is in ${}^{ip}\underline{\underline{\mathcal{C}}}$.*

(2) *For given two internal pre-categories $M = (M,O;\sigma,\tau,\iota)$ and $M' = (M',O;\sigma',\tau',\iota')$ in $\underline{\underline{\mathcal{C}}}$, the cotensor of a right comodule $M$ and a left comodule $M'$ gives an internal pre-category $M\,\square_O\,M' = (M\,\square_O\,M',O;\sigma'',\tau'',\iota'')$ in $\underline{\underline{\mathcal{C}}}$ as the equalizer of $1{\otimes}\sigma'$ and $\tau{\otimes}1$:*

$$M\,\square_O\,M' \dashrightarrow M{\otimes}M' \underset{\tau\otimes 1}{\overset{1\otimes\sigma'}{\rightrightarrows}} M{\otimes}O{\otimes}M'.$$

(3) *For any internal pre-category $M = (M,O;\sigma,\tau,\iota)$, the equalizers $M\,\square_O\,O \to M{\otimes}O$ and $O\,\square_O\,M \to O{\otimes}M$ are naturally equivalent to $\tau : M \to M{\otimes}O$ and $\sigma : M \to O{\otimes}M$, respectively (see [2]).*

(4) *For two internal pre-functors $f = (f,\phi) : (M_1,O_1;\sigma_1,\tau_1,\iota_1) \to (M_2,O_2;\sigma_2,\tau_2,\iota_2)$ and $f' = (f',\phi) : (M'_1,O_1;\sigma_2,\tau_2,\iota'_1) \to (M'_2,O_2;\sigma'_2,\tau'_2,\iota'_2)$ in $\underline{\underline{\mathcal{C}}}$, the cotensor of a right equivariant map $f$ and a left equivariant map $f'$ gives an internal pre-functor $f\,\square_\phi\,f' : M_1\,\square_{O_1}\,M'_1 \to M_2\,\square_{O_2}\,M'_2$ in $\underline{\underline{\mathcal{C}}}$:*

$$\begin{array}{ccccc}
M_1\,\square_{O_1}\,M'_1 & \longrightarrow & M_1{\otimes}M'_1 & \underset{\tau_1\otimes 1}{\overset{1\otimes\sigma'_1}{\rightrightarrows}} & M_1{\otimes}O_1{\otimes}M'_1. \\
\Big\downarrow{\scriptstyle f\,\square_\phi\, f'} & & \Big\downarrow{\scriptstyle f\otimes f'} & & \Big\downarrow{\scriptstyle f\otimes\phi\otimes f'} \\
M_2\,\square_{O_2}\,M'_2 & \dashrightarrow & M_2{\otimes}M'_2 & \underset{\tau_2\otimes 1}{\overset{1\otimes\sigma'_2}{\rightrightarrows}} & M_2{\otimes}O_2{\otimes}M'_2.
\end{array}$$

(5) *For an internal pre-category $M = (M,O;\sigma,\tau,\iota)$ in $\underline{\underline{\mathcal{C}}}$, we obtain an internal pre-category $\square^n_O\,M = (\square^n_O\,M,O;\iota_n,\sigma_n,\tau_n)$ with $\square^1_O\,M = M$ in $\underline{\underline{\mathcal{C}}}$ by induction on $n$.*

(6) *For an internal pre-functor $f = (f,\phi) : (M,O) \to (M',O')$ in $\underline{\underline{\mathcal{C}}}$, we obtain an internal pre-functor $\square^n_\phi\,f : (\square^n_O\,M,O) \to (\square^n_{O'}\,M',O')$ in $\underline{\underline{\mathcal{C}}}$ with $\square^1_\phi\,f = f$ by induction on $n$.*

4.3. **Internal multiplication and internal action.** Let $X = (X,O) = (X,O;\sigma,\tau,\iota)$ be an internal pre-category in $\underline{\underline{\mathcal{C}}}$ with an internal multiplication $\mu : X\,\square_O\,X \to X$ in $\underline{\mathcal{C}}$ in $\underline{\underline{\mathcal{C}}}$.

**Definition 4.6.** *Let $X = (X,O;\sigma,\tau,\iota)$ be an internal pre-category in $\underline{\underline{\mathcal{C}}}$.*

(1) *Let $X = (X, O; \sigma, \tau, \iota)$ be a bicomodule in $\underline{\underline{\mathcal{C}}}$. If $X$ is equipped with a pre-functor $\mu : X \mathbin{\Box_O} X \to X$, then $\overline{\overline{X}} = (X, \mu)$ is called an 'internal semi-category' in $\underline{\underline{\mathcal{C}}}$, and the pre-functor $\mu$ is called an 'internal multiplication' of $\overline{\overline{X}}$.*

(2) *An internal semi-category $X = (X, \mu)$ is an 'internal h-category' in $\underline{\underline{\mathcal{C}}}$, if an internal multiplication $\mu : \Box_O^2 X \to X$ satisfies*

$$\mu\circ(\iota \mathbin{\Box_O} 1_X) = 1_X = \mu\circ(1_X \mathbin{\Box_O} \iota),$$

*where we regard $O \mathbin{\Box_O} X = X = X \mathbin{\Box_O} O$. We denote such an internal h-category by $(X, \mu)$ or simply by $X$.*

We remark that an internal h-category $M = (M, \mu)$ is an internal category or a 'monad' in the sense of Aguiar [2], if the internal multiplication $\mu$ satisfies the strict associativity condition:

$$\mu\circ(1_X \mathbin{\Box_O} \mu) = \mu\circ(\mu \mathbin{\Box_O} 1_X).$$

It would be natural to extend these ideas slightly more: for internal pre-categories $X = (X, O; \sigma_X, \tau_X, \iota_X)$, $Y = (Y, O; \sigma_Y, \tau_Y, \iota_Y)$ and $Z = (Z, O; \sigma_Z, \tau_Z, \iota_Z)$ and internal homomorphisms $p : Y \to X$ and $q : Z \to X$ in $\underline{\underline{\mathcal{C}}}$, we call an internal homomorphism $\mu : Y \Box_O Z \to X$ with $\mu\circ(1_Y \Box_O \iota_Z) = p$ and $\mu\circ(\iota_Y \Box_O 1_Z) = q$ an internal pairing with axes $(p, q)$, or left axis $p$ and right axis $q$ in $\underline{\underline{\mathcal{C}}}$.

We call an internal homomorphism $\mu' : Y \Box_O X \to Y$ with right axis $q : X \to Y$ an internal right pairing of $X$ on $Y$ along $q$, and an internal homomorphism $\mu'' : X \Box_O Z \to Z$ with left axis $p : X \to Z$ an internal left pairing of $X$ on $Z$ along $p$. In each case, we don't care about the other axis in our definition.

**Definition 4.7.** *Let $X = (X, O; \sigma_X, \tau_X, \iota_X)$, $Y = (Y, O; \sigma_Y, \tau_Y, \iota_Y)$ and $Z = (Z, O; \sigma_Z, \tau_Z, \iota_Z)$ be internal pre-categories in the category $\underline{\underline{\mathcal{C}}}$.*

(1) *For an internal pre-functor $q : X \to Y$ in $\underline{\underline{\mathcal{C}}}$, we call $(Y, \mu', X)$ an internal (right) action of $X$ on $Y$ along $q$ in $\underline{\underline{\mathcal{C}}}$, if $\mu' : Y \otimes_O X \to Y$ is an internal right pairing in $\underline{\underline{\mathcal{C}}}$ such that*

$$\mu'\circ(1_Y \mathbin{\Box_O} \iota_X) = 1_Y, \tag{4.1}$$

$$\mu'\circ(\iota_Y \mathbin{\Box_O} 1_X) = q, \tag{4.2}$$

*where we regard $Y \mathbin{\Box_O} O = Y$ and $O \mathbin{\Box_O} X = X$. We denote such an internal action by $(Y, \mu', X)$ or simply by $(Y, X)$.*

(2) *For an internal pre-functor $p : X \to Z$ in $\underline{\underline{\mathcal{C}}}$, we call $(X, \mu'', Z)$ an internal (left) action of $X$ on $Z$ along $k$ in $\underline{\underline{\mathcal{C}}}$, if $\mu'' : X \mathbin{\Box_O} Z \to Z$ is an internal left pairing in $\underline{\underline{\mathcal{C}}}$ such that*

$$\mu''\circ(\iota \mathbin{\Box_O} 1_Z) = 1_Z, \tag{4.3}$$

$$\mu''\circ(1_X \mathbin{\Box_O} \iota_Z) = p, \tag{4.4}$$

*where we regard $O \mathbin{\Box_O} Z = Z$ and $X \mathbin{\Box_O} O = X$. We denote such an internal action by $(X, \mu'', Z)$ or simply by $(X, Z)$.*

**Remark 4.8.** *Even if we drop the condition on $q$ or $p$ to be an internal pre-functor, each becomes an internal pre-functor by the above definition, since so does $\mu'$ or $\mu''$.*

**Example 4.9.** *An internal h-category $(X,\mu)$ gives an internal right and left actions of $X$ on $X$ along the identity internal functor in $\underline{\underline{\mathcal{C}}}$.*

## 5. $A_\infty$ forms for multiplication

5.1. $A_\infty$ **form for internal multiplication.** We introduce a notion of an $A_m$ form ($1{\leq}m{\leq}\infty$) for an internal multiplication in $\underline{\underline{\mathcal{T}}}$. Let $X = (X,\mu)$ be an internal pre-category with an internal multiplication $\mu : X \times_O X \to X$ in $\underline{\underline{\mathcal{T}}}$.

**Definition 5.1.** *We call $\{a(n); 1{\leq}n{\leq}m\}$ ($a(1) = 1_X$) an $A_m$ form for $\mu$, if $a(n) : K(n) \times \prod_O^n X \to X$ satisfies the following formulas for all $(\rho,\sigma) \in K(r){\times}K(s)$, $n = r{+}s{-}1$ and $\mathbf{x} = (x_1, \cdots, x_n) \in \prod_O^n X$, $n{\geq}2$.*

$$a(2) = \mu, \tag{5.1}$$

$$a(n)(\partial_k(\rho,\sigma);\mathbf{x}) = a(r)(\rho; a_k(s)(\sigma;\mathbf{x})), \tag{5.2}$$

*where $a_k(s)(\sigma;\mathbf{x})$ is given by*

$$(x_1, \cdots, x_{k-1}, a_s(\sigma; x_k, \cdots, x_{k+s-1}), x_{k+s}, \cdots, x_n).$$

An internal category with the above $A_m$ form for an internal multiplication might be called an internal $A_m$ category *without unit.* When $O$ is the one-point set, an internal $A_m$ category $X$ *without unit* may be called an $A_m$ space *without unit.*

5.2. **Internal** $A_\infty$ **category with unit.** Now we define an internal $A_m$ category for $1 \leq m \leq \infty$ in $\underline{\underline{\mathcal{T}}}$.

**Definition 5.2.** *Let $\{a(n); 1{\leq}n{\leq}m\}$ ($a(1) = 1_X$) be an $A_\infty$ form for $\mu : X \times_O X \to X$.*

1. *We call an internal $A_m$ category $X = (X; \{a(n)\})$ an "internal $A_m$ category with hopf-unit", if $X$ further satisfies the following hopf-unit condition.*

$$a(2)(1_X{\times_O}\iota_X) = 1_X = a(2)(\iota_X{\times_O}1_X) \tag{5.3}$$

2. *We call an internal $A_m$ category $X = (X; \{a(n)\})$ an "internal $A_m$ category with strict-unit", if $X$ satisfies the following strict-unit condition.*

$$\begin{aligned} a(n)(1_{K(n)}{\times}((\textstyle\prod_O^{j-1} 1_X){\times_O}\iota_X{\times_O}(\prod_O^{n-j} 1_X))) \\ = a(n{-}1)(d_j^K{\times}(\textstyle\prod_O^{n-1} 1_X)),\ 1{\leq}j{\leq}n, \end{aligned} \tag{5.4}$$

If an internal pre-category is an internal $A_m$ category with *strict-(or hopf-)unit* in $\underline{\underline{\mathcal{T}}}$ for all $m \geq 2$, then it is called an internal $A_\infty$ category with *strict-(or hopf-)unit* in $\underline{\underline{\mathcal{T}}}$. When $m = 2$, an internal $A_2$ category with *strict-(or hopf-)unit* in $\underline{\underline{\mathcal{T}}}$ is an internal h-category in $\underline{\underline{\mathcal{T}}}$.

When $O$ is the one-point set, then an internal $A_m$ category $X$ with *strict-(or hopf-)unit* in $\underline{\underline{\mathcal{T}}}$ is called an $A_m$ space with *strict-(or hopf-)unit*. When further $m = 2$, an $A_2$ space with *strict-(or hopf-)unit* is nothing but an h-space. Let us introduce one more definition of a unit: a space $X = (X, \{e\})$ with a based multiplication $\mu : X{\times}X \to X$ is called an h-space with *h-unit*, if $\mu(x, e) \sim x \sim \mu(e, x)$, in other words, the restriction of $\mu$ to $X{\vee}X \subset X{\times}X$ is (based) homotopic to the folding map $\nabla_X : X{\vee}X \to X$. So we call $X$ an $A_m$ space with *h-unit*, if $X$ is an $A_m$ space *without unit* and, at the same time, $X$ is an h-space with *h-unit* with its $A_2$ form.

An internal $A_1$ category is nothing but an internal pre-category and an internal $A_1$ space is nothing but a based space.

**Example 5.3.** (1) *A topological group is an $A_\infty$ space with strict-unit.*
(2) *A topological monoid homotopy equivalent to a CW complex is a loop-like $A_\infty$ space with strict-unit.*
(3) *A space homotopy equivalent to an $A_m$ space with strict-(or hopf-)unit in the category of well-pointed spaces is also an $A_m$ space with strict-(or hopf-)unit.*
(4) *The loop space $\Omega X$ of any simply-connected CW complex $X$ is not actually an $A_\infty$ space with hopf-unit but an $A_\infty$ space with h-unit.*

**Theorem 5.4** (Stasheff [17])**.** *An $A_\infty$ space with strict-unit is homotopy equivalent to a topological monoid.*

5.3. $A_\infty$ **form for internal homomorphism.** We introduce a notion of an $A_m$ form ($1{\leq}m{\leq}\infty$) for an internal homomorphism in $\underline{\underline{\mathcal{T}}}$. We assume that $(X, \{a(n)\})$ and $(X', \{b(n)\})$ be internal $A_m$ categories *without unit* in $\underline{\underline{\mathcal{T}}}$, $1{\leq}m{\leq}\infty$. Let $f : X \to X'$ be an internal homomorphism in $\underline{\underline{\mathcal{T}}}$.

**Definition 5.5.** *We call $\{h(n); 1{\leq}n{\leq}m\}$ an $A_m$ form for $f$, if internal homomorphisms $h(n) : J(n) \times \prod_O^n X \to X'$ satisfy the following formulas for all $(\rho, \sigma) \in J(r){\times}K(s)$, $n = r{+}s{-}1$, $(\tau; \rho_1, \cdots, \rho_t) \in K(t) \times J(r_1){\times}\cdots{\times}J(r_t)$ and $\mathbf{x} = (x_1, \cdots, x_n) \in \prod_O^n X$:*

$$h(1) = f, \tag{5.5}$$

$$h(n)(\delta_k(\rho, \sigma); \mathbf{x}) = h(r)(\rho; a_k(s)(\sigma; \mathbf{x})), \tag{5.6}$$

$$h(n)(\delta(\tau; \rho_1, \cdots, \rho_t); \mathbf{x}) = b(t)(\tau; h(r_1, \cdots, r_t)(\rho_1, \cdots, \rho_t; \mathbf{x})), \tag{5.7}$$

*where $h(r_1, \cdots, r_t)(\rho_1, \cdots, \rho_t; \mathbf{x})$ is given by*

$$(h(r_1)(\rho_1; x_1, \cdots, x_{r_1}), \cdots, h(r_t)(\rho_t; x_{n-r_t+1}, \cdots, x_n)).$$

A internal homomorphism with the above $A_m$ form might be called an internal $A_m$ functor *disregarding unit*. When $O$ is the one-point set, then an internal $A_m$ functor *disregarding unit* may be called an $A_m$ map disregarding base-point.

5.4. **Internal $A_\infty$ functor.** We now define an internal $A_m$ functor in $\underline{\underline{\mathcal{T}}}$ for $1 \leq m \leq \infty$.

Let $(X, \{a(n)\})$ and $(X', \{b(n)\})$ be internal $A_m$ categories with *hopf-units* and let $f : X \to X'$ be an internal homomorphism with an $A_m$ form $\{h(n)\}$ for $f$ *disregarding units.*

**Definition 5.6.** *We call $f = (f, \{h(n)\})$ an "internal $A_m$ functor regarding hopf-units", if $f$ is an internal pre-functor, i.e.,*

$$f \circ \iota_X = \iota_{X'} \circ \phi \tag{5.8}$$

If an internal homomorphism is an internal $A_m$ functor regarding *hopf-unit* in $\underline{\underline{\mathcal{T}}}$ for any $m \geq 1$, then it is an internal pre-functor and is called an internal $A_\infty$ functor regarding *hopf-units*. When $m = 1$, an internal $A_1$ functor regarding *hopf-units* is nothing but an internal pre-functor in $\underline{\underline{\mathcal{T}}}$.

Let $(X, \{a(n)\})$ and $(X', \{b(n)\})$ be internal $A_m$ categories with *strict-units*, $m \geq 2$ and let $f = (f, \phi) : X = (X, O) \to (X', O') = X'$ be an internal homomorphism with an $A_m$ form $\{h(n)\}$ for $f$ *disregarding units.*

**Definition 5.7.** *We call $f = (f, \{h(n)\})$ an "internal $A_m$ functor with strict-unit", if $f$ satisfies the following condition with $h(0) = \iota_X \circ \phi$.*

$$\begin{aligned} h(n)(1_{J(n)} \times ((\textstyle\prod_O^{j-1} 1_X) \times_O \iota_X \times_O (\prod_O^{n-j} 1_X))) \\ = h(n-1)(d_j^J \times (\textstyle\prod_O^{n-1} 1_X)), \ 1 \leq j \leq n, \end{aligned} \tag{5.9}$$

If an internal homomorphism is an internal $A_m$ functor regarding *strict-unit* in $\underline{\underline{\mathcal{T}}}$ for any $m \geq 1$, then it is an internal pre-functor and is called an internal $A_\infty$ functor regarding *strict-unit* in $\underline{\underline{\mathcal{T}}}$. When $m = 1$, an internal $A_1$ functor regarding *strict-unit* in $\underline{\underline{\mathcal{T}}}$ is nothing but an internal pre-functor in $\underline{\underline{\mathcal{T}}}$.

When both $O$ and $O'$ are one-point sets, then an internal $A_m$ functor regarding *strict-(or hopf-)unit* in $\underline{\underline{\mathcal{T}}}$ is called an $A_m$ map regarding *strict-(or hopf-)unit*. When further $m = 1$, an $A_1$ map regarding *strict-(or hopf-)unit* is nothing but a map regarding base-points.

## 6. $A_\infty$ FORMS FOR ACTIONS

6.1. $A_\infty$ **form for internal action.** We introduce a notion of an $A_m$ form for an internal right (or left) pairing along an internal homomorphism in $\underline{\underline{\mathcal{T}}}$, $1 \leq m \leq \infty$. We assume that $(X, \{a(n); 1 \leq n \leq m-1\})$ $(a(1) = 1_X)$ be an internal $A_{m-1}$ category *without unit* in $\underline{\underline{\mathcal{T}}}$.

Let $\mu' : Y \times_O X \to Y$ be an internal right pairing along an internal homomorphism $p : X \to Y$ in $\underline{\underline{\mathcal{T}}}$.

**Definition 6.1.** *We call* $\{a'(n); 1{\le}n{\le}m\}$ *an* $A_m$ *form for the internal right pairing* $\mu'$ *in* $\underline{\underline{\mathcal{T}}}$*, if* $a'(n) : K(n) \times (Y \times_O \prod_O^{n-1} X) \to Y$ *satisfies the following formulas for all* $(\rho, \sigma) \in K(r){\times}K(s)$*,* $r{+}s = n{+}1$ *and* $\mathbf{x} = (y; x_2, \cdots, x_n) \in Y {\times_O} \prod_O^{n-1} X$ *with* $a'(1) = 1_Y$*.*

$$a'(2) = \mu', \tag{6.1}$$

$$a'(n)(\partial_k(\rho, \sigma); \mathbf{x}) = a'(r)(\rho; a'_k(s)(\sigma; \mathbf{x})), \tag{6.2}$$

*where* $a'_k(s)(\sigma; \mathbf{x})$ *is given by*

$$\begin{cases} (a'(s)(\sigma; y, x_2, \cdots, x_s), \cdots, x_n), & k{=}1,\ r{+}s = n{+}1, \\ (y, x_2, \cdots, a(s)(\sigma; x_k, \cdots, x_{k+s-2}), \cdots, x_n), & 1{<}k. \end{cases}$$

A pair of internal pre-categories with the above $A_m$ form for a right pairing might be called an internal (right) $A_m$ action *without unit*. When $O$ is the one-point set, an internal (right) $A_m$ action *without unit* may be called a (right) $A_m$ action *without unit*.

Let $\mu'' : X \times_O Z \to Z$ be an internal left pairing along an internal homomorphism $q : X \to Z$ in $\underline{\underline{\mathcal{T}}}$.

**Definition 6.2.** *We call* $\{a''(n); 1{\le}n{\le}m\}$ *an* $A_m$ *form for the internal left pairing* $\mu''$ *in* $\underline{\underline{\mathcal{T}}}$*, if* $a''(n) : K(n{+}1) \times (\prod_O^n X {\times_O} Z) \to Z$ *satisfies the following formulas for all* $(\rho, \sigma) \in K(r){\times}K(s)$*,* $n = r{+}s{-}1$ *and* $\mathbf{x} = (x_1, \cdots, x_n; z) \in \prod_O^n X {\times_O} Z$ *with* $a''(1) = 1_Z$*.*

$$a''(2) = \mu'', \tag{6.3}$$

$$a''(n)(\partial_k(\rho, \sigma); \mathbf{x}) = a''(r)(\rho; a''_k(s)(\sigma; \mathbf{x})), \tag{6.4}$$

*where* $a''_k(s)(\sigma; \mathbf{x})$ *is given by*

$$\begin{cases} (x_1, \cdots, a(s)(\sigma; x_k, \cdots, x_{k+s-1}), \cdots, x_n, z), & k{<}r{=}n{-}s{+}2, \\ (x_1, \cdots, a''(s)(\sigma; x_{n-s+2}, \cdots, x_n, z)), & k{=}r. \end{cases}$$

A pair of internal pre-categories with the above $A_m$ form for a left pairing might be called an internal (left) $A_m$ action *without unit*. When $O$ is the one-point set, an internal (left) $A_m$ action *without unit* may be called a (left) $A_m$ action *without unit*.

6.2. $A_\infty$ **action of an internal** $A_\infty$ **category.** Now, we define an internal $A_m$ action of an internal $A_m$ category in $\underline{\underline{\mathcal{T}}}$, $2{\le}m{\le}\infty$. We assume that $X$, $Y$ and $Z$ are internal pre-categories with the same 'object' $O$ in $\underline{\underline{\mathcal{T}}}$, and $p : X \to Y$ and $q : X \to Z$ be internal pre-functors. We also assume that $(X, \{a(n)\})$ is an internal $A_{m-1}$ category with *hopf-unit* in $\underline{\underline{\mathcal{T}}}$.

Let $(Y, X) = (Y, X; \{a'(n)\})$ be an internal right $A_m$ action of $X$ on $Y$ along an internal pre-functor $p$ *without unit* and let $(X, Z) =$

$(X, Z; \{a''(n)\})$ be an internal left $A_m$ action of $X$ on $Z$ along an internal pre-functor $q$ *without unit.*

**Definition 6.3.** (1) *We call $(Y, X)$ an "internal right $A_m$ action with hopf-unit", if $(Y, X)$ satisfies the following condition.*

$$a'(2) \text{ has axes } (1_Y, p) \tag{6.5}$$

(2) *We call $(X, Z)$ an "internal left $A_m$ action with hopf-unit", if $(X, Z)$ satisfies the following condition.*

$$a''(2) \text{ has axes } (q, 1_Z) \tag{6.6}$$

If an action of an internal $A_m$ category in $\underline{\underline{\mathcal{T}}}$ is an $A_m$ action with *hopf-unit* for any $m \geq 2$, then it is called an $A_m$ action with *hopf-unit* of an internal $A_m$ category in $\underline{\underline{\mathcal{T}}}$.

**Definition 6.4.** *Let $(Y, X) = (Y, X; \{a'(n)\})$ be an internal right $A_m$ action of $X$ on $Y$ along an internal pre-functor $p$ without unit, and let $(X, Z) = (X, Z; \{a''(n)\})$ be an internal left $A_m$ action of $X$ on $Z$ along an internal pre-functor $q$ without unit.*

(1) *We call $(Y, X)$ an "internal right $A_m$ action with strict-unit", if $(Y, X)$ satisfies the following strict-unit condition.*

$$\begin{aligned} a'(n)(1_{K(n)} \times (1_Y \times_O (\textstyle\prod_O^{j-2} 1_X) \times_O \iota_X \times_O (\prod_O^{n-j} 1_X))) \\ = a'(n-1)(d_j^K \times (1_Y \times_O (\textstyle\prod_O^{n-1} 1_X))),\ 1 < j \leq n, \end{aligned} \tag{6.7}$$

(2) *We call $(X, Z)$ an "internal left $A_m$ action with strict-unit", if $(X, Z)$ satisfies the following strict-unit condition.*

$$\begin{aligned} a''(n)(1_{K(n)} \times ((\textstyle\prod_O^{j-1} 1_X) \times_O \iota_X \times_O (\prod_O^{n-j-1} 1_X) \times_O 1_Z)) \\ = a''(n-1)(d_j^K \times ((\textstyle\prod_O^{n-1} 1_X) \times_O 1_Z)),\ 1 \leq j < n, \end{aligned} \tag{6.8}$$

If an action of an internal $A_m$ category in $\underline{\underline{\mathcal{T}}}$ is an $A_m$ action with *strict-unit* for any $m \geq 2$, then it is called an $A_m$ action with *strict-unit* of an internal $A_m$ category in $\underline{\underline{\mathcal{T}}}$. When $m = 2$, an $A_2$ action with *hopf-unit* or with *strict-unit* of an internal $A_\infty$ category in $\underline{\underline{\mathcal{T}}}$ is nothing but an action of an internal pre-category in $\underline{\underline{\mathcal{T}}}$.

6.3. $A_\infty$ **equivariant form for internal homomorphism.** We introduce a notion of an $A_m$ equivariant form for an internal homomorphism between internal $A_m$ actions *without units*, $2 \leq m \leq \infty$.

We assume that $X$, $Y$, $Z$ are internal pre-categories with an 'object' $O$, $X'$, $Y'$ and $Z'$ be internal pre-categories with another 'object' $O'$, and $q : X \to Y$, $p : X \to Z$, $q' : X' \to Y'$, $p' : X' \to Z'$, $f = (f, \phi) : X = (X, O) \to (X', O') = X'$ be internal pre-functor in $\underline{\underline{\mathcal{T}}}$.

We also assume that $(f, \{h(n)\}) : (X, \{a(n)\}) \to (X', \{b(n)\})$ is an internal $A_{m-1}$ functor *disregarding units* between internal $A_{m-1}$ categories *without units*.

Let $(Y, X; \{a'(n)\})$ and $(Y', X'; \{b'(n)\})$ be internal right $A_m$ pairings *without units* and $g : Y \to Y'$ be an internal homomorphism in $\underline{\underline{\mathcal{T}}}$.

**Definition 6.5.** *We call* $\{h'(n); 1{\le}n{\le}m\}$ *an* $A_m$ *equivariant form for* $(g,f)$, *if* $h'(n) : J(n) \times (Y{\times_O}\prod_O^{n-1} X) \to Y'$ *satisfies the following formulas for all* $(\rho,\sigma) \in J(r){\times}K(s)$, $n = r{+}s{-}1$, $(\tau;\rho_1,\cdots,\rho_t) \in K(t) \times J(r_1){\times}\cdots{\times}J(r_t)$ *and* $\mathbf{x} = (y,x_2,\cdots,x_n) \in Y{\times_O}\prod_O^{n-1} X$*:*

$$h'(1) = g,$$
$$h'(n)(\delta_k(\rho,\sigma);\mathbf{x}) = h'(r)(\rho;a'_k(s)(\sigma;\mathbf{x})),$$
$$h'(n)(\delta(\tau;\rho_1,\cdots,\rho_t);\mathbf{x}) = b'(t)(\tau;h'(r_1,\cdots,r_t)(\rho_1,\cdots,\rho_t;\mathbf{x})),$$

*where* $h'(r_1,\cdots,r_t)(\rho_1,\cdots,\rho_t;\mathbf{x})$ *is given by*

$$(h'(r_1)(\rho_1;y,x_2,\cdots,x_{r_1}),h(r_2)(\rho_2;x_{r_1+1},\cdots),\cdots,h(r_t)(\rho_t;\cdots,x_n)).$$

A pair of internal homomorphisms with the above $A_m$ equivariant form might be called an internal (right) $A_m$ equivariant functor *disregarding units.* When $O$ and $O'$ are one-point sets, an internal (right) $A_m$ equivariant functor *disregarding units* may be called a (right) $A_m$ equivariant map *disregarding units.*

Let $(X,Z;\{a''(n)\})$ and $(X',Z';\{b''(n)\})$ be internal left $A_m$ pairings *without units* and $\ell : Z \to Z'$ be an internal homomorphism in $\underline{\underline{\mathcal{T}}}$.

**Definition 6.6.** *We call* $\{h''(n); 1{\le}n{\le}m\}$ *an* $A_m$ *equivariant form for* $(\ell,f)$, *if* $h''(n) : J(n) \times (\prod_O^{n-1} X{\times_O}Z) \to Z'$ *satisfies the following formulas for all* $(\rho,\sigma) \in J(r){\times}K(s)$, $n = r{+}s{-}1$, $(\tau;\rho_1,\cdots,\rho_t) \in K(t) \times J(r_1){\times}\cdots{\times}J(r_t)$ *and* $\mathbf{x} = (x_1,\cdots,x_{n-1},z) \in \prod_O^{n-1} X{\times_O}Z$*:*

$$h''(1) = \ell,$$
$$h''(n)(\delta_k(\rho,\sigma);\mathbf{x}) = h''(r)(\rho;a''_k(s)(\sigma;\mathbf{x})),$$
$$h''(n)(\delta(\tau;\rho_1,\cdots,\rho_t);\mathbf{x}) = b''(t)(\tau;h''(r_1,\cdots,r_t)(\rho_1,\cdots,\rho_t;\mathbf{x})),$$

*where* $h''(r_1,\cdots,r_t)(\rho_1,\cdots,\rho_t;\mathbf{x})$ *is given by*

$$(h(r_1)(\rho_1;x_1,\cdots),\cdots,h(r_{t-1})(\rho_2;\cdots),h''(r_t)(\rho_t;\cdots,x_{n-1},z)).$$

A pair of internal homomorphisms with the above $A_m$ equivariant form might be called an internal $A_m$ equivariant functor *disregarding units.* When $O$ and $O'$ are one-point sets, an internal $A_m$ equivariant functor *disregarding units* may be called a $A_m$ equivariant map *disregarding units.*

6.4. **Internal $A_\infty$ equivariant functor.** Now we define an internal $A_m$ equivariant functor between internal $A_m$ actions in $\underline{\underline{\mathcal{T}}}$, $2{\le}m{\le}\infty$.

We assume that $X$, $Y$, $Z$ are internal pre-categories with an 'object' $O$, $X'$, $Y'$ and $Z'$ be internal pre-categories with another 'object' $O'$, and $q : X \to Y$, $p : X \to Z$, $q' : X' \to Y'$, $p' : X' \to Z'$, $f = (f,\phi) : X = (X,O) \to (X',O') = X'$ be internal pre-functor in $\underline{\underline{\mathcal{T}}}$.

Let $(f,\{h(n)\}) : (X,\{a(n)\}) \to (X',\{b(n)\})$ be an internal $A_{m-1}$ functor regarding *hopf-units* between internal $A_{m-1}$ categories with *hopf-units.* Let $(g,f;\{h'(n)\}) : (Y,X;\{a'(n)\}) \to (Y',X';\{b'(n)\})$ be an internal $A_m$ equivariant functor *disregarding units* between right $A_m$ actions with *hopf-units* and let $(f,\ell;\{h''(n)\}) : (X,Z;\{a''(n)\}) \to (X',Z';\{b''(n)\})$ be an internal $A_m$ equivariant functor *disregarding units* between left $A_m$ actions with *hopf-units*.

**Definition 6.7.** (1) *We call* $(g,f) = (g,f;\{h'(n)\})$ *an "internal right* $A_m$ *equivariant functor regarding hopf-units" in* $\underline{\underline{\mathcal{T}}}$*, if* $g$ *and* $f$ *are internal pre-functors, i.e.,*

$$f\circ\iota_X = \iota_{X'}\circ\phi, \quad g\circ\iota_Y = \iota_{Y'}\circ\phi$$

(2) *We call* $(f,\ell) = (f,\ell;\{h''(n)\})$ *an "internal left* $A_m$ *equivariant functor regarding hopf-units" in* $\underline{\underline{\mathcal{T}}}$*, if* $\ell$ *and* $f$ *are internal pre-functors, i.e.,*

$$f\circ\iota_X = \iota_{X'}\circ\phi, \quad \ell\circ\iota_Z = \iota_{Z'}\circ\phi$$

If an internal pre-functor is a right (or left) $A_m$ equivariant functor regarding *hopf-units* for any $m \geq 1$, then it is called a right (or left) $A_\infty$ equivariant regarding *hopf-units*.

Let $(g,f;\{h'(n)\}) : (Y,X;\{a'(n)\}) \to (Y',X';\{b'(n)\})$ be an internal $A_m$ equivariant functor *disregarding units* between right $A_m$ actions with *strict-units* and let $(f,\ell;\{h''(n)\}) : (X,Z;\{a''(n)\}) \to (X',Z';\{b''(n)\})$ be an internal $A_m$ equivariant functor *disregarding units* between left $A_m$ actions with *strict-units*.

**Definition 6.8.** (1) *We call* $(g,f) = (g,f;\{h'(n)\})$ *an "internal right* $A_m$ *equivariant functor regarding strict-units", if* $g$ *and* $f$ *are internal pre-functors, i.e.,*

$$\begin{aligned} &h'(n)(1_{J(n)}\times(1_Y\times_O(\textstyle\prod_O^{j-2} 1_X)\times_O\iota_X\times_O(\prod_O^{n-j} 1_X))) \\ &\quad = h'(n-1)(d_j^J\times(1_Y\times_O(\textstyle\prod_O^{n-1} 1_X))), \ 1<j\leq n, \end{aligned}$$

(2) *We call* $(f,\ell) = (f,\ell;\{h''(n)\})$ *an "internal left* $A_m$ *equivariant functor regarding strict-units", if* $\ell$ *and* $f$ *are internal pre-functors, i.e.,*

$$\begin{aligned} &h''(n)(1_{J(n)}\times((\textstyle\prod_O^{j-1} 1_X)\times_O\iota_X\times_O(\prod_O^{n-j-1} 1_X)\times_O 1_Z)) \\ &\quad = h''(n-1)(d_j^J\times((\textstyle\prod_O^{n-1} 1_X)\times_O 1_Z)), \ 1\leq j<n, \end{aligned}$$

If an internal pre-functor is a right (or left) $A_m$ equivariant functor regarding *strict-units* for any $m \geq 1$, then it is called a right (or left) $A_\infty$ equivariant regarding *strict-units*.

## 7. $A_\infty$ OPERADIC CATEGORIES

7.1. **$A_\infty$ operadic categories for objects.** We introduce $A_\infty$ operadic categories for objects as small enriched categories.

Now, we define a $\underline{\underline{\mathcal{T}}}$-enriched small category as follows:

**Definition 7.1.** *Let $\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}})$ be the $\underline{\underline{\mathcal{T}}}$-enriched small category consisting of $\mathcal{O}(\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}}))$ the set of objects and $\mathcal{M}(\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}}))$ the set of morphisms:*

**(objects):** $\mathcal{O}(\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}})) = \{\underline{0}, \underline{1}, \underline{2}, \cdots\} = \bar{\mathbb{N}}$ *the set of non-negative integers,*

**(morphisms):** *For any two non-negative integers $m$ and $n$,*

$$\mathcal{M}(\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m}, \underline{n}) = \coprod_{\substack{1 \le a_0, \cdots, a_{m+1} \\ a_0 + \cdots + a_{m+1} = n+2}} K(a_0) \times \cdots \times K(a_{m+1}).$$

*For any $(\rho_0, \cdots, \rho_{\ell+1}) : \underline{\ell} \to \underline{m}$, $\rho_i \in K(r_i)$, $r_0 + \cdots + r_{\ell+1} = m+2$ and $(\sigma_0, \cdots, \sigma_{m+1}) : \underline{m} \to \underline{n}$, $\sigma_j \in K(a_j)$, $a_0 + \cdots + a_{m+1} = n+2$, the composition*

$$(\tau_0, \cdots, \tau_{\ell+1}) = (\sigma_0, \cdots, \sigma_{m+1}) \circ (\rho_0, \cdots, \rho_{\ell+1})$$

*is given by*

$$\tau_i = \partial_1(\sigma_0^{(i)}) \circ \partial_2(\sigma_1^{(i)}) \circ \cdots \circ \partial_{r_i}(\sigma_{r_i}^{(i)})(\rho_i),$$

*where $\sigma_j^{(i)} = \sigma_{r_0 + \cdots + r_{i-1} + j}$, $a_j^{(i)} = a_{r_0 + \cdots + r_{i-1} + j}$ for $1 \le j \le r_i$.*

Truncating the $\underline{\underline{\mathcal{T}}}$-enriched small category $\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}})$, we obtain a series of $\underline{\underline{\mathcal{T}}}$-enriched small categories as follows.

**Definition 7.2.** *For each $m \ge 0$, we define a $\underline{\underline{\mathcal{T}}}$-enriched small category $\underline{\underline{\mathcal{K}}}_m(\underline{\underline{\mathcal{T}}})$ as the full-subcategory of $\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}})$, whose set of objects is $\{\underline{0}, \underline{1}, \cdots, \underline{m}\} \approx \{0, 1, \cdots, m\}$.*

7.2. $A_\infty$ **operadic categories for morphisms.** We also introduce $A_\infty$ operadic categories for morphisms as small enriched categories.

Now, we define a $\underline{\underline{\mathcal{T}}}$-enriched small category as follows:

**Definition 7.3.** *Let $\underline{\underline{\mathcal{J}}}_\infty(\underline{\underline{\mathcal{T}}})$ be the $\underline{\underline{\mathcal{T}}}$-enriched small category consisting of $\mathcal{O}(\underline{\underline{\mathcal{J}}}_\infty(\underline{\underline{\mathcal{T}}}))$ the set of objects and $\mathcal{M}(\underline{\underline{\mathcal{J}}}_\infty(\underline{\underline{\mathcal{T}}}))$ the set of morphisms:*

**(objects):** $\mathcal{O}(\underline{\underline{\mathcal{J}}}_\infty(\underline{\underline{\mathcal{T}}})) = \{\underline{0}, \underline{0}', \underline{1}, \underline{1}', \underline{2}, \underline{2}', \cdots\} \approx \bar{\mathbb{N}} \times C_2,$

**(morphisms):** *For any two non-negative integers $m$ and $n$,*

$$\mathcal{M}(\underline{\underline{\mathcal{J}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m}, \underline{n}) = \mathcal{M}(\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m}, \underline{n}),$$

$$\mathcal{M}(\underline{\underline{\mathcal{J}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m}', \underline{n}') = \mathcal{M}(\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m}', \underline{n}'),$$

$$\mathcal{M}(\underline{\underline{\mathcal{J}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m}', \underline{n}) = \emptyset \text{ and}$$

$$\mathcal{M}(\underline{\underline{\mathcal{J}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m}, \underline{n}') = \coprod_{\substack{1 \le a_0, \cdots, a_{m+1} \le n+2 \\ a_0 + \cdots + a_{m+1} = n+2}} J(a_0) \times \cdots \times J(a_{m+1})$$

*with the following relations:*

(1) *For any $(\rho_0, \cdots, \rho_{\ell+1}) : \underline{\ell} \to \underline{m}'$, $\rho_i \in J(r_i)$ ($r_0 + \cdots + r_{\ell+1} = m+2$) and $(\sigma_0, \cdots, \sigma_{m+1}) : \underline{m}' \to \underline{n}'$ with $\sigma_j \in K(a_j)$ ($a_0 + \cdots + a_{m+1} = n+2$), the composition*

$$(\tau_0, \cdots, \tau_{\ell+1}) = (\sigma_0, \cdots, \sigma_{m+1}) \circ (\rho_0, \cdots, \rho_{\ell+1})$$

*is given by*

$\tau_i = \delta_1(\sigma_0^{(i)})\circ\cdots\circ\delta_{r_i+1}(\sigma_{r_i}^{(i)})(\rho_i)$,
*where* $\sigma_j^{(i)} = \sigma_{r_0+\cdots+r_{i-1}+j}$, $a_j^{(i)} = a_{r_0+\cdots+r_{i-1}+j}$ *for* $1\le j\le r_i$.

(2) *For any* $(\rho_0,\cdots,\rho_{\ell+1}) : \underline{\ell} \to \underline{m}$, $\rho_i \in K(r_i)$ $(r_0+\cdots+r_{\ell+1} = m+2)$ *and* $(\sigma_0,\cdots,\sigma_{m+1}) : \underline{m} \to \underline{n}'$ *with* $\sigma_j \in J(a_j)$ $(a_0+\cdots+a_{m+1} = n+2)$*, the composition*
$(\tau_0,\cdots,\tau_{\ell+1}) = (\sigma_0,\cdots,\sigma_{m+1})\circ(\rho_0,\cdots,\rho_{\ell+1})$
*is given by*
$\tau_i = \delta(\sigma_0^{(i)},\cdots,\sigma_{r_i}^{(i)})(\rho_i)$,
*where* $\sigma_j^{(i)} = \sigma_{r_0+\cdots+r_{i-1}+j}$, $a_j^{(i)} = a_{r_0+\cdots+r_{i-1}+j}$ *for* $1\le j\le r_i$.

**Remark 7.4.** *There are two inclusion functors* $Bj, Bj' : \underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}}) \to \underline{\underline{\mathcal{J}}}_\infty(\underline{\underline{\mathcal{T}}})$ *between* $\underline{\underline{\mathcal{T}}}$*-enriched small categories determined by*

$$Bj(\underline{n}) = \underline{n},\ \text{and } Bj'(\underline{n}) = \underline{n}',$$

Truncating the $\underline{\underline{\mathcal{T}}}$-enriched small category $\underline{\underline{\mathcal{J}}}_\infty(\underline{\underline{\mathcal{T}}})$, we obtain a series of $\underline{\underline{\mathcal{T}}}$-enriched small categories as follows.

**Definition 7.5.** *For each* $m \ge 0$*, we define the* $\underline{\underline{\mathcal{T}}}$*-enriched small category* $\underline{\underline{\mathcal{J}}}_m(\underline{\underline{\mathcal{T}}})$ *as the full-subcategory of* $\underline{\underline{\mathcal{J}}}_\infty(\underline{\underline{\mathcal{T}}})$*, whose set of objects is* $\{\underline{0},\underline{0}',\underline{1},\underline{1}',\cdots,\underline{m},\underline{m}'\} \approx \{0,1,\cdots,m\}\times C_2$.

**Remark 7.6.** *By restricting* $Bj$ *and* $Bj'$*, we obtain two inclusion functors* $Bj_m, Bj'_m : \underline{\underline{\mathcal{K}}}_m(\underline{\underline{\mathcal{T}}}) \to \underline{\underline{\mathcal{J}}}_m(\underline{\underline{\mathcal{T}}})$ *are obtained.*

7.3. $A_\infty$ **operadic categories with units for objects.** Now, we define a $\underline{\underline{\mathcal{T}}}$-enriched small category as follows.

**Definition 7.7.** *Let* $\underline{\underline{\widetilde{\mathcal{K}}}}_\infty(\underline{\underline{\mathcal{T}}})$ *be the* $\underline{\underline{\mathcal{T}}}$*-enriched small category consisting of* $\mathcal{O}(\underline{\underline{\widetilde{\mathcal{K}}}}_\infty(\underline{\underline{\mathcal{T}}}))$ *the set of objects and* $\mathcal{M}(\underline{\underline{\widetilde{\mathcal{K}}}}_\infty(\underline{\underline{\mathcal{T}}}))$ *the set of morphisms:*

**(objects):** $\mathcal{O}(\underline{\underline{\widetilde{\mathcal{K}}}}_\infty(\underline{\underline{\mathcal{T}}})) = \{\underline{0},\underline{1},\underline{2},\underline{3},\cdots\} = \bar{\mathbb{N}}$ *the set of all non-negative integers,*

**(morphisms):** *For any two non-negative integers* $m$ *and* $n$,

$$\mathcal{M}(\underline{\underline{\widetilde{\mathcal{K}}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m},\underline{n}) = \coprod_{\substack{1\le i_1<\cdots<i_{m-\ell}\le m\\ 1\le\ell\le m}} \mathcal{M}(\underline{\underline{\widetilde{\mathcal{K}}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{\ell},\underline{n})\times\{(i_1,\cdots,i_{m-\ell})\}$$

*is the set of all formal compositions of elements of the finite set*
$\{(i_1,\cdots,i_{m-\ell}) \mid 1\le i_1<\cdots<i_{m-\ell}\le m\}$
*followed by elements of the topological space*

$$\mathcal{M}(\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{\ell},\underline{n}) = \coprod_{\substack{1\le a_0,\cdots,a_{\ell+1}\\ a_0+\cdots+a_{\ell+1}=n+2}} K(a_0)\times\cdots\times K(a_{\ell+1})$$

*for some* $\ell$*, with the following additional composition formulas.*

(1) *For any* $(i_1,\cdots,i_{m-\ell}) : \underline{m} \to \underline{\ell}$ *with* $1\le i_1<\cdots<i_{m-\ell}\le m$, $\ell < m$ *and* $(j_1,\cdots,j_{n-m}) : \underline{n} \to \underline{m}$ *with* $1\le j_1<\cdots<j_{n-m}\le n$, $m < n$*, the composition*

$(k_1,\cdots,k_{n-\ell}) = (i_1,\cdots,i_{m-\ell})\circ(j_1,\cdots,j_{n-m})$
*is given by*
$\{k_1,\cdots,k_{n-\ell}\} = \{j_1,\cdots,j_{n-m}\} \cup \{i'_1,\cdots,i'_{m-\ell}\}$,
*where* $i'_a - i_a = b$ *is determined by* $j_b{-}b{+}1 \le i_a < j_{b+1}{-}b$ *for each* $a$.

(2) *For any* $(i) : \underline{n} \to \underline{n{-}1}$, $1 \le i \le n$ *and* $(\tau_0,\cdots,\tau_{m+1}) : \underline{m} \to \underline{n}$ *with* $\tau_j \in K(r_j{+}1)$, $r_0{+}\cdots{+}r_{m+1} = n{-}m$, *the following equation holds.*

$$(i)\circ(\tau_0,\cdots,\tau_{m+1}) = \begin{cases} (\tau_0,\cdots,d^K_{i'}(\tau_j),\cdots,\tau_{m+1}), & r_j > 1, \\ (\tau_0,\cdots,\tau_{j-1},\tau_{j+1},\cdots,\tau_{m+1})\circ(j), & r_j = 1, \end{cases}$$

*where* $i'$ *and* $j$ *are determined by* $i{+}1 = r_0{+}\cdots{+}r_{j-1}{+}i'$, $1 \le i' \le r_j$.

7.4. $A_\infty$ **operadic categories with units for morphisms.** To follow the original definitions due to Stasheff, we define here some small categories provided from the topological $A_\infty$ operads with degeneracies for objects.

Now, we define a $\underline{\underline{\mathcal{T}}}$-enriched small category as follows.

**Definition 7.8.** *Let* $\widetilde{\underline{\underline{\mathcal{J}}}}_\infty(\underline{\underline{\mathcal{T}}})$ *be the* $\underline{\underline{\mathcal{T}}}$*-enriched small category consisting of* $\mathcal{O}(\widetilde{\underline{\underline{\mathcal{J}}}}_\infty(\underline{\underline{\mathcal{T}}}))$ *the set of objects and* $\mathcal{M}(\widetilde{\underline{\underline{\mathcal{J}}}}_\infty(\underline{\underline{\mathcal{T}}}))$ *the set of morphisms:*

**(objects):** $\mathcal{O}(\widetilde{\underline{\underline{\mathcal{J}}}}_\infty(\underline{\underline{\mathcal{T}}})) = \{\underline{0},\underline{0}',\underline{1},\underline{1}',\underline{2},\underline{2}',\cdots\} \approx \bar{\mathbb{N}} \times C_2$,

**(morphisms):** *For any two non-negative integers* $m$ *and* $n$,

$\mathcal{M}(\widetilde{\underline{\underline{\mathcal{J}}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m},\underline{n}) = \mathcal{M}(\widetilde{\underline{\underline{\mathcal{K}}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m},\underline{n})$,
$\mathcal{M}(\widetilde{\underline{\underline{\mathcal{J}}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m}',\underline{n}') = \mathcal{M}(\widetilde{\underline{\underline{\mathcal{K}}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m}',\underline{n}')$,
$\mathcal{M}(\widetilde{\underline{\underline{\mathcal{J}}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m}',\underline{n}) = \emptyset$ *and*

$$\mathcal{M}(\widetilde{\underline{\underline{\mathcal{J}}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{m},\underline{n}') = \coprod_{\substack{1\le i_1<\cdots<i_{m-\ell}\le m \\ 1\le \ell\le m}} \mathcal{M}(\widetilde{\underline{\underline{\mathcal{J}}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{\ell},\underline{n}')\times\{(i_1,\cdots,i_{m-\ell})\}$$

*is the set of all formal compositions of elements of the finite set* $\{(i_1,\cdots,i_{m-\ell}) \,|\, 1{\le}i_1{<}\cdots{<}i_{m-\ell}{\le}m\}$ *followed by elements of the coalgebra*

$$\mathcal{M}(\widetilde{\underline{\underline{\mathcal{J}}}}_\infty(\underline{\underline{\mathcal{T}}}))(\underline{\ell},\underline{n}') = \coprod_{\substack{1\le a_0,\cdots,a_{\ell+1} \\ a_0+\cdots+a_{\ell+1}=n+2}} J(a_0)\times\cdots\times J(a_{\ell+1}),$$

*for some* $\ell$, *with the following additional composition formulas.*

(1) *For any* $(i) : \underline{n}' \to \underline{n{-}1}'$, $1 \le i \le n$ *and* $(\rho_0,\cdots,\rho_{m+1}) : \underline{m} \to \underline{n}'$ *with* $\rho_j \in J(a_j)$, $a_0{+}\cdots{+}a_{m+1} = n{+}2$, *the following equation holds.*

$(i)\circ(\rho_0,\cdots,\rho_{m+1})$

$$= \begin{cases} (\rho_0, \cdots, d^J_{i'}(\rho_j), \cdots, \rho_{m+1}), \ a_j > 1, \\ (\rho_0, \cdots, \rho_{j-1}, \rho_{j+1}, \cdots, \rho_{m+1}) \circ (j), \ a_j = 1, \end{cases}$$

*where $i'$ and $j$ are determined by $i+1 = a_0 + \cdots + a_{j-1} + i'$, $1 \leq i' \leq a_j$.*

## 8. Bar construction of an internal $A_\infty$ category

Let $\underline{\underline{\mathcal{C}}}$ be a monoidal category $\underline{\underline{\mathcal{C}}}$ by a tensor product $\otimes : \underline{\underline{\mathcal{C}}} \times \underline{\underline{\mathcal{C}}} \to \underline{\underline{\mathcal{C}}}$ with unit object 1.

**Definition 8.1.** *An object $O$ in $\underline{\underline{\mathcal{C}}}$ is called 'flat', if $\lim_\lambda (O \otimes A_\lambda) = O \otimes (\lim_\lambda A_\lambda)$ and $\lim_\lambda (A_\lambda \otimes O) = (\lim_\lambda A_\lambda) \otimes O$. Further we say that $\underline{\underline{\mathcal{C}}}$ is 'regular', if every object $O$ in $\underline{\underline{\mathcal{C}}}$ is flat.*

8.1. **Representations of an enriched category.** A category $\underline{\underline{\mathcal{D}}}$ is called a $\underline{\underline{\mathcal{C}}}$-enriched category, if the set of morphisms $\underline{\underline{\mathcal{D}}}(A, B)$ is an object of $\underline{\underline{\mathcal{C}}}$ for any two objects $A, B \in \mathcal{O}(\underline{\underline{\mathcal{D}}})$ such that $\underline{\underline{\mathcal{D}}}(-,-)$ gives a functor from $\underline{\underline{\mathcal{D}}}^{op} \times \underline{\underline{\mathcal{D}}}$ to $\underline{\underline{\mathcal{C}}}$. If a small category $\underline{\underline{\mathcal{D}}}$ is a $\underline{\underline{\mathcal{C}}}$-enriched category, then $\underline{\underline{\mathcal{D}}}$ is called a $\underline{\underline{\mathcal{C}}}$-enriched small category. Let us paraphrase the word '$\underline{\underline{\mathcal{C}}}$-enriched' by 'topological', if $\underline{\underline{\mathcal{C}}} = \underline{\underline{\mathcal{T}}}$.

Because it is technically difficult to treat a functor between enriched categories, we introduce here a notion of a representation of a $\underline{\underline{\mathcal{C}}}$-enriched small category in $\underline{\underline{\mathcal{C}}}$. For a $\underline{\underline{\mathcal{C}}}$-enriched category $\underline{\underline{\mathcal{D}}}$, a left representation $\Phi$ of $\underline{\underline{\mathcal{D}}}$ in $\underline{\underline{\mathcal{C}}}$ is a pair $(\mathcal{O}(\Phi), \mathcal{M}(\Phi))$ of correspondences satisfying the following conditions and is denoted by $\Phi : \underline{\underline{\mathcal{D}}} \rightharpoonup \underline{\underline{\mathcal{C}}}$:

(1) $\mathcal{O}(\Phi) : \mathcal{O}(\underline{\underline{\mathcal{D}}}) \to \mathcal{O}(\underline{\underline{\mathcal{C}}})$ and
(2) $\mathcal{M}(\Phi) : \mathcal{O}(\underline{\underline{\mathcal{D}}}) \otimes \mathcal{O}(\underline{\underline{\mathcal{D}}}) \to \mathcal{M}(\underline{\underline{\mathcal{C}}})$ such that
  a) for any $\underline{a}, \underline{b}$ in $\mathcal{O}(\underline{\underline{\mathcal{D}}})$, $\mathcal{M}(\Phi)(\underline{a}, \underline{b}) : \mathcal{M}(\underline{\underline{\mathcal{D}}})(\underline{a}, \underline{b}) \otimes \mathcal{O}(\Phi)(\underline{a}) \to \mathcal{O}(\Phi)(\underline{b})$,
  b) for any $\underline{a}$ in $\mathcal{O}(\underline{\underline{\mathcal{D}}})$ and $x \in \mathcal{O}(\Phi)(\underline{a})$, $\mathcal{M}(\Phi)(\underline{a}, \underline{a})(1_a, x) = x$ and
  c) for any $\underline{a}$, $\underline{b}$, $\underline{c}$ in $\mathcal{O}(\underline{\underline{\mathcal{D}}})$ and any $x \in \mathcal{O}(\Phi)(\underline{a})$,
  $\mathcal{M}(\Phi)(\underline{b}, \underline{c})(\beta, \mathcal{M}(\Phi)(\underline{a}, \underline{b})(\alpha, x)) = \mathcal{M}(\Phi)(\underline{a}, \underline{c})(\beta \circ \alpha, x)$.

A right representation $\Psi$ of $\underline{\underline{\mathcal{D}}}$ in $\underline{\underline{\mathcal{C}}}$ is a pair $(\mathcal{O}(\Psi), \mathcal{M}(\Psi))$ of correspondences satisfying the following conditions and is denoted by $\Psi : \underline{\underline{\mathcal{D}}} \rightharpoondown \underline{\underline{\mathcal{C}}}$:

(1) $\mathcal{O}(\Psi) : \mathcal{O}(\underline{\underline{\mathcal{D}}}) \to \mathcal{O}(\underline{\underline{\mathcal{C}}})$ and
(2) $\mathcal{M}(\Psi) : \mathcal{O}(\underline{\underline{\mathcal{D}}}) \otimes \mathcal{O}(\underline{\underline{\mathcal{D}}}) \to \mathcal{M}(\underline{\underline{\mathcal{C}}})$ such that
  a) for any $\underline{a}, \underline{b}$ in $\mathcal{O}(\underline{\underline{\mathcal{D}}})$, $\mathcal{M}(\Psi)(\underline{a}, \underline{b}) : \mathcal{O}(\Psi)(\underline{b}) \otimes \mathcal{M}(\underline{\underline{\mathcal{D}}})(\underline{a}, \underline{b}) \to \mathcal{O}(\Psi)(\underline{a})$,
  b) for any $\underline{a}$ in $\mathcal{O}(\underline{\underline{\mathcal{D}}})$ and $x \in \mathcal{O}(\Psi)(\underline{a})$, $\mathcal{M}(\Psi)(\underline{a}, \underline{a})(x, 1_a) = x$ and
  c) for any $\underline{a}$, $\underline{b}$, $\underline{c}$ in $\mathcal{O}(\underline{\underline{\mathcal{D}}})$ and any $x \in \mathcal{O}(\Psi)(\underline{a})$,
  $\mathcal{M}(\Psi)(\underline{b}, \underline{c})(\mathcal{M}(\Psi)(\underline{a}, \underline{b})(x, \alpha), \beta) = \mathcal{M}(\Psi)(\underline{a}, \underline{c})(x, \alpha \circ \beta)$.

**Remark 8.2.** *If the regular bimonoidal category $\underline{\underline{\mathcal{C}}}$ is self-enriched to be a closed monoidal category by tensor product, then a left (or right) representation is nothing but a covariant (or contravariant) functor.*

We give some examples of left representations as follows.

**Examples 8.3.** *The followings are canonical left representations of $A_\infty$ operadic categories in $\underline{\underline{\mathcal{T}}}$.*

(1) *Let $\overline{K}$ be the following representation of $\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}})$:*

$$\overline{K}(\underline{n}) = K(n{+}2),$$
$$\overline{K}(\underline{m},\underline{n})(\tau_0,\cdots,\tau_{m+1};\sigma) = \partial_1(\tau_0)\circ\cdots\circ\partial_{m+2}(\tau_{m+1})(\sigma).$$

(2) *Let $\overline{J}_0$ be the following representation of $\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}})$:*

$$\overline{J}_0(\underline{n}) = J_0(n{+}2),$$
$$\overline{J}_0(\underline{m},\underline{n})(\tau_0,\cdots,\tau_{m+1};\sigma) = \delta_1(\tau_0)\circ\cdots\circ\delta_{m+2}(\tau_{m+1})(\sigma).$$

(3) *Let $\overline{J}$ be the following representation of $\underline{\underline{\mathcal{J}}}_\infty(\underline{\underline{\mathcal{T}}})$:*

$$\overline{J}(\underline{n}) = K(n{+}2), \quad \overline{J}(\underline{n}') = J(n{+}2),$$
$$\begin{cases} \overline{J}(\underline{m},\underline{n})(\tau_0,\cdots,\tau_{m+1};\sigma) = \partial_1(\tau_0)\circ\cdots\circ\partial_{m+2}(\tau_{m+1})(\sigma), \\ \overline{J}(\underline{m},\underline{n}')(\rho_0,\cdots,\rho_{m+1};\tau) = \delta(\tau;\rho_0,\cdots,\rho_{m+1}), \\ \overline{J}(\underline{m}',\underline{n}')(\tau_0,\cdots,\tau_{m+1};\rho) = \delta_1(\tau_0)\circ\cdots\circ\delta_{m+2}(\tau_{m+1})(\rho). \end{cases}$$

*where $\partial_k(\tau)(\rho) = \partial_k(\rho,\tau)$ and $\delta_k(\tau)(\rho) = \delta_k(\rho,\tau)$.*

Similarly, we obtain the following examples.

**Examples 8.4.** *The followings are canonical left representations of $A_\infty$ operadic categories with degeneracies in $\underline{\underline{\mathcal{T}}}$.*

(1) *Let $\overline{\widetilde{K}}$ be the following representation of $\underline{\underline{\widetilde{\mathcal{K}}}}_\infty(\underline{\underline{\mathcal{T}}})$:*

$$\overline{\widetilde{K}}(\underline{n}) = K(n{+}2),$$
$$\begin{cases} \overline{\widetilde{K}}(\underline{m},\underline{n})(\tau_0,\cdots,\tau_{m+1};\sigma) = \partial_1(\tau_0)\circ\cdots\circ\partial_{m+2}(\tau_{m+1})(\sigma), \\ \overline{\widetilde{K}}(\underline{m},\underline{\ell})(j_1,\cdots,j_{m-\ell};\sigma) = d^K_{j_1+1}\circ\cdots\circ d^K_{j_{m-\ell}+1}(\sigma), \end{cases}$$

*for $\ell\le m\le n$.*

(2) *Let $\overline{\widetilde{J}}_0$ be the following representation of $\underline{\underline{\widetilde{\mathcal{K}}}}_\infty(\underline{\underline{\mathcal{T}}})$:*

$$\overline{\widetilde{J}}_0(\underline{n}) = J_0(n{+}2),$$
$$\begin{cases} \overline{\widetilde{J}}_0(\underline{m},\underline{n})(\tau_0,\cdots,\tau_{m+1};\sigma) = \delta_1(\tau_0)\circ\cdots\circ\delta_{m+2}(\tau_{m+1})(\sigma), \\ \overline{\widetilde{J}}_0(\underline{m},\underline{\ell})(j_1,\cdots,j_{m-\ell};\sigma) = d^J_{j_1+1}\circ\cdots\circ d^J_{j_{m-\ell}+1}(\sigma), \end{cases}$$

*for $\ell\le m\le n$.*

(3) *Let* $\overline{\widetilde{J}}$ *be the following representation of* $\underline{\underline{\widetilde{\mathcal{J}}}}_\infty(\underline{\underline{\mathcal{T}}})$*:*

$$\overline{\widetilde{J}}(\underline{n}) = K(n{+}2), \quad \overline{\widetilde{J}}(\underline{n}') = J(n{+}2),$$

$$\begin{cases} \overline{\widetilde{J}}(\underline{m},\underline{n})(\tau_0,\cdots,\tau_{m+1};\sigma) = \partial_1(\tau_0)\circ\cdots\circ\partial_{m+2}(\tau_{m+1})(\sigma), \\ \overline{\widetilde{J}}(\underline{m},\underline{n}')(\rho_0,\cdots,\rho_{m+1};\tau) = \delta(\tau;\rho_0,\cdots,\rho_{m+1}), \\ \overline{\widetilde{J}}(\underline{m}',\underline{n}')(\tau_0,\cdots,\tau_{m+1};\rho) = \delta_1(\tau_0)\circ\cdots\circ\delta_{m+2}(\tau_{m+1})(\rho), \\ \overline{\widetilde{J}}(\underline{m},\underline{\ell})(j_1,\cdots,j_{m-\ell};\rho) = d^J_{j_1+1}\circ\cdots\circ d^J_{j_{m-\ell}+1}(\rho), \\ \overline{\widetilde{J}}(\underline{m}',\underline{\ell}')(j_1,\cdots,j_{m-\ell};\rho) = d^J_{j_1+1}\circ\cdots\circ d^J_{j_{m-\ell}+1}(\rho), \end{cases}$$

*for* $\ell \le m \le n$.

8.2. **Hom and tensor of representations.** In this section, we introduce two natural constructions of an object from two representations.

Firstly we introduce a natural hom set of two left representations. Let $\Phi, \Phi' : \underline{\underline{\mathcal{D}}} \rightharpoonup \underline{\underline{\mathcal{C}}}$ be two left representations of a $\underline{\underline{\mathcal{C}}}$-enriched small category $\underline{\underline{\mathcal{D}}}$ in a regular monoidal category $\underline{\underline{\mathcal{C}}}$.

**Definition 8.5.** *We define* $\mathrm{Hom}_{\underline{\underline{\mathcal{D}}}}(\Phi,\Phi')$ *the set of natural homomorphisms between two left representations* $\Phi$ *and* $\Phi'$ *over* $\underline{\underline{\mathcal{D}}}$*, which consists of a family of maps* $\{f_X \,|\, X \in \mathcal{O}(\underline{\underline{\mathcal{D}}})\}$ *in the category* $\underline{\underline{\mathcal{C}}}$*, such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{M}(\underline{\underline{\mathcal{D}}})(A,B)\otimes\mathcal{O}(\Phi)(B) & \xrightarrow{\mathcal{M}(\Phi)} & \mathcal{O}(\Phi)(A) \\ \Big\downarrow{\scriptstyle 1\otimes f_B} & & \Big\downarrow{\scriptstyle f_A} \\ \mathcal{M}(\underline{\underline{\mathcal{D}}})(A,B)\otimes\mathcal{O}(\Phi')(B) & \xrightarrow{\mathcal{M}(\Phi')} & \mathcal{O}(\Phi')(A), \end{array}$$

*where* $A$ *and* $B$ *run over the set of objects* $\mathcal{O}(\underline{\underline{\mathcal{D}}})$ *of the category* $\underline{\underline{\mathcal{D}}}$*.*

A natural homomorphism between two right representations $\Psi$ and $\Psi'$ is defined similarly, and we also obtain $\mathrm{Hom}_{\underline{\underline{\mathcal{D}}}}(\Psi,\Psi')$ the set of natural homomorphisms between $\Psi$ and $\Psi'$.

Secondly we introduce a tensor product of left and right representations. Since the ordinary bar construction of a group can be regarded by co-equalizer, we also use a co-equalizer here. Let $\Phi : \underline{\underline{\mathcal{D}}} \rightharpoonup \underline{\underline{\mathcal{C}}}$ and $\Psi : \underline{\underline{\mathcal{D}}} \rightharpoondown \underline{\underline{\mathcal{C}}}$ be left and right representations of a $\underline{\underline{\mathcal{C}}}$-enriched small category $\underline{\underline{\mathcal{D}}}$ in a regular monoidal category $\underline{\underline{\mathcal{C}}}$.

**Definition 8.6.** *Let the tensor product* $\Psi\otimes_{\underline{\underline{\mathcal{D}}}}\Phi$ *of two representations* $\Psi$ *and* $\Phi$ *over* $\underline{\underline{\mathcal{D}}}$ *be the co-equalizer of the following morphisms in* $\underline{\underline{\mathcal{C}}}$*:*

- $\displaystyle\bigoplus_{A,B} \mathcal{O}(\Psi)(A)\otimes\mathcal{M}(\underline{\underline{\mathcal{D}}})(A,B)\otimes\mathcal{O}(\Phi)(B)$

$$\xrightarrow{\bigoplus_{A,B} 1\otimes\mathcal{M}(\Phi)(A,B)} \bigoplus_{A} \mathcal{O}(\Psi)(A)\otimes\mathcal{O}(\Phi)(A),$$

- $\bigoplus_{A,B} \mathcal{O}(\Psi)(A)\otimes\mathcal{M}(\underline{\underline{\mathcal{D}}})(A,B)\otimes\mathcal{O}(\Phi)(B)$
$$\xrightarrow{\bigoplus_{A,B} \mathcal{M}(\Psi)(A,B)\otimes 1} \bigoplus_{B} \mathcal{O}(\Psi)(B)\otimes\mathcal{O}(\Phi)(B),$$

*where $A$ and $B$ run over the set of objects $\mathcal{O}(\underline{\underline{\mathcal{D}}})$ of the category $\underline{\underline{\mathcal{D}}}$.*

8.3. **Two-sided bar construction of an internal $A_\infty$ category with *hopf-unit*.** We define two-sided bar construction with *hopf-unit* using co-equalizer in $\underline{\underline{\mathcal{T}}}$.

Let $X = (X, \{a(n)\})$ be an internal $A_\infty$ category with *hopf-unit* in $\underline{\underline{\mathcal{T}}}$ with a right $A_\infty$ action $(Y, X; \{a'(n)\})$ with *hopf-unit* of $X$ on $Y$ and a left $A_\infty$ action $(X, Z; \{a''(n)\})$ with *hopf-unit* of $X$ on $Z$ in $\underline{\underline{\mathcal{T}}}$.

**Definition 8.7.** *The left and right $A_\infty$ actions of $X$ on $Y$ and $Z$ induces a right representation $\underline{B(Y,X,Z)}$ of the $A_\infty$ operadic category $\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}})$ defined for $n \geq 0$, $\tau_k \in K(t_k)$, $0 \leq k \leq m{+}1$ and $(y, x_1, \cdots, x_m, z) \in Y\times_O(\prod_O^m X)\times_O Z$ as follows:*

$$\underline{B(Y,X,Z)}(\underline{n}) = Y\times_O(\prod_O^n X)\times_O Z,$$
$$\underline{B(Y,X,Z)}(\underline{m},\underline{n})(\tau_0,\cdots,\tau_{m+1}; y, x_1,\cdots,x_n, z)$$
$$= (a'(\tau_0), a(\tau_1),\cdots, a(\tau_m), a''(\tau_{m+1})),$$

*where $a'(\tau_0)$, $a(\tau_k)$ $(1 \leq k \leq m)$, $a''(\tau_{m+1})$ are given by*

$$a'(\tau_0) = a'(t_0)(\tau_0; y, x_1,\cdots,x_{t_0-1}),$$
$$a(\tau_k) = a(t_k)(\tau_k; x_{t_0+\cdots+t_{k-1}},\cdots,x_{t_0+\cdots+t_k-1}), \quad 1 \leq k \leq m,$$
$$a''(\tau_{m+1}) = a''(t_{m+1})(\tau_{m+1}; x_{t_0+\cdots+t_m},\cdots,x_{t_0+\cdots+t_{m+1}-2}, z).$$

**Definition 8.8.** *$B(Y,X,Z)$ and $B'(Y,X,Z)$ the two-sided Bar constructions of $(Y,X,Z)$ in $\underline{\underline{\mathcal{T}}}$ are defined as follows:*

$$B(Y,X,Z) = \underline{B(Y,X,Z)} \otimes \overline{K},$$
$$B'(Y,X,Z) = \underline{B(Y,X,Z)} \otimes \overline{J_0},$$

*where $\overline{K}$ and $\overline{J_0}$ denote the canonical left representations of $\underline{\underline{\mathcal{K}}}_\infty(\underline{\underline{\mathcal{T}}})$.*

Then we can show that the two-sided bar constructions $B(Y,X,Z)$ and $B'(Y,X,Z)$ are always well-defined.

**Theorem 8.9.** *$B(Y,X,Z)$ and $B'(Y,X,Z)$ are well-defined and homeomorphic to each other in $\underline{\underline{\mathcal{T}}}$.*

*Proof*: We first define subspaces $B_n$, $n \geq 0$ defined inductively on $n$, which give a filtration of $B(Y,X,Z)$.

$$B_0 = Y \times_O Z$$
$$B_n = B_{n-1} \cup K(n{+}2) \times (Y \times_O (\prod_O^n X) \times_O Z), \quad n \geq 1,$$

where $K(n{+}2)\times(Y\times_O(\prod_O^n X)\times_O Z)$ is attached to $B_{n-1}$ by

$$K(n{+}2)\times(Y\times_O(\prod_O^n X)\times_O Z) \leftarrow \coprod_{\substack{1\leq k\leq r,\ 2\leq r\leq n+1\\ r+s=n+3}} K_k(r,s)\times(Y\times_O(\prod_O^n X)\times_O Z)$$
$$\to \coprod_{2\leq r\leq n+1} K(r)\times(Y\times_O(\prod_O^{r-2} X)\times_O Z) \to \coprod_{2\leq r\leq n+1} B_{r-2} \to B_{n-1},$$

which is given by the structure maps of $A_\infty$ structures followed by the canonical projection. Since a wedge-sum is compatible with a colimit, $B(Y,X,Z)$ is also a colimit of $B_n$'s, and hence is well-defined. By replacing $K(n{+}2)$ by $J_0(n{+}2)$ and $K_k(r,s)$ by $J(r,s)_0$, we obtain that $B'(Y,X,Z)$ is also well-defined. Then the homeomorphism $\omega_n : K(n) \to J_0(n)$ introduced in Definition 3.19 of §3.4 gives a homeomorphism between $B(Y,X,Z)$ and $B'(Y,X,Z)$. □

**Remark 8.10.** *We denote by $B_n(Y,X,Z)$ the $n$-th filtration $B_n$ of $B(Y,X,Z)$ and by $B'_n(Y,X,Z)$ for $B'(Y,X,Z)$.*

Let $(f,\{h(n)\}) : (X,\{a(n)\}) \to (X',\{b(n)\})$ be an internal $A_\infty$ functor regarding *hopf-units* between internal $A_\infty$ categories with *hopf-units*. Let $(g,f;\{h'(n)\}) : (Y,X;\{a'(n)\}) \to (Y',X';\{b'(n)\})$ be an internal $A_\infty$ equivariant functor regarding *hopf-units* between right $A_\infty$ actions with *hopf-units* and let $(f,\ell;\{h''(n)\}) : (X,Z;\{a''(n)\}) \to (X',Z';\{b''(n)\})$ be an internal $A_m$ equivariant functor regarding *hopf-units* between left $A_m$ actions with *hopf-units*. Then $(g,f,\ell)$ induces a map $B(g,f,\ell) : B(Y,X,Z) \approx B'(Y,X,Z) \to B(Y',X',Z')$ by

$$B(g,f,\ell)([\sigma;y,x_2,\cdots,x_{n-1};z]) = [\tau;h'(\rho_1),h(\rho_2)\cdots,h(\rho_{t-1};h''(\rho_t))],$$
$$h'(\rho_1) = h'(\rho_1;y;x_2,\cdots,x_{r_1})$$
$$h(\rho_j) = h(\rho_j;x_{r_1+\cdots+r_{j-1}+1},\cdots,x_{r_1+\cdots r_j}) \quad (1<j<t)$$
$$h''(\rho_t) = h''(\rho_t;x_{r_1+\cdots+r_{t-1}+1},\cdots,x_{n-1};z)$$

where $\omega_n(\sigma) = \delta(t;r_1,\cdots,r_t)(\tau;\rho_1,\cdots,\rho_t)$.

8.4. **Two-sided bar construction of an internal $A_\infty$ category with *strict-unit*.** We define another two-sided bar construction with *strict-unit* using co-equalizer in $\underline{\underline{\mathcal{T}}}$.

Let $X = (X,\{a(n)\})$ be an internal $A_\infty$ category with *strict-unit*, with a right $A_\infty$ action $(Y,X;\{a'(n)\})$ with *strict-unit* of $X$ on $Y$ and a left $A_\infty$ action $(X,Z;\{a''(n)\})$ with *strict-unit* of $X$ on $Z$ in $\underline{\underline{\mathcal{T}}}$.

**Definition 8.11.** *The left and right* $A_\infty$ *actions of* $X$ *on* $Y$ *and* $Z$ *induces a right representation* $\underline{\widetilde{B}(Y,X,Z)}$ *of the* $A_\infty$ *operadic category* $\underline{\underline{\widetilde{\mathcal{K}}}}_\infty(\underline{\underline{\mathcal{T}}})$ *defined for* $n \geq 0$*,* $\tau_k \in K(t_k)$*,* $0 \leq k \leq m{+}1$ *and* $(y, x_1, \cdots, x_m, z) \in Y \times_O (\prod_O^m X) \times_O Z$ *as follows:*

$$\underline{\widetilde{B}(Y,X,Z)}(\underline{n}) = Y \times_O (\prod_O^n X) \times_O Z,$$
$$\underline{\widetilde{B}(Y,X,Z)}(\underline{m},\underline{n})(\tau_0,\cdots,\tau_{m+1}; y, x_1,\cdots,x_n,z)$$
$$= (a'(\tau_0), a(\tau_1),\cdots,a(\tau_m),a''(\tau_{m+1})),$$

*where* $a'(\tau_0)$*,* $a(\tau_k)$ $(1 \leq k \leq m)$*,* $a''(\tau_{m+1})$ *are given by*

$$a'(\tau_0) = a'(t_0)(\tau_0; y, x_1,\cdots,x_{t_0-1}),$$
$$a(\tau_k) = a(t_k)(\tau_k; x_{t_0+\cdots+t_{k-1}},\cdots,x_{t_0+\cdots+t_k-1}), \quad 1 \leq k \leq m,$$
$$a''(\tau_{m+1}) = a''(t_{m+1})(\tau_{m+1}; x_{t_0+\cdots+t_m},\cdots,x_{t_0+\cdots+t_{m+1}-2}, z).$$

**Definition 8.12.** $\widetilde{B}(Y,X,Z)$ *and* $\widetilde{B}'(Y,X,Z)$ *the two-sided Bar constructions of* $(Y,X,Z)$ *in* $\underline{\underline{\mathcal{T}}}$ *of* $(Y,X,Z)$ *is defined as follows:*

$$\widetilde{B}(Y,X,Z) = \underline{\widetilde{B}(Y,X,Z)} \otimes \overline{\overline{\widetilde{K}}}, \quad \widetilde{B}'(Y,X,Z) = \underline{\widetilde{B}(Y,X,Z)} \otimes \overline{\overline{\widetilde{J}}}_0,$$

*where* $\overline{\overline{\widetilde{K}}}$ *denotes the canonical left representation of* $\underline{\underline{\widetilde{\mathcal{K}}}}_\infty(\underline{\underline{\mathcal{T}}})$*.*

Then we can show that the two-sided bar constructions $\widetilde{B}(Y,X,Z)$ and $\widetilde{B}'(Y,X,Z)$ are always well-defined:

**Theorem 8.13.** $\widetilde{B}(Y,X,Z)$ *and* $\widetilde{B}(Y,X,Z)$ *are well-defined and homeomorphic to each other in* $\underline{\underline{\mathcal{T}}}$*.*

*Proof*: We first define subspaces $\widetilde{B}_n$, $n \geq 0$ defined inductively on $n$, which give a filtration of $\widetilde{B}(Y,X,Z)$.

$$\widetilde{B}_0 = Y \times_O Z$$
$$\widetilde{B}_n = \widetilde{B}_{n-1} \cup K(n{+}2) \times (Y \times_O (\prod_O^n X) \times_O Z), \quad n \geq 1,$$

where $K(n{+}2) \times (Y \times_O (\prod_O^n X) \times_O Z)$ is attached to $B_{n-1}$ by the map

$$K(n{+}2) \times (Y \times_O (\prod_O^n X) \times_O Z) \leftarrow \coprod_{\substack{1 \leq k \leq r,\ 2 \leq r \leq n+1 \\ r+s=n+3}} K_k(r,s) \times (Y \times_O (\prod_O^n X) \times_O Z)$$
$$\amalg \coprod_{i=1}^{n} K(n{+}2) \times (Y \times_O (\prod_O^{i-1} X \times_O O \times_O \prod_O^{n-i} X) \times_O Z)$$
$$\rightarrow \coprod_{2 \leq r \leq n+1} K(r) \times (Y \times_O (\prod_O^{r-2} X) \times_O Z) \rightarrow \coprod_{2 \leq r \leq n+1} \widetilde{B}_{r-2} \rightarrow \widetilde{B}_{n-1},$$

which is given by structure maps of $A_\infty$ structures followed by the canonical projection. Since a wedge-sum is compatible with a colimit, $\widetilde{B}(Y,X,Z)$ is also a colimit of $\widetilde{B}_n$'s, and hence is well-defined. By replacing $K(n{+}2)$ by $J_0(n{+}2)$ and $K_k(r,s)$ by $J(r,s)_0$, we obtain that $\widetilde{B}'(Y,X,Z)$ is also well-defined. Then the homeomorphism $\omega_n : K(n) \to J_0(n)$ introduced in Definition 3.19 of §3.4 gives a homeomorphism between $\widetilde{B}(Y,X,Z)$ and $\widetilde{B}'(Y,X,Z)$. □

**Remark 8.14.** *We denote by $\widetilde{B}_n(Y,X,Z)$ the $n$-th filtration $\widetilde{B}_n$ of $\widetilde{B}(Y,X,Z)$ and similarly by $\widetilde{B}'_n(Y,X,Z)$ for $\widetilde{B}'(Y,X,Z)$.*

Let $(f,\{h(n)\}) : (X,\{a(n)\}) \to (X',\{b(n)\})$ be an internal $\widetilde{A}_\infty$ functor regarding *strict-units* between internal $\widetilde{A}_\infty$ categories with *strict-units*. Let $(g,f;\{h'(n)\}) : (Y,X;\{a'(n)\}) \to (Y',X';\{b'(n)\})$ be an internal $\widetilde{A}_\infty$ equivariant functor regarding *strict-units* between right $\widetilde{A}_\infty$ actions with *strict-units* and let $(f,\ell;\{h''(n)\}) : (X,Z;\{a''(n)\}) \to (X',Z';\{b''(n)\})$ be an internal $\widetilde{A}_m$ equivariant functor regarding *strict-units* between left $\widetilde{A}_m$ actions with *strict-units*. Then $(g,f,\ell)$ induces a map $\widetilde{B}(g,f,\ell) : \widetilde{B}(Y,X,Z) \approx \widetilde{B}'(Y,X,Z) \to \widetilde{B}(Y',X',Z')$ by

$$\widetilde{B}(g,f,\ell)([\sigma;y,x_2,\cdots,x_{n-1};z]) = [\tau;h'(\rho_1),h(\rho_2)\cdots,h(\rho_{t-1};h''(\rho_t))],$$

where $h'(\rho_1)$, $h(\rho_j)$ and $h''(\rho_t)$ are defined as before.

## 9. Unit conditions in an $A_\infty$ form

We adopt here a completely different approach from the original one to consider about the equivalence of two definitions given in [17]. In this section, we give a bar construction for an $A_\infty$ space with *h-unit*.

### 9.1. $A_\infty$ space with *h-unit*.

First we define a slightly weaker version of an $A_m$ forms ($m{\le}\infty$) for a topological space.

**Definition 9.1.** *We call $(X;\{a(n);1{\le}n{\le}m\})$ ($a(1) = 1_X$) an '$A_m$ space with h-unit', if based maps $a(n) : K(n) \times \prod^n X \to X$ satisfy the following formulas for all $n \le m$ and $\mathbf{x} = (x_1,\cdots,x_n) \in \prod^n X$:*

$$a(2)|_{\{*\}\times(X\vee X)} \sim \nabla_X \text{ (based homotopic)}, \tag{9.1}$$

$$a(n)(\partial_k(\rho,\sigma);\mathbf{x}) = a(r)(\rho;a_k(s)(\sigma;\mathbf{x})), \tag{9.2}$$

*where $a_k(s)(\sigma;\mathbf{x})$ is given by*

$$(x_1,\cdots,x_{k-1},a(s)(\sigma;x_k,\cdots,x_{k+s-1}),x_{k+s},\cdots,x_n).$$

If a space is an $A_m$ space with *h-unit* for any $m \ge 1$, then it is called an $A_\infty$ space with *h-unit*. It is easy to see that an $A_2$ space with *h-unit* is just an h-space. Then an $A_m$ space with *hopf-unit* is an $A_m$ space with *h-unit* and the converse is also true by using homotopy extension property (HEP) of $(X,e)$, the space with non-degenerate base point.

9.2. $A_\infty$ **map regarding *h-units*.** First we define a version of an $A_m$ forms ($m{\leq}\infty$) for a map between $A_m$ spaces with *h-units*.

**Definition 9.2.** *We call* $(f : X \to Y, \{h(n); 1{\leq}n{\leq}m\})$ $(h(1) = f)$ *an "$A_m$ map regarding h-units", if* $h(n) : J(n) \times \prod^n X \to Y$ *satisfies the following formulas for all* $n \leq m$ *and* $\mathbf{x} = (x_1, \cdots, x_n) \in \prod^n X$*:*

(9.3) $f(e_X)$ *and* $e_Y$ *lie in the same connected component,*

(9.4) $h(n)(\delta_k(\rho, \sigma); \mathbf{x}) = h(r)(\rho; a_k(s)(\sigma; \mathbf{x}))$,

(9.5) $h(n)(\delta(\tau; \rho_1, \cdots, \rho_t); \mathbf{x}) = b(t)(\tau; h(r_1, \cdots, r_t)(\rho_1, \cdots, \rho_t; \mathbf{x}))$,

*where* $h(r_1, \cdots, r_t)(\rho_1, \cdots, \rho_t; \mathbf{x})$ *is given by*

$$(h(r_1)(\rho_1; x_1, \cdots, x_{r_1}), \cdots, h(r_t)(\rho_t; x_{r_1+\cdots+r_{t-1}+1}, \cdots, x_{r_1+\cdots+r_t})).$$

If a map is an $A_m$ map regarding *h-units* for any $m \geq 1$, then it is called an $A_\infty$ map regarding *h-units*.

9.3. **Projective spaces of an $A_\infty$ space with *h-unit*.** Let $X$ be an $A_m$ space, i.e, $X$ has an $A_m$ form $\{a(r) : K(r){\times}X^r \to X \mid 1{\leq}r{\leq}m\}$.

**Definition 9.3.** *Let $Y$ and $Z$ be either $X$ or $* = \{*\}$.*

(1) *We define a map* $a'(s) : K(s){\times}Y{\times}X^{s-1} \to Y$ *by*

$$\begin{cases} a'(s)(\sigma; y, \chi) = a(s)(\sigma; y, \chi), & Y = X,\ s \leq m, \\ a'(s)(\sigma; *, \chi) = *, & Y = *,\ s \leq m{+}1, \end{cases}$$

*where* $\sigma \in K(s)$*,* $y \in Y$ *and* $\chi \in X^{s-1}$*.*

(2) *We define a map* $a''(s) : K(s){\times}X^{s-1}{\times}Z \to Z$ *by*

$$\begin{cases} a''(s)(\sigma; \chi, z) = a(s)(\sigma; \chi, z), & Z = X,\ s \leq m, \\ a''(s)(\sigma; \chi, *) = *, & Z = *,\ s \leq m{+}1, \end{cases}$$

*where* $\sigma \in K(s)$*,* $z \in Z$ *and* $\chi \in X^{s-1}$*.*

(3) *For any* $\sigma \in K(s)$*,* $r, s \geq 2$ *and* $1 \leq k \leq r$*, we define a map* $\bar{a}_k(\sigma) : Y{\times}X^{r+s-3}{\times}Z \to Y{\times}X^{r-2}{\times}Z$ *by the following formula.*

$$\bar{a}_k(\sigma)(y, \chi, z) = \begin{cases} (a'(s)(\sigma; y, x_2, \cdots, x_s), \cdots, x_{r+s-2}, z), & k{=}1, \\ (y, x_2, \cdots, a(s)(\sigma; x_k, \cdots, x_{k+s-1}), \cdots, x_{r+s-2}, z), & 1{<}k{<}r, \\ (y, x_2, \cdots, a''(s)(\sigma; x_r, \cdots, x_{r+s-2}, z)), & k{=}r, \end{cases}$$

*where* $y \in Y$*,* $z \in Z$ *and* $\chi = (x_2, \cdots, x_{r+s-2}) \in X^{r+s-3}$*.*

We give here the following definition of $A_m$ structure for $X$:

**Definition 9.4.** *Using the above actions of $X$ on $Y$ and $Z$, we define $E^n$, $P^n$ and $D^n$ for $0 \leq n \leq m$ inductively as follows.*

(1) $E^0 = \emptyset$*,* $P^0 = K(2){\times}{*}{\times}{*} \approx *$ *and* $D^0 = K(2){\times}\{e\}{\times}{*} \approx *$*.*

(2) $E^{n+1} = B_n(X, X, *) = (E^n \coprod (K(n{+}2) \times X \times X^n \times *)) / \sim$, *where* $(\partial_k(\sigma)(\rho), x, \chi, *) \sim (\rho, \bar{a}_k(\sigma)(x, \chi, *))$ *for* $n \geq 1$.

(3) $P^n = B_n(*, X, *) = \big(P^{n-1} \coprod (K(n{+}2) \times * \times X^n \times *)\big) /\sim$, *where* $(\partial_k(\sigma)(\rho), *, \chi, *) \sim (\rho, \bar{a}_k(\sigma)(*, \chi, *))$ *for* $n \geq 1$,

(4) $D^n = (E^n \coprod (K(n{+}2) \times \{e\} \times X^n \times *)) /\sim$, *where* $(\partial_k(\sigma)(\rho), e, \chi, *) \sim [\rho, \bar{a}_k(\sigma)(e, \chi, *)] \in E^{r-1} \subset E^n$ *for* $n \geq 1$, *in which* $(e, \chi, *)$ *is regarded as an element in* $X \times X^n \times *$.

**Remark 9.5.** *Since* $E^0 = \emptyset$, *we have* $E^1 = K(2) \times X \times * \approx X$.

*We also have obvious projections* $p^X_n : E^n \to P^{n-1}$ *and* $q^X_n : (D^n, E^n) \to (P^n, P^{n-1})$ *with* $q^X_n|_{E^n} = p^X_{n-1}$ *and* $p^X_n|_{D^n} = q^X_n$, $1 \leq n \leq m$ *given by*

$$p^X_n([\tau, x, \chi, *]) = [\tau, *, \chi, *] \quad \text{and} \quad q^X_n([\tau, e, \chi, *]) = [\tau, *, \chi, *].$$

Then we show the following proposition to obtain an $A_m$ structure.

**Proposition 9.6.** (1) $p^X_n$ *is a quasi-fibration for all* $n$.
(2) *The inclusion* $E^n \hookrightarrow D^n$ *is null-homotopic.*

*Proof*: To show (1), let $\alpha(n{+}2) = (0, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{n+2}{2})$. Then $K(n{+}2)$ can be described as a union of subsets $A = \overline{K_2(2, n{+}1) * \{\alpha(n{+}2)\}} \subset K(n{+}2)$ (join construction) and $B = \overline{(K(n{+}2) \smallsetminus A)} \subset K(n{+}2)$, where $(A, \operatorname{im} \partial_2(2, n{+}1))$ and $(B, \overline{(\partial K(n{+}2) \smallsetminus \operatorname{im} \partial_2(2, n{+}1))})$ are DR-pairs in the sense of Whitehead [21]. Thus $P^n$ can be described as a union of images $V \subset P^n$ and $W \subset P^n$ of $A \times * \times X^n \times *$ and $B \times * \times X^n \times *$, respectively, on which the projection $p^X_n$ is a quasi-fibration. We can also observe that, on the intersection of $V$ and $W$, $p^X_n$ is also a quasi-fibration and a fibre on $V \cap W$ mapped to a corresponding fibre on $V$ or $W$ by a right or left translation. Since $X$ is loop-like, left or right translation is a homotopy equivalence and hence we can deduce that $p^X_n$ is a quasi-fibration (see also Proof for Theorem 5 of [17] or the arguments in Chapter 7 of Mimura [15] using Corollary 1.8 and Lemma 1.13 in Chapter 5 of Mimura-Toda [16] for more detailed arguments).

To show (2), we introduce a series of spaces $\widehat{D}^n$ inductively on $n$:

(1) $\widehat{D}^0 = K(2) \times \{e\} \times * \approx *$.

(2) $\widehat{D}^n = \Big(\widehat{D}^{n-1} \coprod (K(n{+}2) \times \{e\} \times X^n \times *)\Big) /\sim$, $n \geq 1$,

where $(\partial_k(\sigma)(\rho), e, \chi, *) \sim [\rho, \bar{a}_k(\sigma)(e, \chi, *)] \in \widehat{D}^r \subset \widehat{D}^{n-1}$ for $k > 1$, and also $(\partial_1(\sigma)(\rho), e, \chi, *) \sim [\rho, \bar{a}_1(\sigma)(e, \chi, *)] \in E^{r-1} \subset E^n$ for $k = 1$ and $r \leq n$. Then the only free-face $\partial_1(n{+}1, 2) : K(n{+}1) \hookrightarrow K(n{+}2)$ induces a canonical map $\hat{i}_n : E^n \to \widehat{D}^n$. We can further obtain that $\hat{i}_n$ is an inclusion by induction on $n$. Apparently, the inclusion $E^n \hookrightarrow D^n$ is a composition of $\hat{i}_n : E^n \hookrightarrow \widehat{D}^n$ with the identification map $\widehat{D}^n \twoheadrightarrow D^n$.

Let us remark here that $\widehat{D}^1 = \big(D^0 \coprod K(3) \times \{e\} \times X \times *\big) /\sim \ \approx \widehat{C}X$ the unreduced cone of $X$ and $D^1 = (X \coprod (K(3) \times \{e\} \times X \times *) /\sim \ \approx$

$\widehat{C}X/(e \sim *) \simeq S^1$. Thus $\widehat{D}^1$ is contractible while $D^1$ is not. So, we are left to show that $\widehat{D}^n$ is contractible for all $n \geq 2$.

Let $L'(n{+}2) = \partial K(n{+}2) \smallsetminus K_1(n{+}1, 2)$ to obtain $(K(n{+}2), L'(n{+}2))$ a DR-pair. Since $e$ is a non-degenerate base point of a CW complex $X$, the pair $(X, \{e\})$ is a NDR-pair in the sense of [21]. Thus the pair $(K, L) = (K(n{+}2), L'(n{+}2)){\times}(X, \{e\}){\times}X^n{\times}*$ is a DR-pair.

Since the identification map $(K, L) \to (\widehat{D}^n, \widehat{D}^{n-1})$ gives a relative homeomorphism, the pair $(\widehat{D}^n, \widehat{D}^{n-1})$ is also a DR-pair, and hence so is $(\widehat{D}^n, \widehat{D}^0)$, for any $n \geq 1$. On the other hand by definition, $\widehat{D}^0$ is nothing but a one-point-set which is contractible. Thus $\widehat{D}^n$ is contractible for all $n \geq 0$. It completes the proof of Proposition 9.6. □

This implies that the above data gives an $A_m$ structure for $X$ in the sense of Stasheff. Thus by Lemma 0.1, we obtain the following.

**Proposition 9.7.** *If a CW complex h-space $X$, whose $\pi_0$ is a loop, has an $A_m$ form with h-unit, then there is a homotopy-equivalence $A_m$ map $j : X \hookrightarrow X'$ such that $X'$ has an $A_m$ form with strict-unit.*

In case when $X$ is a connected CW complex, James shear-map argument shows that the right and left translations of $X$ are homotopy equivalences and thus no further assumption is needed: let $X$ be a connected CW complex which has an $A_m$ form $\{a(n), n{\leq}m\}$ with *h-unit*. Then we may assume that $X$ is a loop-like h-space with *h-unit*. Hence we can apply Proposition 9.7 to obtain a homotopy-equivalence inclusion map $j : X \hookrightarrow X'$ which has an $A_m$ form regarding *h-units*, where $X'$ has an $A_m$ form $\{\hat{a}(n), n{\leq}m\}$ with *strict-unit*. So we have a homotopy-equivalence $A_m$ map $j : X \hookrightarrow X'$ in the sense of [11].

Let us assume that there is a deformation of $A_{\ell-1}$ form $\{\hat{h}_t(n), n{<}\ell\}$ with *strict-unit*, $\ell \leq m$, where $\hat{h}_t(n)$ is given by maps $\hat{h}_t(n) : K(n){\times}X^n \to X'$ obtained by taking adjoint of $h(n) : [0,1]{\times}K(n){\times}X^n \to X'$ which satisfies the following.

$$\hat{h}_0(n) = \hat{a}(n), \text{ and} \tag{9.6}$$

$$\hat{h}_1(n)(K(n){\times}X^n) \subset X, \quad n < \ell. \tag{9.7}$$

Then, by using $\{\hat{h}_t(n), n{<}\ell\}$, we obtain a map

$$h'(\ell) : \{0\}{\times}K(\ell){\times}X^\ell \cup [0,1]{\times}\partial K(\ell){\times}X^\ell \cup K(\ell){\times}X^{[\ell]} \to X'$$

given as follows: for $(\rho, \sigma) \in K(r){\times}K(s)$ $(1 \leq k \leq r,\ 2 \leq r, s < \ell,\ r + s = \ell{+}1)$, $\tau \in K(\ell)$ and $(x_1, \cdots, x_\ell) \in X^\ell$, we define

(1) $h'(\ell)(0; \tau; x_1, \cdots, x_\ell) = \hat{a}(\ell)(\tau; x_1, \cdots, x_\ell)$

(2) $h'(\ell)(t; \partial_k(\sigma)(\rho); x_1, \cdots, x_\ell)$
$= \hat{h}_t(r)(\rho; x_1, \cdots, x_{k-1}, \hat{h}_t(s)(\sigma; x_k, \cdots, x_{k+s-1}), x_{k+s}, \cdots, x_\ell),$

(3) $h'(\ell)(t;\tau;x_1,\cdots,x_{j-1},e,x_{j+1}\cdots,x_\ell)$
$= \hat{h}_t(\ell{-}1)(d^K_k(\tau);x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_\ell)$,

which coincide on their intersection with each other, by the relations given in (5.1), (5.2) and (5.4).

Since $([0,1],\{0\})\times(K(\ell),\partial K(\ell))\times(X^\ell,X^{[\ell]})$ is a DR-pair, we can extend $h'(\ell)$ to a homotopy $h'(\ell):[0,1]\times K(\ell)\times X^\ell\to X'$ and thus obtain a map $\hat{h'}_1(\ell)=h'(\ell)|_{\{1\}\times K(\ell)\times X^\ell}:K(\ell)\times X^\ell\to X'$ which satisfies the following: for $(\rho,\sigma)\in K(r)\times K(s)$ $(1\le k\le r,\ 2\le r,s<\ell,\ r+s=\ell{+}1)$, $\tau\in K(\ell)$ and $(x_1,\cdots,x_\ell)\in X^\ell$, we have

(1) $\hat{h'}_1(\ell):(K(\ell),\partial K(\ell))\times(X^\ell,X^{[\ell]})\to(X',X)$,

(2) $\hat{h'}_1(\ell)(\partial(\sigma)(\rho);x_1,\cdots,x_\ell)$
$=\hat{h}_1(r)(\rho;x_1,\cdots,\hat{h}_1(s)(\sigma;x_k,\cdots,x_{k+s-1}),x_{k+s},\cdots,x_\ell)$,

(3) $\hat{h'}_1(\ell)(\tau;x_1,\cdots,x_{j-1},e,x_{j+1}\cdots,x_\ell)$
$=\hat{h}_1(\ell{-}1)(d^K_k(\tau);x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_\ell)$.

Since $(X',X)$ is a DR-pair, we can further compress $\hat{h'}_1(n{+}1)$ into $X$, and hence we get a deformation $h(n{+}1):[0,1]\times K(n{+}1)\times X^{n+1}\to X'$ which satisfies the following: for $(\rho,\sigma)\in K(r)\times K(s)$ $(1\le k\le r,\ 2\le r,s<\ell,\ r+s=\ell{+}1)$, $\tau\in K(\ell)$ and $(x_1,\cdots,x_\ell)\in X^\ell$, we have

(1) $h(\ell)(0;\tau;x_1,\cdots,x_{n+1})=\hat{a}(n{+}1)(\tau;x_1,\cdots,x_{n+1})$ and
$h(\ell)(1;\tau;x_1,\cdots,x_{n+1})\in X$

(2) $h(\ell)(t;\partial(\sigma)(\rho);x_1,\cdots,x_{n+1})$
$=\hat{h}_t(r)(\rho;x_1,\cdots,\hat{h}_t(s)(\sigma;x_k,\cdots,x_{k+s-1}),x_{k+s},\cdots,x_{n+1})$,

(3) $\hat{h}(\ell)(t;\tau;x_1,\cdots,x_{j-1},e,x_{j+1}\cdots,x_n)$
$=\hat{h}_t(\ell{-}1)(d^K_k(\tau);x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_\ell)$.

Let $h_t(\ell):K(\ell)\times X^\ell\to X'$ be the map obtained by taking adjoint of $\hat{h}(\ell):[0,1]\times K(\ell)\times X^\ell\to X'$:

$$h_t(\ell)(\tau;x_1,\cdots,x_\ell)=\hat{h}(\ell)(t;\tau;x_1,\cdots,x_\ell).$$

Then the $A_{\ell-1}$ form $\{h_t(n),n{<}\ell\}$ together with $h_t(\ell)$ gives a deformation of $A_\ell$ form with *strict-unit*. Then by induction, we obtain a deformation of $A_m$ form $\{h_t(n),n{\le}m\}$ with *strict-unit*. Thus we obtain an $A_m$ form with *strict-unit* for $X$.

A similar argument gives us an $A_\infty$ form regarding *h-units* for $j$.

## Appendix A. Proof of Lemma 0.1

Assume that a CW complex $X$ admits an $A_m$ structure in the sense of Stasheff. Then by definition, there is a sequence $\{q^X_n,n{\le}m\}$ of maps $q^X_n:(D^n,E^n)\to(P^n,P^{n-1})$ such that $p^X_n=q^X_n|_{E^n}:E^n\to P^{n-1}$ is a quasi-fibration and $E^n$ is contractible in $D^n$.

We replace a quasi-fibration $p^X_m:E^m\to P^{m-1}$ by a Hurewicz fibration $\tilde{p}:\tilde{E}\to P^{m-1}$ with fibre $\tilde{X}$. Then there is a homotopy-equivalence inclusion map $\tilde{j} : (E^m, X) \to (\tilde{E}, \tilde{X})$. Let $j = \tilde{j}|_X : X \hookrightarrow \tilde{X}$ and let $\tilde{E}^n = (\tilde{p})^{-1}|_{P^{n-1}}$ and $\tilde{p}_n = \tilde{p}|_{\tilde{E}^n} : \tilde{E}^n \to P^{n-1}$. Then by combining the arguments given in Theorem 5 of [17] or [15] with [9] or [11], we can construct an $A_n$ form for $\tilde{X}$ together with a commutative ladder between $A_n$ structures in the sense of Stasheff, inductively on $n \leq m$.

We remark that we can also proceed to show the existence of an $A_m$ form of the inclusion $j$ regarding *h-units* by [11] which uses the same method due to Stasheff (see [17]).

## Appendix B. Proof of Theorem 0.4

It is a little bit tricky idea to consider an $A_\infty$ space *without unit*, because any $(X, e)$ a space $X$ with a base point $e$ has a sequence of maps $\{a(n)\,;\, n{\geq}1\}$ given by $a(n) : K(n) \times X^n \to \{e\} \hookrightarrow X$, which should give an $A_\infty$ form *without unit*. Anyway, we will give a proof of Theorem 0.4: First, we define $M$ by

$$M = \left(\bigcup_{n\geq 1} K(n{+}1) \times X^n\right) / \sim,$$

where the equivalence relation '$\sim$' is defined as follows.

$$(\partial_{k+1}(\sigma)(\rho); x_1, \cdots, x_n) \sim (\rho; x_1, \cdots, a(s)(\sigma; x_k, \cdots, x_{k+s-1}), \cdots, x_n),$$

for $\rho \in K(r{+}1)$, $\sigma \in K(s)$, $1 \leq k \leq r$ and $r + s{-}1 = n$.

Second, we observe that $M$ has an associative multiplication '$\cdot$' given by

$$[\rho; x_1, \cdots, x_r]\cdot[\sigma; y_1, \cdots, y_s] = [\rho\cdot\sigma; x_1, \cdots, x_r, y_1, \cdots, y_s],$$

where $\rho \in K(r{+}1)$ and $\sigma \in K(s{+}1)$ with $r + s = n$, and $\rho\cdot\sigma = \partial_1(\rho)(\sigma) = \partial_1(s{+}1, r{+}1)(\sigma, \rho) \in K(r{+}s{+}1) = K(n{+}1)$. Then by the Stasheff's boundary formulas, we obtain the following proposition, which shows that the multiplication '$\cdot$' is well-defined on $M$.

**Proposition B.1.** *For any $1 \leq k \leq r$ and $\rho \in K(r{+}1)$, $\sigma \in K(s)$ and $\tau \in K(t{+}1)$, we have the following relation.*

(1) $(\partial_{k+1}(\sigma)(\rho))\cdot\tau = \partial_{k+1}(\sigma)(\rho\cdot\tau)$
(2) $\tau\cdot(\partial_{k+1}(\sigma)(\rho)) = \partial_{k+t}(\sigma)(\tau\cdot\rho)$

Again by the Stasheff's boundary formulas for $\rho \in K(r{+}1)$, $\sigma \in K(s{+}1)$ and $\tau \in K(t{+}1)$, we obtain

$$\begin{aligned}(\rho\cdot\sigma)\cdot\tau &= (\partial_1(\rho)(\sigma))\cdot\tau = \partial_1(\partial_1(\rho)(\sigma))(\tau) = \partial_1(\partial_1(s{+}1, r{+}1)(\sigma, \rho))(\tau)\\ &= \partial_1(t, r{+}s{+}1)(\tau, \partial_1(s{+}1, r{+}1)(\sigma, \rho))\\ &= \partial_1(t{+}s{+}1, r)(\partial_1(t{+}1, s{+}1)(\tau, \sigma), \rho)\\ &= \partial_1(t{+}s{+}1, r)(\partial_1(\sigma)(\tau), \rho)\\ &= \partial_1(\rho)(\partial_1(\sigma)(\tau)) = \partial_1(\rho)(\sigma\cdot\tau) = \rho\cdot(\sigma\cdot\tau),\end{aligned}$$

which implies that $M$ has an associative multiplication *without unit*. Let $j : X \hookrightarrow M$ be as follows.

$$j(x) = [\alpha_2; x],\ \alpha_2 = (0,1) \in K(2).$$

By using a homeomorphism $\eta^1_n : [0,1] \times K(n) \to K(n{+}1)$, we can define a homotopy $g_n : [0,1] \times K(n{+}1) \to [0,1] \times [0,1] \times K(n) \to [0,1] \times K(n) \to K(n{+}1)$ by the following formula:

$$g_n = \eta^1_n \circ \kappa_n \circ (1 \times \eta^1_n)^{-1}, \quad \kappa_n(s,t,\rho) = (st, \rho).$$

Then we have $g_n(1,\tau) = \tau$ and $g_n(0,\tau) \in \operatorname{im}\partial_2(2,n) = K_2(2,n) \subset K(n{+}1)$. Since $\eta^1_n$ commutes with face operators, $g_n$ induces a deformation $G_n : [0,1] \times M \to M$ such that

$$G_n(1,x) = x \text{ and } G_n(0,x) \in X \subset M,$$

which implies that $X$ is a deformation retract of $M$. Further we define a sequence of maps $h(n) : K(n{+}1) \times X^n \to M$ which gives an $A_\infty$ form $\{h(n); n \geq 1\}$ in (our version of) the sense of Stasheff (see Appendix C) for the inclusion $j : X \hookrightarrow M$ as follows.

$$h(n)(\tau; x_1, \cdots, x_n) = [\tau; x_1, \cdots, x_n]$$

which satisfies the condition of an $A_\infty$ form for the inclusion $j : X \hookrightarrow M$. We leave the details to the readers.

If further $\{a(n)\,;\, n \geq 1\}$ the $A_\infty$ form with *strict unit* $e \in X$, we can replace $M$ by the following monoid $\hat{M}$ defined by the same way:

$$\hat{M} = \left( \bigcup_{n \geq 1} K(n{+}1) \times X^n \right) / \simeq,$$

where '$\simeq$' the equivalence relation for $G$ is defined as follows.

$$(\partial_{k+1}(\sigma)(\rho); x_1, \cdots, x_n) \simeq (\rho; x_1, \cdots, a(s)(\sigma; x_k, \cdots, x_{k+s-1}), \cdots, x_n),$$
$$(\tau; x_1, \cdots, x_{j-1}, e, x_j, \cdots, x_n) \simeq (d^K_{j+1}(\tau); x_1, \cdots, x_n).$$

Then $\hat{e} = [\alpha_2; e]$ gives the unit of the monoid $\hat{M}$. The inclusion $\hat{j} : X \to \hat{M}$ and homotopy $\hat{H}_n : [0,1] \times \hat{M} \to \hat{M}$ are defined similarly. Since $\eta^1_n$ commutes with degeneracy operations other than $d^K_1$, $\hat{H}_n$ is also well-defined and $X$ is a deformation retract of $\hat{M}$. Similar to the case for $M$, we can observe that $j$ is an $A_\infty$ map regarding *strict-units*.

## Appendix C. $A_\infty$ form from an $A_\infty$ space to a monoid

We will give here a slightly different formulation in [18] from the original given of an $A_m$ form for a map from an $A_m$ space *without unit* to a space with an associative multiplication.

Let $(X, \{a(n)\,;\, n{\leq}m\})$ be an $A_m$ space with *strict-unit* $* \in X$. Then it satisfies the following equations for any $\tau \in K(n)$, $(\rho, \sigma) \in$

$K(r)\times K(s)$ with $r+s-1 = n \geq 2$, $2 \leq r \leq n-1$ and $1 \leq k \leq r$.

$$a(n)(\partial_k(\sigma)(\rho); x_1, \cdots, x_n) = a(r)(\rho; x_1, \cdots, a(s)(\sigma; x_k, \cdots), \cdots, x_n),$$

$$a(n)(\tau; x_1, \cdots, x_{j-1}, *, x_j, \cdots, x_{n-1}) = a(n-1)(d^K_j(\tau); x_1, \cdots, x_{n-1}).$$

We then define an $A_m$ map from $X$ to $G$ a topological monoid.

**Definition C.1.** *A map $f : X \to G$ is an $A_\infty$ map if there exists an $A_\infty$ form $\{f(n); n{\leq}m\}$, $f(n) : K(n+1) \times X^n \to G$ ($f(1) = f$) satisfying*

$$f(n)(\partial_{k+1}(\sigma)(\rho); x_1, \cdots, x_n) = f(r)(\rho; x_1, \cdots, a(s)(\sigma; x_k, \cdots), \cdots, x_n),$$

$$f(n)(\partial_1(\rho_1)(\rho_2); x_1, \cdots, x_n) = f(r_1)(\rho_1; x_1, \cdots x_{r_1}) \cdot f(r_2)(\rho_2; \cdots, x_n),$$

$$f(n)(\tau; x_1, \cdots, x_{j-1}, *, x_j, \cdots, x_{n-1}) = f(n-1)(d^K_j(\tau); x_1, \cdots, x_{n-1})$$

*and $f(*) = e$, for any $\tau \in K(n+1)$, $(\rho, \sigma) \in K(r+1)\times K(s)$ with $r+s-1 = n \geq 1$, $2 \leq r \leq n-1$ and $1 \leq k \leq r$, and $(\rho_1, \rho_2) \in K(r_1+1)\times K(r_2+1)$ with $r_1+r_2 = n \geq 1$.*

Since $\partial K(n+1) \subset K(n+1) \smallsetminus \operatorname{Int} J(n)$ is a deformation retract, the existence of the above map implies that of an $A_\infty$ form $\{h(n), n{\leq}m\}$ for $f$, where $h(n)$ is given as a map $h(n) : J(n) \times X^n \to G$.

If we disregard units, then we shall obtain the following definition. Let $(X, \{a(n)\,;\, n{\geq}1\})$ be an $A_m$ space *without unit*. Then it just satisfies the following equation for any $(\rho, \sigma) \in K(r)\times K(s)$ with $r+s-1 = n \geq 2$, $2 \leq r \leq n-1$ and $1 \leq k \leq r$.

$$a(n)(\partial_k(\sigma)(\rho); x_1, \cdots, x_n) = a(r)(\rho; x_1, \cdots, a(s)(\sigma; x_k, \cdots), \cdots, x_n).$$

We then define an $A_m$ map *disregarding units* from $X$ to $G$ a topological space with associative multiplication.

**Definition C.2.** *A map $f : X \to G$ is an $A_m$ map* disregarding units *if there exists an $A_m$ form $\{f(n); n{\leq}m\}$, $f(n) : K(n+1) \times X^n \to G$ ($f(1) = f$) satisfying*

$$f(n)(\partial_{k+1}(\sigma)(\rho); x_1, \cdots, x_n) = f(r)(\rho; x_1, \cdots, a(s)(\sigma; x_k, \cdots), \cdots, x_n),$$

$$f(n)(\partial_1(\rho_1)(\rho_2); x_1, \cdots, x_n) = f(r_1)(\rho_1; x_1, \cdots x_{r_1}) \cdot f(r_2)(\rho_2; \cdots, x_n),$$

*for any $(\rho, \sigma) \in K(r+1)\times K(s)$ with $r+s-1 = n \geq 1$ and $1 \leq k \leq r$ and $(\rho_1, \rho_2) \in K(r_1+1)\times K(r_2+1)$ with $r_1+r_2 = n \geq 1$.*

Similarly, the existence of the above map implies the existence of an $A_m$ form *disregarding units* for $f$, $\{h(n), n{\leq}m\}$, where $h(n)$ is given as a map $h(n) : J(n) \times X^n \to G$.

## Appendix D. $A_\infty$ homomorphism

We usually call a map $f : X \to Y$ of $A_m$ spaces $(X, \{a(n)\,;\, 2 \leq n \leq m\})$ and $(Y, \{b(n)\,;\, 2 \leq n \leq m\})$ an $A_m$ homomorphism, if it satisfies the following equation:

$$f{\circ}a(n)(\tau; x_1, \cdots, x_n) = b(n)(\tau; f(x_1), \cdots, f(x_n)).$$

**Theorem D.1.** *An $A_m$ homomorphism of $A_m$ spaces is an $A_m$ map for $1{\leq}m{\leq}\infty$.*

To see this, we define maps among triples $(J^a(n), \partial J^a(n), J^a_0)$, $0 \leq a < 1$, where $J^a_0 = J^a_0(n) \smallsetminus \mathrm{Int}\, J^a(n; 1, \cdots, 1)$. Then we can easily see

$$(J^0(n), \partial J^0(n), J^0_0(n)) = (K(n), \partial K(n), K_1(n)).$$

We define a map $f_{a,b} : (J^a(n), \partial J^a(n), J^a_0(n)) \to (J^b(n), \partial J^b(n), J^b_0(n))$ for any $0 < a < 1$ and $0 \leq b \leq 1$ by the following formula:

$$f_{a,b} \circ \delta^{\varepsilon}(\rho_1, \cdots, \rho_t) = \delta^{\varepsilon \frac{b}{a}}(f_{a,b}(\rho_1), \cdots, f_{a,b}(\rho_t)),$$

where $0 \leq \varepsilon \leq a$. The well-definedness and the continuity of $f_{a,b}$ is obtained by a straight-forward argument, so we skip the proof. We adopt the following notation in the case when $0 < a < 1$ and $b = 0$:

$$\pi_n = f_{a,0} : (J^a(n), \partial J^a(n), J^a_0(n)) \to (K(n), \partial K(n), K_1(n)),$$

where $0 < a < 1$.

**Lemma D.2.** *Let $0 < a < 1$ and $0 \leq b \leq 1$.*

(1) $f_{a,b}{\circ}\delta^a_k(\sigma) = \delta^b_k(\sigma){\circ}f_{a,b}$, *where $\sigma \in K(s)$,*
(2) $f_{a,b} \circ \delta^a(\rho_1, \cdots, \rho_t) = \delta^b(f_{a,b}(\rho_1), \cdots, f_{a,b}(\rho_t))$, *where $\rho_i \in J(r_i)$,*
(3) $d^{J,b}_k{\circ}f_{a,b} = f_{a,b}{\circ}d^{J,a}_k$,

The proof of this proposition is directly obtained and left to the reader. This immediately implies the following.

**Proposition D.3.** (1) $\pi_a{\circ}\delta^a_k(\sigma) = \partial_k(\sigma){\circ}\pi_a$, *where $\sigma \in K(s)$,*
(2) $\pi_a \circ \delta^a(\beta_1, \cdots, \beta_1)(\tau) = \tau$, *where $J^a(1) = \{\beta_1\}$ and $\tau \in K(n)$,*
(3) $d^K_k{\circ}\pi_a = \pi_a{\circ}d^{J,a}_k$,

These equations imply Theorem D.1.

## Appendix E. Associahedra and Multiplihedra

E.1. **Shadows of trivalent trees and Associahedra.** Boardman and Vogt gave in [5] an alternative description of Stasheff's Associahedra $K(n)$ as the convex hull of the set of trivalent trees each of which has one root and $n$ top-branches. A branching point is called a node, from which one edge is going down (to left *or* right) and two edges are going up (to left *and* right). Hence, a trivalent tree $t$ with one root and $n$ top-branches has exactly $n{-}1$ nodes.

For a trivalent tree $t$, we give an order to top-branches of $t$ from the left as 1-st top-branch, 2-nd top-branch, $\cdots$, $n$-th top-branch. Then we count the number of nodes lying on the straight line going down to left from the $k$-th top-branch and denote it by $a_k(t)$.


Similarly, we denote by $b_k(t)$ the number of nodes lying on the straight line going down to right from the $k$-th top-branch of $t$.

The sequence of numbers $a(t) = (a_1(t), a_2(t), \cdots, a_n(t))$ is in $\mathbb{Z}_+^n \subset \mathbb{Z}^n$, where $\mathbb{Z}_+$ is the set of non-negative integers, and is satisfying

$$\begin{aligned} &a_1(t) = 0, \quad a_2(t) \le 1, \quad a_3(t) \le 2 - a_2(t), \\ &\quad \cdots \quad a_k(t) \le k{-}1 - (a_2(t) + \cdots + a_{k-1}(t)), \quad (1 < k < n) \\ &\qquad \cdots \quad a_n(t) = n{-}1 - (a_2(t) + \cdots + a_{n-1}(t)), \end{aligned}$$

since $a_1(t)+\cdots+a_k(t)$ is at most $k{-}1$ for all $k$ and $a_1(t)+\cdots+a_n(t) = n{-}1$ the total number of nodes. Hence $a(t)$ is in the set

$$K_L(n) = \left\{ (a_1, \cdots, a_n) \in \mathbb{Z}_+^n \;\middle|\; \begin{array}{l} a_j \le \sum_{i=1}^{j-1}(1{-}a_i),\ 1{\le}j{<}n \\ a_n = \sum_{i=1}^{n-1}(1{-}a_i) \end{array} \right\}$$

Similarly, $b(t) = (b_1(t), b_2(t), \cdots, b_n(t))$ is in the set

$$K'_L(n) = \left\{ (b_1, \cdots, b_n) \in \mathbb{Z}_+^n \;\middle|\; \begin{array}{l} b_j \le \sum_{i=j+1}^{n}(1{-}b_i),\ 1{<}j{\le}n \\ b_1 = \sum_{i=2}^{n}(1{-}b_i) \end{array} \right\}$$

Then we can easily see the following.

**Proposition E.1.** (1) $\#K_L(n) = C_{n-1}$, *the Catalan number*

(2) $K_L(n) = \left\{ a(t) \in \mathbb{Z}_+^n \;\middle|\; \begin{array}{l} t \text{ is a trivalent tree with one} \\ \text{root and } n \text{ top-branches} \end{array} \right\}$.

(3) $K'_L(n) = \left\{ (b_1, \cdots, b_n) \in \mathbb{Z}_+^n \;\middle|\; (b_n, \cdots, b_1) \in K_L(n) \right\}$.

Conversely assume that $(a_1, \cdots, a_n)$ is in $K_L(n)$. Then we can construct a trivalent tree $t$ with one root and $n$ top-branches such that $a(t) = (a_1, \cdots, a_n)$ using the information that $t$ must have exactly $a_k$ nodes on the line going down to left from the $k$-th top-branch. Thus $K_L(n)$ is in one-to-one correspondence with $K'_L(n)$.

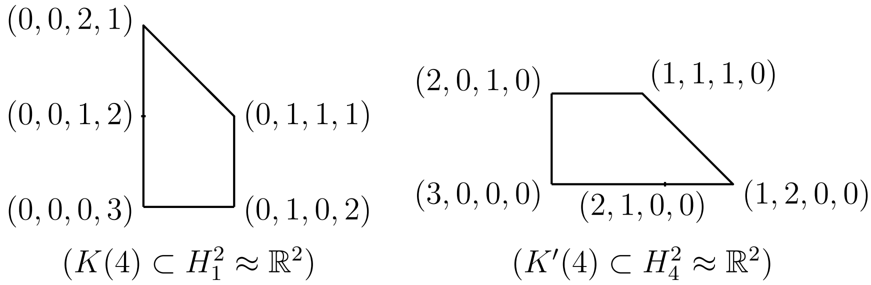
We now introduce two more definitions similarly to our $K(n)$:

**Definition E.2.** (1) *Let $K'(n)$ be the convex hull of $K'_L(n)$.*

We then obtain the following proposition.

**Proposition E.3.** $K'(n) = \{(t_1, \cdots, t_n) \mid (t_n, \cdots, t_1) \in K(n)\}$.

We can easily observe that the face operators for $K'(n)$ is nothing but $\partial'$ introduced in §1.1. In $\mathbb{R}^n$, we take hyper planes $H^{n-1} : x_1 + \cdots + x_n = n-1$, $R_1^{n-1} : x_1 = 0$ and $R_n^{n-1} : x_n = 0$. Let us define $H_1^{n-2} = H^{n-1} \cap R_1^{n-1} \approx \mathbb{R}^{n-2}$ and $H_n^{n-2} = H^{n-1} \cap R_n^{n-1} \approx \mathbb{R}^{n-2}$. Then we can easily observe that $K(n) \subset H_1^{n-2}$ and $K'(n) \subset H_n^{n-2}$:




Let us summarize properties of $K(n)$ family.

**Proposition E.4.** *Let* $C_n = \dfrac{{}_{2n}\mathrm{C}_n}{n+1}$ *the Catalan number.*
(1) $\#K_L(n) = \#K'_L(n) = C_{n-1}$.
(2) $K(n)$ *is a convex hull of* $K_L(n)$.
(3) $K'(n)$ *is a convex hull of* $K'_L(n)$.
(4) $K(n) \approx K'(n)$ *as polytopes.*
(5) $K(n) \cap L = K_L(n)$, *if we ignore first and last coordinates.*
(6) $K'(n) \cap L = K'_L(n)$, *if we ignore first and last coordinates.*

$K(n)$ and $K'(n)$ are easy to manipulate and mirror images to each other, and are constructed directly by taking shadows of trivalent trees on the integral lattice, where we can play our games.

E.2. **Language of bearded trees and Multiplihedra.** First, we introduce a language of trees in terms of (Reverse) Polish Notation. For any trivalent tree $t$ with one root and $n$ top-branches, $n \geq 1$, we define a word $w(t)$ of a tree $t$ by the following way:

(1) assign a word '$x_i$' to the $i$-th top-branch from the left.
(2) if the two upper branches of a node is assigned by a word '$w_1$' and '$w_2$', then assign a word '$w_1 w_2 @$' to its lower branch.
(3) if the root branch is assigned by a word '$w$', we define $w(t)$ the word of a tree $t$ to be $w$, i.e, $w(t) = w$.

This defines the set $\mathrm{W}(n)$ of all words $w(t)$ of trivalent trees $t$ with one root and $n$ top-branches:

$$\mathrm{W}(n) = \left\{ w(t) \;\middle|\; \begin{array}{l} t \text{ is a trivalent tree with one} \\ \text{root and } n \text{ top-branches} \end{array} \right\}$$

Similarly, we obtain another word $w'(t)$ for $t$.

(1) assign a word '$x_i$' to the $i$-th top-branch from the left.

(2) if the two upper branches of a node is assigned by a word '${w'}_1$' and '${w'}_2$', then assign a word '$@{w'}_1{w'}_2$' to its lower branch.
(3) if the root branch is assigned a word '$w'$', we define $w'(t)$ the word of a tree $t$ to be $w'$, i.e, $w'(t) = w'$.

This defines another set $\mathrm{W}'(n)$ of all words $w'(t)$ of trivalent trees $t$ with one root and $n$ top-branches.

Since the number of at-marks ('@') between $x_i$ and $x_{i+1}$ in the word '$w(t)$' gives the number of nodes in the down-to-left line from the $i$-th top-branch of $t$ a trivalent tree with one root and $n$ top-branches:

$$a_i(t) = \left\langle \begin{matrix}\text{the number of at-marks appearing be-}\\ \text{tween } x_i \text{ and } x_{i+1} \text{ in the word `}w(t)\text{'}\end{matrix} \right\rangle, \ i < n,$$
$$a_n(t) = \left\langle \begin{matrix}\text{the number of at-marks appearing after}\\ x_n \text{ in the word `}w(t)\text{'}\end{matrix} \right\rangle.$$

Thus we can identify $K_L(n)$ with $\mathrm{W}(n)$. Similarly, we obtain

$$b_i(t) = \left\langle \begin{matrix}\text{the number of at-marks appearing be-}\\ \text{tween } x_{i-1} \text{ and } x_i \text{ in the word `}w'(t)\text{'}\end{matrix} \right\rangle, \ i > 1,$$
$$b_1(t) = \left\langle \begin{matrix}\text{the number of at-marks appearing be-}\\ \text{fore } x_1 \text{ in the word `}w'(t)\text{'}\end{matrix} \right\rangle.$$

Thus we can also identify $K'_L(n)$ with $\mathrm{W}'(n)$.

Second, we extend the idea to the one for Multiplihedra. Let us consider a 'bearded tree' which is a trivalent tree with one root, $n$ top-branches and several beards each of which comes out from just below a node or the top-edge of a top-branch, and every way from a top-edge down to the root meets exactly one beard. Since a node is on a way down to the root from a top-edge, we may call it upper or lower, if it is upper a beard or lower a beard, resp. For any bearded tree $\check{t}$ of one root and $n$ top-branches, we define $w(\check{t})$ a word of $\check{t}$ as follows:

(1) assign a word '$x_i$' to the $i$-th top-branch from the left, if the branch has no beard.
(2) assign a word '$x_i\natural$' to the $i$-th top-branch from the left, if the branch has a beard.
(3) if the two upper branches of a node is assigned by a word '$w_1$' and '$w_2$' and its lower branch has no beard, then assign a word '$w_1w_2\sharp$' to the lower branch.
(4) if the two upper branches of a node is assigned by a word '$w_1$' and '$w_2$' and its lower branch has a beard, then assign a word '$w_1w_2\sharp\natural$' to the lower branch.
(5) if the root branch is assigned by a word '$w$', we define $w(\check{t})$ the word of a tree $\check{t}$ to be $w$, i.e, $w(\check{t}) = w$.

This defines the set $\mathrm{E}(n)$ of all extended words $w(\check{t})$ of bearded trees $\check{t}$ with one root and $n$ top-branches:

$$\mathrm{E}(n) = \left\{ w(\check{t}) \,\middle|\, \begin{array}{l} \check{t} \text{ is a bearded tree with one} \\ \text{root and } n \text{ top-branches} \end{array} \right\}$$

For a word $w$ in $\mathrm{E}(n)$, we obtain an $n$-tuple $(v_1(\check{t}), \cdots, v_n(\check{t}))$ of half integers as follows:

$$v_i(\check{t}) = \begin{cases} k, & \text{if } w(\check{t}) \text{ contains } x_i \sharp^k x_{i+1}, \\ k + \dfrac{\ell+1}{2}, & \text{if } w(\check{t}) \text{ contains } x_i \sharp^k \natural \flat^\ell x_{i+1}, \end{cases} \quad i < n,$$

$$v_n(\check{t}) = \begin{cases} k, & \text{if } w(\check{t}) \text{ ends as } x_n \sharp^k, \\ k + \dfrac{\ell+1}{2}, & \text{if } w(\check{t}) \text{ ends as } x_n \sharp^k \natural \flat^\ell, \end{cases} \quad i = n.$$

Let $u \geq 0$ and $\ell \geq 0$ be the total numbers of upper and lower nodes, respectively, and $k \geq 1$ be the number of beards. Then we have

$$v_1(\check{t}) + \cdots + v_n(\check{t}) = u - \frac{k+\ell}{2}, \quad u + \ell = n-1,$$

where $\check{t}$ is a bearded tree with one root and $n$ top-branches. Then we have the following proposition.

**Proposition E.5.** $v_1(\check{t}) + \cdots + v_n(\check{t}) = n - \dfrac{1}{2}$.

To prove this, we need to show the following lemma.

**Lemma E.6.** *The number of nodes below beards is one less than the total number of beards of* $\check{t}$.

*Proof*: Firstly, because $\check{t}$ is bearded, there is at least one beard. In the case when the $\check{t}$ has just one beard, we see that there must not be any node under the only beard. If there is a node under the beard, the other upper branch of the node under the beard does not have any beard which contradicts to the hypothesis that every top-branch meets exactly one beard on the way down to the root. So we have done for the case when $\check{t}$ has the just one beard, and we are left to show the lemma in the case when the number of beards of $\check{t}$ has more than 1.

Secondly, we show that there is a node each upper branch of which has a beard, by induction on the total number of nodes: we may assume that one beard is on a branch which is one of upper branches of a node. By the hypothesis on beards of a bearded tree, we can find out no beard under the node. If the total number of nodes is 1, then our claim is clear, and so we may assume that the whole upper part of the other branch of the node gives a smaller bearded tree which must satisfy our claim. Thus we can find out a node in $\check{t}$ such that each of the two upper branches has a beard.

Finally, we show the lemma by induction on the number of beards: we fix a node in $\check{t}$, each upper branch of which has a beard. Let $\check{t}'$ be a bearded tree by removing two beard from two upper branches of the node and by adding one beard to the lower branch of the node. Then, by the induction hypothesis, $\check{t}'$ satisfies the lemma and $\check{t}$ has one more beard with one node changed from upper to lower. This completes the proof of the lemma. □

*Proof of Proposition E.5*: Let $k \geq 1$ be the number of beard in a bearded tree $\check{t}$ with one root and $n$ top-branches, so that the number of lower node is $k{-}1$ and hence the number of upper nodes is $n{-}k$. Then we have

$$v_1(\check{t}) + \cdots + v_n(\check{t}) = (n{-}k) + \frac{k + (k{-}1)}{2} = n - \frac{1}{2}. \qquad \square$$

A similar consideration yields $v_1(\check{t}) + \cdots + v_i(\check{t}) \leq i - \frac{1}{2}$, which immedeately implies that $(v_1(\check{t}), \cdots, v_n(\check{t})) \in J(n)$.

**Definition E.7.** *For $n \geq 1$, we define a set $J_L(n)$ on the half-lattice.*

$$J_L(n) = \left\{ (v_1(\check{t}), \cdots, v_n(\check{t})) \;\middle|\; \begin{array}{l} \check{t} \textit{ is a bearded tree with one} \\ \textit{root and } n \textit{ top-branches} \end{array} \right\},$$

*where each entry of an element of $J_L(n)$ is a half integer.*

Since $J_L(1) = J(1)$, we can show by induction on $n$ that $J_L(n)$ gives the set of all vertices of $J(n)$, and hence we have

**Proposition E.8.** *$J(n)$ is the convex hull of $J_L(n)$.*

Similarly to the above, for any bearded tree $\check{t}$ of one root and $n$ top-branches, we define $w'(\check{t})$ a word of $\check{t}$ as follows:

(1) assign a word '$x_i$' to the $i$-th top-branch from the left, if the branch has no beard.
(2) assign a word '$\natural x_i$' to the $i$-th top-branch from the left, if the branch has a beard.
(3) if the two upper branches of a node is assigned by a word '$w'_1$' and '$w'_2$' and its lower branch has no beard, then assign a word '$\sharp w'_1 w'_2$' to the lower branch.
(4) if the two upper branches of a node is assigned by a word '$w'_1$' and '$w'_2$' and its lower branch has a beard, then assign a word '$\natural\sharp w'_1 w'_2$' to the lower branch.
(5) if the root branch is assigned by a word '$w'$', we define $w'(\check{t})$ the word of a tree $\check{t}$ to be $w'$, i.e, $w'(\check{t}) = w'$.

We then define $\mathrm{E}(n)$ as the set of all words $w'(t)$ of bearded trees $t$ with one root and $n$ top-branches:

$$\mathrm{E}'(n) = \left\{ w'(\check{t}) \;\middle|\; \begin{array}{l} \check{t} \text{ is a bearded tree with one} \\ \text{root and } n \text{ top-branches} \end{array} \right\}$$

For a word $w$ in $\mathrm{E}'(n)$, we obtain an $n$-tuple $(u_1(\check{t}), \cdots, u_n(\check{t}))$ of half integers as follows:

$$u_i(\check{t}) = \begin{cases} k, & \text{if } w(\check{t}) \text{ contains } x_{i-1}\sharp^k x_i, \\ k + \dfrac{\ell+1}{2}, & \text{if } w(\check{t}) \text{ contains } x_{i-1}\flat^\ell\natural\sharp^k x_i, \end{cases} \quad i > 1,$$

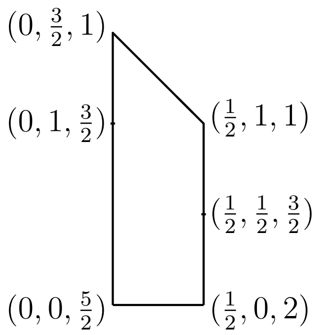
$$u_1(\check{t}) = \begin{cases} k, & \text{if } w(\check{t}) \text{ starts as } \sharp^k x_1, \\ k + \dfrac{\ell+1}{2}, & \text{if } w(\check{t}) \text{ ends as } \flat^\ell\natural\sharp^k x_1, \end{cases} \quad i = 1.$$

**Definition E.9.** *For $n \geq 1$, we define a set $J'_L(n)$ on the half-lattice.*

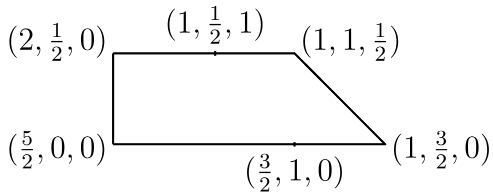
$$J'_L(n) = \left\{ (u_1(\check{t}), \cdots, u_n(\check{t})) \;\middle|\; \begin{array}{l} \check{t} \text{ is a bearded tree with one} \\ \text{root and } n \text{ top-branches} \end{array} \right\},$$

*where each entry of an element of $J'_L(n)$ is a half integer. Further, we define $J'(n)$ as the mirror image of $J(n)$ by taking convex hull of $J'_L(n)$.*

In $\mathbb{R}^n$, we take another hyper plane $H_0^{n-1} : x_1 + \cdots + x_n = n - \frac{1}{2}$. Then we can easily observe that $J(n) \subset H_0^{n-2}$ and $J'(n) \subset H_0^{n-2}$:




$(J(3) \subset H_0^2 \approx \mathbb{R}^2)$




$(J'(3) \subset H_0^2 \approx \mathbb{R}^2)$

Let us summarize the properties of $J(n)$ family.

**Proposition E.10.** (1) *$\# J_L(n) = \# J'_L(n)$.*
(2) *$J(n)$ is a convex hull of $J_L(n)$.*
(3) *$J'(n)$ is a convex hull of $J'_L(n)$.*
(4) *$J(n) \approx J'(n)$ as polytopes.*

By induction on $n$, we can show that there is a combinatorial homeomorphism $J(n) \approx J'(n)$ using bijection between $J_L(n)$ and $J'_L(n)$. $J(n)$ and $J'(n)$ are easy to manipulate and mirror images to each other, and are constructed directly by using the language of bearded trees with one root and $n$ top-branches.

## References

[1] J. F. Adams, *Infinite loop spaces*, Ann. of Math. Studies **90**, Princeton Univ. press, Princeton, N. J., 1978.

[2] M. Aguiar, *Internal categories and quantum groups*, Ph.D. Thesis, Cornell University, Ithaca, 1997.
[3] H. Asashiba and D. Tamaki, in preparation.
[4] J. M. Boardman, *Homotopy structures and the language of trees*, Proc. Symp. in Pure Math. **22**, Amer. Math. Soc. (1971), 37–58.
[5] J. M. Boardman, R.M. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lecture Notes in Math. **347**, Springer Verlag, Berlin, 1973.
[6] A. Dold and R. Thom: *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. (2) **67** (1958), 285-305.
[7] S. Forcey: *Convex hull realization of the multiplihedra*, Topology and its appl. **156** (2008), 326–347.
[8] M. Haiman: *Constructing the Associahedron*, preprint, 1984, unpublished.
[9] N. Iwase, *On the ring structure of* $K^*(XP^n)$ (Japanese), Master Thesis, Kyushu University, Fukuoka, 1983. (http://hdl.handle.net/2324/20353)
[10] N. Iwase, *On the* $A_n$*-structures of mappings* (Japanese), Research on unstable homotopy theory (Japanese) (Kyoto 1983), Sūri Kaiseki Kenkyūsho Kōkyūroku, **505** (1983), 63–75.
[11] N. Iwase and M. Mimura, *Higher homotopy associativity*, Algebraic Topology, (Arcata CA 1986), 193–220, Lecture Notes in Math. **1370**, Springer Verlag, Berlin, 1989.
[12] I. M. James, *On H-spaces and their homotopy groups*, Quart. J. Math. Oxford (2), **11** (1960), 161–179.
[13] F. Müller-Hoissen, J. M. Pallo, J. Stasheff, *Associahedra, Tamari Lattices and Related Structures*, Birkhäuser, Basel, 2012.
[14] S. Mau and C. T. Woodward, *Geometric realizations of the Multiplihedra*, Compos. Math. **146** (2010), 1002–1028.
[15] M. Mimura, "Hopf Spaces" (Japanese), Kinokuniya Shoten, Tokyo, 1986.
[16] M. Mimura and H. Toda, "Homotopy Theory" (Japanese), Kinokuniya Shoten, Tokyo, 1975.
[17] J. D. Stasheff, *Homotopy associativity of H-spaces, I & II*, Trans. Amer. Math. Soc. **108** (1963), 275–292; 293–312.
[18] J. D. Stasheff, *H-spaces from a Homotopy Point of View*, Lecture Notes in Math. **161**, Springer Verlag, Berlin, 1970.
[19] D. Tamari, *Monoïdes préordonnés et chaînes de Malcev*, Thèse de Mathématique, Université de Paris, Paris, 1951.
[20] A. Zabrodsky, "Hopf spaces", Notas de Matemática (59), North-Holland Publ. Co., Amsterdam, North-Holland Mathematics Studies, **22** (1976).
[21] G. W. Whitehead, "Elements of Homotopy Theory", Graduate Texts in Mathematics, **61**, Springer Verlag, Berlin, 1978.

*E-mail address*: `iwase@math.kyushu-u.ac.jp`

Faculty of Mathematics, Kyushu University, Motooka 744, Fukuoka 819-0395, Japan