ON THE CELLULAR DECOMPOSITION AND THE
LUSTERNIK-SCHNIRELMANN CATEGORY OF $\text{Spin}(7)$

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Abstract. We give a cellular decomposition of the compact connected Lie
group $\text{Spin}(7)$. We also determine the L-S categories of $\text{Spin}(7)$ and $\text{Spin}(8)$.

1. Introduction

In this paper, we assume that a space has the homotopy type of a CW-complex.
The Lusternik-Schnirelmann category $\text{cat} X$ of a space $X$ is the least integer $n$
such that $X$ is the union of $(n + 1)$ open subsets, each of which is contractible
in $X$. G. Whitehead [15] showed that $\text{cat} X \leq n$ if and only if the diagonal map
$\Delta_{n+1} : X \rightarrow \prod^{n+1} X$ is homotopic to some composition map

$$X \rightarrow T^{n+1}(X) \rightarrow \prod^{n+1} X,$$

where $T^{n+1}(X)$ is the fat wedge and $T^{n+1}(X) \rightarrow \prod^{n+1} X$ is the inclusion map.

The weak Lusternik-Schnirelmann category $\text{weakcat} X$ is the least integer $n$
such that the reduced diagonal map $\bar{\Delta}_{n+1} : X \rightarrow \wedge^{n+1} X$ is trivial. Then it is easy to
see that $\text{weakcat} X \leq \text{cat} X$, since $\wedge^{n+1} X = \prod^{n+1} X/T^{n+1}(X)$.

The strong Lusternik-Schnirelmann category $\text{Cat} X$ is the least integer $n$
such that there exist a space $X'$ which is homotopy equivalent to $X$ and is covered by
$(n + 1)$ open subsets contractible in themselves. $\text{Cat} X$ is closely related with $\text{cat} X$,
and Ganea and Takens [14] showed that

$$\text{cat} X \leq \text{Cat} X \leq \text{cat} X + 1.$$

Ganea [3] showed that $\text{Cat} X$ is equal to the invariant which is the least integer $n$
such that there is a cofibre sequence

$$A_i \rightarrow X_{i-1} \rightarrow X_i$$

where $X_0$ is a point and $X_n$ is homotopy equivalent to $X$.

The Lusternik-Schnirelmann category for some Lie groups are determined, such as
$\text{cat}(U(n)) = n$ and $\text{cat}(SU(n)) = n - 1$ by Singhof [11], $\text{cat}(Sp(2)) = 3$ by
Schweitzer [10], $\text{cat}(Sp(3)) = 5$ by Fernández-Suárez, Gómez-Tato, Strom and
Tanré [2], and Iwase and Mimura [6], $\text{cat}(SO(2)) = 1$, $\text{cat}(SO(3)) = 3$, $\text{cat}(SO(4)) =$

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4. cat(\text{SO}(5)) = 8 by James and Singhof [7]. Some general argument about the Lusternik-Schnirelmann category implies that cat(\text{G}_2) = 4 (see for example [6]).

As is well-known, we have the following isomorphisms:

\[
\text{Spin}(3) \cong S^3, \quad \text{Spin}(4) \cong S^3 \times S^3, \quad \text{Spin}(5) \cong \text{Sp}(2), \quad \text{Spin}(6) \cong \text{SU}(4).
\]

Thus \text{Spin}(7) is the first non-trivial case in determining the cellular decomposition and the Lusternik-Schnirelmann category as well; it is our purpose in this paper.

**Theorem 1.1.** We have \(\text{wcat}(\text{Spin}(7)) = \text{cat}(\text{Spin}(7)) = \text{Cat}(\text{Spin}(7)) = 5\).

Since \text{Spin}(8) is homeomorphic to \(\text{Spin}(7) \times S^7\), we obtain the following corollary.

**Corollary 1.2.** We have \(\text{wcat}(\text{Spin}(8)) = \text{cat}(\text{Spin}(8)) = \text{Cat}(\text{Spin}(8)) = 6\).

The paper is organized as follows. In Section 2 we give a cellular decomposition of \text{Spin}(7) such that \text{Spin}(7) contains a subgroup \(\text{SU}(4)\), which turns out to be useful for determining the Lusternik-Schnirelmann category of \text{Spin}(7). In Section 3 we give a cone-decomposition of \(\text{SU}(4)\), which gives rise to the Lusternik-Schnirelmann category of \text{Spin}(7) in Section 4.

2. The cellular decomposition of \text{Spin}(7)

In this section, we use the notation in [9]. Let \(\mathcal{C}\) be the Cayley algebra. \text{SO}(8) acts on \(\mathcal{C}\) naturally since \(\mathcal{C} \cong \mathbb{R}^8\) as \(\mathbb{R}\)-module. We regard \text{SO}(7) as the subgroup of \text{SO}(8) fixing \(e_0\), the unit of \(\mathcal{C}\). As is well known, the exceptional Lie group \(\text{G}_2\) is defined by

\[
\text{G}_2 = \{g \in \text{SO}(7) \mid g(x)g(y) = g(xy), x, y \in \mathcal{C}\} = \text{Aut}(\mathcal{C}).
\]

According to [19], the group \text{Spin}(7) is the set of the elements \(\tilde{g} \in \text{SO}(8)\) such that \(g(x)g(y) = \tilde{g}(xy)\) for any \(x, y \in \mathcal{C}\), where \(g \in \text{SO}(7)\) is uniquely determined by \(\tilde{g}\):

\[
\text{Spin}(7) = \{\tilde{g} \in \text{SO}(8) \mid g(x)g(y) = \tilde{g}(xy), g \in \text{SO}(7), x, y \in \mathcal{C}\}.
\]

It is easy to see that \(\text{G}_2\) is the subgroup of \text{Spin}(7). Observe that the algebra generated by \(e_1\) in \(\mathcal{C}\) is isomorphic to \(\mathcal{C}\). \text{SU}(4) acts on \(\mathcal{C}\) naturally, since as \(\mathcal{C}\)-module \(\mathcal{C} \cong \mathbb{C}^4\) whose basis is \(\{e_0, e_2, e_4, e_6\}\). We regard \text{SU}(3) as the subgroup of \text{SU}(4) fixing \(e_0\) and also as the subgroup of \(\text{G}_2\) fixing \(e_1\).

Let \(D^i\) be the \(i\)-dimensional disc. We define four maps:

\[
\begin{align*}
A : D^3 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\} \rightarrow \text{SO}(8), \\
B : D^2 &= \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq 1\} \rightarrow \text{SO}(8), \\
C : D^1 &= \{z_1 \in \mathbb{R} \mid z_1^2 \leq 1\} \rightarrow \text{SO}(8), \\
D : D^2 &= \{(w_1, w_2) \in \mathbb{R}^2 \mid w_1^2 + w_2^2 \leq 1\} \rightarrow \text{SO}(8),
\end{align*}
\]
as follows:

\[
A(x_1, x_2, x_3) = \begin{pmatrix}
1 \\
1 \\
1 \\
1 - 2X^2 & -2x_1X & -2x_2X & -2x_3X \\
2x_1X & 1 - 2X^2 & 2x_2X & -2x_2X \\
2x_2X & -2x_3X & 1 - 2X^2 & 2x_1X \\
2x_3X & 2x_2X & -2x_1X & 1 - 2X^2 \\
\end{pmatrix},
\]

\[
B(y_1, y_2) = \begin{pmatrix}
1 \\
y_1 - y_2 & -Y & 0 \\
y_2 & y_1 & 0 & -Y \\
Y & 0 & y_1 & y_2 \\
0 & Y & -y_2 & y_1 \\
1 \\
\end{pmatrix},
\]

\[
C(z_1) = \begin{pmatrix}
1 \\
z_1 & 0 & -Z \\
0 & 1 & 0 \\
Z & 0 & z_1 \\
1 \\
z_1 & 0 & -Z \\
0 & 1 & 0 \\
Z & 0 & z_1 \\
\end{pmatrix},
\]

\[
D(w_1, w_2) = \begin{pmatrix}
w_1 & -w_2 & -W \\
w_2 & w_1 & -W \\
W & w_1 & w_2 \\
W & -w_2 & w_1 \\
1 \\
1 \\
1 \\
\end{pmatrix},
\]

where we put for simplicity

\[
X = \sqrt{1 - x_1^2 - x_2^2 - x_3^2}, \quad Y = \sqrt{1 - y_1^2 - y_2^2}, \quad Z = \sqrt{1 - z_1^2}, \quad W = \sqrt{1 - w_1^2 - w_2^2}.
\]

**Lemma 2.1.** The elements \(A(x_1, x_2, x_3), B(y_1, y_2), C(z_1)\) and \(D(w_1, w_2)\) belong to Spin(7).
Proof. Apparently the elements \( A(x_1, x_2, x_3) \), \( B(y_1, y_2) \) and \( C(z_1) \) belong to \( G_2 \). In the proof, we denote \( D(w_1, w_2) \) simply by \( D \). Let \( D' \) be the matrix

\[
\begin{pmatrix}
1 & 1 \\
1 & w_1 - w_2 & W \\
w_2 & w_1 & -W \\
-W & w_1 - w_2 & W
\end{pmatrix}.
\]

Then we can show by a tedious calculation that \( D' x D y = D (x y) \) for any \( x, y \in \mathfrak{c} \), which gives us the result. \( \Box \)

Let \( \varphi_3, \varphi_5, \varphi_6 \) and \( \varphi_7 \) be maps

\[
\begin{align*}
\varphi_3 & : D^3 \longrightarrow \text{Spin}(7), \\
\varphi_5 & : D^3 \times D^2 \longrightarrow \text{Spin}(7), \\
\varphi_6 & : D^3 \times D^2 \times D^1 \longrightarrow \text{Spin}(7), \\
\varphi_7 & : D^3 \times D^2 \times D^2 \longrightarrow \text{Spin}(7)
\end{align*}
\]

respectively defined by the equalities

\[
\begin{align*}
\varphi_3(x) & = A(x), \\
\varphi_5(x, y) & = B(y) A(x) B(y)^{-1}, \\
\varphi_6(x, y, z) & = C(z) B(y) A(x) B(y)^{-1} C(z)^{-1}, \\
\varphi_7(x, y, w) & = D(w) B(y) A(x) B(y)^{-1} D(w)^{-1},
\end{align*}
\]

where \( x = (x_1, x_2, x_3) \), \( y = (y_1, y_2) \), \( z = (z_1) \) and \( w = (w_1, w_2) \). We define sixteen cells \( e^j \) for \( j = 0, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 21 \) respectively as follows:

\[
\begin{align*}
e^0 & = \{ 1 \}, \\
e^3 & = \text{Im } \varphi_3, \\
e^5 & = \text{Im } \varphi_5, \\
e^6 & = \text{Im } \varphi_6, \\
e^7 & = \text{Im } \varphi_7, \\
e^8 & = e^6 e^3, \\
e^9 & = e^6 e^3, \\
e^{10} & = e^7 e^3, \\
e^{11} & = e^6 e^5, \\
e^{12} & = e^7 e^5, \\
e^{13} & = e^6 e^7, \\
e^{14} & = e^6 e^5 e^3, \\
e^{15} & = e^7 e^5 e^3, \\
e^{16} & = e^6 e^7 e^3, \\
e^{18} & = e^6 e^7 e^5, \\
e^{21} & = e^6 e^7 e^5 e^3.
\end{align*}
\]

Let \( S^7 \) be the unit sphere of \( \mathfrak{c} \). Then we have a principal bundle over it:

\[
SU(3) \longrightarrow SU(4) \overset{p_0}{\longrightarrow} S^7,
\]

where \( p_0(g) = g e_0 \).

**Lemma 2.2.** Let \( V^7 = D^3 \times D^2 \times D^2 \). Then the composite map \( p_0 \varphi_7 : (V^7, \partial V^7) \longrightarrow (S^7, e_0) \) is a relative homeomorphism.
Proof. We express the map \((p_0\varphi_7)|_{V_7\setminus 0V^7}\) as follows:
\[
\begin{pmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5 \\
  a_6 \\
  a_7
\end{pmatrix} = D(w)B(y)A(x)B(y)^{-1}D(w)^{-1}c_0 = \begin{pmatrix}
  1 - 2X^2Y^2W^2 \\
  2x_1XY^2W^2 \\
  2(w_1X - x_1w_2)XY^2W \\
  -2(w_2X + x_1w_1)XY^2W \\
  2(-y_1X + x_1y_2)XYW \\
  2(y_2X + x_1y_1)XYW \\
  2x_2XYW \\
  2x_3XYW
\end{pmatrix}.
\]
and hence we have
\[
\begin{pmatrix}
  1 - a_0 \\
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5 \\
  a_6 \\
  a_7
\end{pmatrix} = 2XYW
\begin{pmatrix}
  XYW \\
  x_1YW \\
  (w_1X - x_1w_2)Y \\
  -(w_2X + x_1w_1)Y \\
  -y_1X + x_1y_2 \\
  y_2X + x_1y_1 \\
  x_2 \\
  x_3
\end{pmatrix}.
\]
Since \(X > 0, Y > 0, W > 0\) and \(1 - a_0 > 0\), an easy calculation as for the first component in the above equation gives the following equation:
\[
XYW = \frac{\sqrt{1 - a_2}}{\sqrt{2}},
\]
from which we easily obtain
\[
x_2 = \frac{a_6}{\sqrt{2(1 - a_2)}}, \quad x_3 = \frac{a_7}{\sqrt{2(1 - a_2)}}.
\]
Further we obtain three more equalities from the above equalities:
\[
(1 - a_0)^2 + a_1^2 = 4X^2Y^4W^4(x_1^2 + X^2),
\]
\[
a_2^2 + a_3^2 = 4X^2Y^4W^2(w_1^2 + w_2^2)(x_1^2 + X^2) = 4X^2Y^4W^2(1 - W^2)(x_1^2 + X^2),
\]
\[
a_4^2 + a_5^2 = 4X^2Y^2W^2(y_1^2 + y_2^2)(x_1^2 + X^2) = 4X^2Y^2W^2(1 - Y^2)(x_1^2 + X^2).
\]
Using these three equalities, we obtain
\[
y^2 = \frac{(1 - a_0)^2 + a_1^2 + a_2^2 + a_4^2}{(1 - a_0)^2 + a_1^2 + a_4^2 + a_5^2 + a_4^2 + a_5^2},
\]
\[
W^2 = \frac{(1 - a_0)^2 + a_3^2}{(1 - a_0)^2 + a_4^2 + a_5^2}.
\]
It follows from (2.1), (2.3) and (2.4) that
\[
X^2 = \frac{(1 - a_0)((1 - a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)}{2((1 - a_0)^2 + a_1^2)}.
\]
It follows also from (2.2) and (2.5) that

\[ x_1^2 = \frac{a_1^2((1 - a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)}{2(1 - a_0)((1 - a_0)^2 + a_1^2)}. \]  

Since \( Y, W, X \) are positive, (2.3), (2.4), (2.5) imply respectively

\[ Y = \frac{\sqrt{(1 - a_0)^2 + a_1^2 + a_2^2 + a_3^2}}{\sqrt{(1 - a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2}}, \]
\[ W = \frac{\sqrt{(1 - a_0)^2 + a_1^2}}{\sqrt{(1 - a_0)^2 + a_1^2 + a_2^2 + a_3^2}}, \]
\[ X = \frac{\sqrt{(1 - a_0)((1 - a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)}}{\sqrt{2((1 - a_0)^2 + a_1^2)}}. \]

Since the signs of \( x_1 \) and \( a_1 \) are the same, (2.6) implies that

\[ x_1 = \frac{a_1\sqrt{(1 - a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2}}{\sqrt{2((1 - a_0)^2 + a_1^2)}}. \]

Now we determine \( y_1 \); we have

\[-a_3X + a_5x_1 = 2XYW(x_1^2 + X^2)y_2.\]

Substituting the equations (2.1), (2.9) and (2.10) in the above equation, we obtain

\[ y_1 = \frac{a_1a_5 - (1 - a_0)a_4}{\sqrt{((1 - a_0)^2 + a_1^2)((1 - a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)}}. \]

We determine \( y_2 \); we have

\[ a_4x_1 + a_5X = 2XYW(x_1^2 + X^2)y_2. \]

Substituting the equations (2.1), (2.9) and (2.10) in the above equation, we obtain

\[ y_2 = \frac{a_1a_4 + (1 - a_0)a_5}{\sqrt{((1 - a_0)^2 + a_1^2)((1 - a_0)^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)}}. \]

We determine \( w_1 \); we have

\[ a_2X - a_3x_1 = 2XY^2W(x_1^2 + X^2)w_1. \]

Substituting the equations (2.1), (2.7), (2.9) and (2.10) in the above equation, we obtain

\[ w_1 = \frac{(1 - a_0)a_2 - a_1a_3}{\sqrt{((1 - a_0)^2 + a_1^2)((1 - a_0)^2 + a_1^2 + a_2^2 + a_3^2)}}. \]

Finally we determine \( w_2 \); we have

\[-a_2x_1 - a_3X = 2XY^2W(x_1^2 + X^2)w_2. \]

Substituting the equations (2.1), (2.7), (2.9) and (2.10) in the above equation, we obtain

\[ w_2 = \frac{-a_1a_2 - (1 - a_0)a_3}{\sqrt{((1 - a_0)^2 + a_1^2)((1 - a_0)^2 + a_1^2 + a_2^2 + a_3^2)}}. \]
Thus we have expressed $x_1, x_2, x_3, y_1, y_2, w_1, w_2$ in terms of $a_0, \cdots, a_7$, that is, the inverse map has been constructed, which completes the proof.

In a similar way to that of Section 3 of [9], we can obtain the following theorem, which is essentially the same as Yokota’s decomposition [17].

**Proposition 2.3.** $e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}$ thus obtained is a cellular decomposition of $SU(4)$.

**Proof.** First we show that $e^i \cap e^j = \emptyset$ if $i \neq j$. We consider the following three cases:

1. For the case where $i, j \in \{0, 3, 5, 8\}$; both cells $e^i$ and $e^j$ are in $SU(3)$ and $e^0 \cup e^3 \cup e^5 \cup e^8$ is a cellular decomposition of $SU(3)$. Then we have $e^i \cap e^j = \emptyset$ if $i \neq j$.

2. For the case where $i \in \{0, 3, 5, 8\}$ and $j \in \{7, 10, 12, 15\}$; we have $p_0(e^i) = \{e_0\}$ and $p_0(e^j) = S^7 \setminus \{e_0\}$. Then we have $e^i \cap e^j = \emptyset$.

3. For the case where $i, j \in \{7, 10, 12, 15\}$; suppose that $A \in e^i \cap e^j$. Since $e^7 = e^7 e^{-7}$ and $e^{15} = e^{15} e^{-15}$, we can put $A = A_1 A_2 = A'_1 A'_2$ where $A_1, A'_1 \in e^7, A_2, A'_2 \in e^{15}$. We have $A_1 = A'_1$, since $p_0(A_1) = p_0(A_2) = p_0(A'_1) = p_0(A'_2)$ and $p_0|e^i$ is monic. Then we have $A_2 = A'_2$ and the first case shows that $i - 7 = j - 7$, that is, $i = j$. Thus $e^i \cap e^j = \emptyset$ if $i \neq j$.

Next, we will check that the boundaries of the cells are included in the lower dimensional cells. In the proof of Proposition 3.2 [9], it is proved that the boundaries $e^3, e^5$ and $e^8$ are included in the lower dimensional cells. Observe that the boundary $e^7$ is the union of the following three sets:

$$\{DBAB^{-1}D^{-1} | A \in A(D^3), B \in (D^2), D \in D(D^2)\},$$
$$\{DBAB^{-1}D^{-1} | A \in A(D^3), B \in (D^2), D \in D(D^2)\},$$
$$\{DBAB^{-1}D^{-1} | A \in A(D^3), B \in (D^2), D \in D(D^2)\}.
$$

The first set contains only the identity element, since $A$ is the identity element. It is easy to see that the second set is contained in $e^3$ and that the third set is contained in $e^5$. We have $e^{10} = e^7 e^3 \cup e^5 e^3 \subset e^7 e^0 \cup e^5 e^3 = e^7 \cup e^8$. We also have $e^{12} = e^7 e^5 \cup e^5 e^3 \subset e^5 e^3 \cup e^7 e^3 = e^8 \cup e^{10}$, and $e^{15} = e^7 e^5 e^3 \cup e^7 e^5 e^3 \cup e^7 e^5 e^3 \subset e^7 e^5 e^3 \cup e^7 e^5 e^3 \cup e^7 e^5 e^3 = e^8 \cup e^{10} \cup e^{12}$.

Finally, we will show that the inclusion map $e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15} \to SU(4)$ is epic. Let $g \in SU(4)$. If $p_0(g) = e_0$, then $g$ is contained in $SU(3) = e^0 \cup e^3 \cup e^5 \cup e^8$. Suppose that $p_0(g) \neq e_0$. There is an element $h \in e^7$ such that $p_0(h) = p_0(g)$. Thus we have $h^{-1} g \in SU(3) = e^0 \cup e^3 \cup e^5 \cup e^8$. Since $p_0(h^{-1} g) = e_0$. Therefore we have $g \in (e^0 \cup e^3 \cup e^5 \cup e^8) \subset e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}$. □

**Remark 2.4.** (1) We regard $SO(6)$ as the subgroup of $SO(7)$ fixing $e_1$. Let $\pi : Spin(6) \to SO(6)$ be the double covering. Then, according to the Proof of Lemma 2.1, $\pi(SU(4)) \subset SO(6)$ so that $\pi|_{SU(4)} : SU(4) \to SO(6)$ is the double covering.
(2) For \(1 \leq n \leq 3\), the subcomplex \(e^0 \cup e^3 \cup \cdots \cup e^{2n+1}\) is homeomorphic to \(\Sigma \mathbb{C} P^n\), which consists of the elements

\[
A \begin{pmatrix} 1 & 1 \\ 1 & e^{2i\theta} \end{pmatrix} A^{-1} \begin{pmatrix} 1 & 1 \\ 1 & e^{-2i\theta} \end{pmatrix}
\]

for any elements \(A\) in \(SU(n+1)\). Moreover, according to Proposition 2.6 of Chapter IV of [13], we have \(e^{2i+1}e^{2j+1} \subset e^{2j+1}e^{2i+1}\) for \(i < j\); in fact we have \(e^{2i+1}e^{2j+1} = e^{2j+1}e^{2i+1}\) (see [19]).

Let \(S^6\) be the unit sphere of \(\mathbb{R}^7\) whose basis \(\{e_i | 1 \leq i \leq 7\}\. We consider the following diagram

\[
\begin{array}{ccc}
SU(3) & \longrightarrow & G_2 \\
\downarrow & & \downarrow \\
SU(4) & \longrightarrow & Spin(7) \\
& \pi & \\
SO(6) & \longrightarrow & SO(7) \\
& & \longrightarrow S^6
\end{array}
\]

where the horizontal lines are principal fibre bundles and \(p(g) = \pi(g)e_1\).

Lemma 4.1 of [9] implies the following lemma immediately.

**Lemma 2.5.** Put \(V^6 = D^3 \times D^2 \times D^1\). Then the composite map \(p \varphi_0 : (V^6, \partial V^6) \rightarrow (S^6, \{e_1\})\) is a relative homeomorphism.

Now we can state one of our main results.

**Theorem 2.6.** The cell complex \(e^0 \cup e^3 \cup e^5 \cup e^6 \cup e^7 \cup e^8 \cup e^9 \cup e^{10} \cup e^{11} \cup e^{12} \cup e^{13} \cup e^{14} \cup e^{15} \cup e^{16} \cup e^{18} \cup e^{21}\) gives a cellular decomposition of \(Spin(7)\).

**Proof.** First we show that \(e^i \cap e^j = \emptyset\) if \(i \neq j\). We consider the following three cases:

1. For the case where \(i, j \in \{0, 3, 5, 7, 8, 10, 12, 15\}\): both cells \(e^i\) and \(e^j\) are in \(SU(4)\) and \(e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^9 \cup e^{10} \cup e^{11} \cup e^{12}\) is a cellular decomposition of \(SU(4)\), whence we have \(e^i \cap e^j = \emptyset\) if \(i \neq j\).

2. For the case where \(i \in \{0, 3, 5, 7, 8, 10, 12, 15\}\) and \(j \in \{6, 9, 11, 13, 14, 16, 18, 21\}\): we have \(p(e^i) = \{e_1\}\) and \(p(e^j) = S^6 \setminus \{e_1\}\), whence we have \(e^i \cap e^j = \emptyset\).

3. For the case where \(i, j \in \{6, 9, 11, 13, 14, 16, 18, 21\}\), suppose that \(A \in e^i \cap e^j\). Since \(e^i = e^6 e^{i-6}\) and \(e^j = e^6 e^{j-6}\), we can put \(A = A_1 A_2 = A'_1 A'_2\), where \(A_1, A'_1 \in e^6, A_2 \in e^{i-6}\) and \(A'_2 \in e^{j-6}\). We have \(A_1 = A'_1\), since \(p(A_1) = p(A_1 A_2) = p(A'_1 A'_2) = p(A'_1)\) and \(p|_{e^6}\) is monic. Then we have \(A_2 = A'_2\) and the first case shows that \(i - 6 = j - 6\), that is, \(i = j\). Thus \(e^i \cap e^j = \emptyset\) if \(i \neq j\).
Next, we will check that the boundaries of the cells are included in the lower dimensional cells. In Proposition 2.3, it is proved that the boundaries of the cells of $SU(4)$ are included in the lower dimensional cells. In Proof of Theorem 4.2 in [9], we showed that $e^6 \subset e^3 \cup e^5$, $e^9 \subset e^6 \cup e^8$, $e^{11} \subset e^5 \cup e^9$ and $e^{14} \subset e^8 \cup e^9 \cup e^{11}$.

By using (2) of Remark 2.4, we also obtain

\[
\begin{align*}
\hat{e}^{13} &= e^6 e^7 \cup e^6 e^7 \subset e^{11} \cup e^{12}, \\
\hat{e}^{16} &= e^6 e^7 e^5 \cup e^6 e^7 e^3 \subset e^{13} \cup e^{14} \cup e^{15}, \\
\hat{e}^{18} &= e^6 e^7 e^5 \cup e^6 e^7 e^3 \subset e^{16} \cup e^{14} \cup e^{15}, \\
\hat{e}^{21} &= e^6 e^7 e^3 \subset e^{18} \cup e^{16} \cup e^{14} \cup e^{15}.
\end{align*}
\]

Finally, we will show that the inclusion map $e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{11} \cup e^{12} \cup e^{13} \cup e^{14} \cup e^{15} \cup e^{16} \cup e^{18} \cup e^{21} \to Spin(7)$ is epic. Let $g \in Spin(7)$. If $p(g) = e_1$, then $g$ is contained in $SU(4) = e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}$. Suppose that $p(g) \neq e_1$. There is an element $h \in e^6$ such that $p(h) = p(g)$. Thus we have $h^{-1}g \in SU(4)$ since $p(h^{-1}g) = e_1$. Therefore we have $g \in h(e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}) \subset e^0 \cup e^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{11} \cup e^{12} \cup e^{13} \cup e^{14} \cup e^{15} \cup e^{16} \cup e^{18} \cup e^{21}$. □

**Remark 2.7.** Araki [1] also gave a cellular decomposition of $Spin(n)$, but the one we have given here is a cellular decomposition with the minimum number of cells, satisfying the Yokota principle ([17], [18], [19]). As will be seen later, it is effectively used to determine the Lusternik-Schnirelmann category.

It is easy to give a cellular decomposition of $Spin(8)$ using a homeomorphism $Spin(8) \to Spin(7) \times S^7$.

3. **The cone-decomposition of SU(4)**

Obviously there is a filtration $F'_0 = \ast \subset F'_1 = SU(4)^{(7)} \subset F'_2 = SU(4)^{(12)} \subset F'_3 = SU(4)$. It is well-known that $F'_1 = \Sigma CP^3 = S^3 \cup e^5 \cup e^7$ and $F'_2 = F'_1 \cup e^8 \cup e^{10} \cup e^{12}$. Thus the integral cohomology $H^n(F'_2; \mathbb{Z})$ is given by

\[
H^n(F'_2; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}(1) & (n = 0) \\
\mathbb{Z}(y_n) & (n = 3, 5, 7, 8, 10, 12) \\
0 & (otherwise).
\end{cases}
\]

The action of the squaring operation $Sq^2$ is given as follows:

\[
Sq^2 y_n = \begin{cases} 
y_{n+2} & \text{for } n = 3, 10, \\
0 & \text{for } n = 5, 7, 8, 12
\end{cases}
\]

where $y_n$ is regarded as an element of the mod 2 cohomology. To give the cone decomposition of $SU(4)$, we use the following homotopy fibration:

\[
(3.1) \quad F \xrightarrow{\Psi} F'_1 \xrightarrow{\iota} F'_2.
\]
Without loss of generality, we may regard this as a Hurewicz fibration over $F'_2$.

Firstly we consider the Serre spectral sequence $(E^r_{pq}, d_r)$ associated with the above fibration, where the generators of $E^r_{pq}$ for $r \leq 7$ are permanent cycles and survive to $E_\infty$-terms. Hence $F$ is 6-connected and the transgression $\tau : H^7(F; \mathbb{Z}) \to H^8(F'_2; \mathbb{Z})$ is an isomorphism to $H^8(F'_2; \mathbb{Z}) \cong \mathbb{Z}(y_8)$. Thus $H^7(F; \mathbb{Z}) \cong \mathbb{Z}(x_7)$ for some $x_7 \in H^7(F; \mathbb{Z})$. Similarly, the generators in $E^3_{2,7} \cong \mathbb{Z}(y_3 \otimes x_7)$ and $E^4_{2,0} \cong H^10(F'_2; \mathbb{Z}) \cong \mathbb{Z}(y_{10})$ must lie in the image of differentials $d_3$ and $d_{10} = \tau : H^9(F; \mathbb{Z}) \to H^{10}(F'_2; \mathbb{Z})$ respectively, and we have that $H^8(F; \mathbb{Z}) = 0$ and $H^9(F; \mathbb{Z}) \cong \mathbb{Z}(x_9) \oplus \mathbb{Z}(x'_9)$, where the elements $x_9$ and $x'_9$ in $H^9(F; \mathbb{Z})$ are corresponding to $x_{10}$ and $y_3 \otimes x_7$ by the transgression $\tau$ and $d_3$ respectively. We remark that the choice of the generator $x'_9$ is not unique. Continuing this process, we have that $H^{10}(F; \mathbb{Z}) = 0$ and $H^{11}(F; \mathbb{Z}) \cong \mathbb{Z}(x_{11}) \oplus \mathbb{Z}(x'_{11}) \oplus \mathbb{Z}(x''_{11}) \oplus \mathbb{Z}(x_{11})$ whose generators are corresponding to $x_{12}$, $y_3 \otimes x_9$, $y_3 \otimes x'_9$ and $y_5 \otimes x_7$ respectively by the transgression $\tau$ and differentials $d_3$, $d_3$ and $d_5$.

Thus the integral cohomology $H^n(F; \mathbb{Z})$ for $0 \leq n \leq 11$ is given by

$$H^n(F; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}(1) & (n = 0) \\ \mathbb{Z}(x_7) & (n = 7) \\ \mathbb{Z}(x_9) \oplus \mathbb{Z}(x'_9) & (n = 9) \\ \mathbb{Z}(x_{11}) \oplus \mathbb{Z}(x'_{11}) \oplus \mathbb{Z}(x''_{11}) & (n = 11) \\ 0 & \text{(otherwise)} \end{cases}$$

where $x_7$, $x_9$ and $x_{11}$ are transgressive generators in $H^*(F; \mathbb{Z})$. Hence $F$ has, up to homotopy, a cellular decomposition $e^0 \cup e^7 \cup e'_1 \cup e''_1 \cup e_{11} \cup \text{cells in dimensions } \geq 11$, where the cells $e^7$, $e^9$ and $e^{11}$ correspond to $x_7$, $x_9$ and $x_{11}$ respectively. Then we obtain a subcomplex $A' = e^0 \cup e^7 \cup e'_1 \cup e''_1 \cup e_{11} \cup e_{11}$ of $F$.

Secondly, we determine the attaching maps $\varphi_1$ and $\varphi'_1$: Let us recall that $\pi_8(S^7) \cong \mathbb{Z}/2(2\eta)$ whose generator $\eta$ can be detected by $Sq^2$, the mod 2 Steenrod operation. Since the action of mod 2 Steenrod operation commutes with the cohomology transgression (see [8, Proposition 6.5]), we see that $Sq^2 x_7$ is transgressive, and hence is $c x_9$ for some $c \in \mathbb{Z}/2$. We know that $\tau x_9 = y_{10} \neq 0$ and $\tau Sq^2 x_7 = Sq^2 \tau x_7 = Sq^2 y_8 = 0$, and hence $Sq^2 x_7$ must be trivial. Thus the attaching maps $\varphi_1$ and $\varphi'_1$ are both null homotopic and $A'$ is homotopy equivalent to $(S^7 \setminus S^0 \setminus S'_1) \cup e_{11}$.

Thirdly we check the composition of projections with the attaching map $\varphi_2 : S^{10} \to S^7 \setminus S^0 \setminus S'_1$ to $S^9$ and $S_0^9$, which can also be detected by $Sq^2$. Again by the commutativity of the action of mod 2 Steenrod operation with the transgression, we see that the composition map $pr_{S^9} \circ \varphi_2 : S^{10} \to S^7 \setminus S^0 \setminus S'_1 \to S^9$ represents a generator of $\pi_9(S^9) \cong \mathbb{Z}/2(2\eta)$, since $Sq^2 : H^8(F'_2; \mathbb{Z}/2) \to H^{10}(F'_2; \mathbb{Z}/2)$ is non-trivial. If the composition map $\varphi_1 = pr_{S'_1} \circ \varphi_2 : S^{10} \to S^7 \setminus S^9 \setminus S'_1 \to S^9$ is non-trivial, we replace $\varphi_2$ by the composition of $\varphi_2$ and the homotopy equivalence
ξ : S^7 ∪ S^6 ∪ S^4 \rightarrow S^7 ∪ S^6 ∪ S^4 where ξ|_{S^7} and ξ|_{S^6} are the identity maps and ξ|_{S^4} is the unique co-H-structure map φ : S^6 ∪ S^6 ∪ S^4; then we obtain that φ_1 is trivial, since 2\eta_0 = 0. Then A' is homotopy equivalent to ((S^7 ∪ S^6) ∪_{\varphi_2} \varepsilon_{11}) ∪ S^6. Let A denote the subcomplex (S^7 ∪ S^6) ∪_{\varphi_2} \varepsilon_{11} of A' and \psi = Ψ|_A : A → F'_2.

Lemma 3.1. F'_2 is homotopy equivalent to F'_1 ∪_ψ CA.

Proof. The image of H^*(A; \mathbb{Z}) in H^*(F'; \mathbb{Z}) under the induced map of the inclusion coincides with the module of transgressive elements with respect to the fibration (3.1) (see [8, Chapter 6]). Thus we may regard that H^{n-1}(A; \mathbb{Z}) = \delta^{-1}(i^*(H^n(F'_1; \mathbb{Z}))) \subset H^{n-1}(F'; \mathbb{Z}):

\[ H^{n-1}(F'; \mathbb{Z}) \xrightarrow{\delta_F} H^n(F'_1, F'; \mathbb{Z}) \xleftarrow{\iota^*_F} H^n(F'_2; \mathbb{Z}) \]

where \iota_F and \delta_F denote the connecting homomorphisms of the long exact sequences for the pairs (F'_1, F) and (F'_2, A), respectively. Thus the image of \delta_A is contained in the image of \iota'_A and we also have

\[ H^n(F'_1, A; \mathbb{Z}) \cong H^n(F'_1 \cup_\psi CA, CA; \mathbb{Z}) \cong H^n(F'_1 \cup_\psi CA, *, \mathbb{Z}). \]

Since the composition map A \xrightarrow{\cong} F'_1 \xrightarrow{i} F'_2 is trivial, we can define a map

\[ f : F'_1 \cup_\psi CA \rightarrow F'_2, \]

by f|_{F'_1} = i : F'_1 → F'_2 and f|_{CA} = *. In order to prove the lemma, we show that f^* : H^n(F'_2; \mathbb{Z}) \cong \mathbb{Z} → H^n(F'_1 \cup_\psi CA; \mathbb{Z}) \cong \mathbb{Z} is an isomorphism for n = 3, 5, 7, 8, 10, 12. We have a commutative diagram

\[ H^n(F'_2; \mathbb{Z}) \xrightarrow{i^*} H^n(F'_1; \mathbb{Z}) \]

\[ f^* \]

\[ H^n(F'_1 \cup CA, F'_1; \mathbb{Z}) \xrightarrow{f^*} H^n(F'_1 \cup CA; \mathbb{Z}) \xrightarrow{\iota^*_F} H^n(F'_1; \mathbb{Z}), \]

where the bottom row is a part of the exact sequence for the pair (F'_1 \cup CA, F'_1). The induced map i^* is an isomorphism for n ≤ 7, since H^n(F'_1 \cup CA, F'_1; \mathbb{Z}) = 0 for n ≤ 7 and since i^* is an isomorphism for n ≤ 7. Then we obtain that f^* is an isomorphism for n ≤ 7. Moreover we can show that j^* : H^n(F'_1 \cup CA, F'_1; \mathbb{Z}) → H^n(F'_1 \cup CA; \mathbb{Z}) is an isomorphism for n ≥ 8, by considering the exact sequence for the pair (F'_1 \cup CA, F'_1), since we have H^n(F'_1) = 0 for n ≥ 8. To perform the other cases for n = 8, 10, 12, it is sufficient to show that f^* is surjective. In fact, we have a commutative diagram
Proof. The cohomology of $H^{n-1}(A; \mathbb{Z})$ implies that $\text{Cat}(F^n) = \text{Cat}(F^n_0)$ is a relative homeomorphism. Then we obtain the cell $\Sigma$ is the suspension isomorphism. Since $j^*$ is an isomorphism for $n \geq 8$, we obtain that $\delta_A$ is an isomorphism for $n \geq 8$. Since the image of $\delta_A$ is contained in the image $i_A^n$, we see that $f^*$ is surjective for $n \geq 8$, and hence $f$ is a homotopy equivalence. □

Proposition 3.2. We have $\text{wcat}(F^n_i) = \text{cat}(F^n_i) = \text{Cat}(F^n_i) = i$.

Proof. The cohomology of $F^n_i$ implies that $\text{wcat}(F^n_i) \geq i$. The cone-decomposition

$F^n_i = \Sigma CP^3$, $F^n_2 \simeq F^n_i \cup CA$, $F^n_3 = F^n_2 \cup CS^{14}$

implies that $\text{Cat}(F^n_i) \leq i$, which completes the proof. □

4. PROOF OF THEOREM 1.1

We define a filtration $F_0 = \ast \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset F_5 = \text{Spin}(7)$ by

$F_1 = SU(4)^{(7)}$, $F_2 = SU(4)^{(12)} \cup e^6$, $F_3 = SU(4) \cup e^6 \cup e^9 \cup e^{11} \cup e^{13}$, $F_4 = \text{Spin}(7)^{(18)}$.

We need the following lemma to prove Theorem 4.2.

Lemma 4.1. We have a homeomorphism of pairs

$(CA_1, A_1) \times (CA_2, A_2) = (C(A_1 \ast A_2), A_1 \ast A_2)$.

(The proof can be found in p.482-483 of [16].)

Now Theorem 1.1 follows from the following theorem.

Theorem 4.2. We have $\text{wcat}(F_i) = \text{cat}(F_i) = \text{Cat}(F_i) = i$.

Proof. The mod 2 cohomology of $F_i$ implies that $\text{wcat}(F_i) \geq i$. Then it is sufficient to show that $\text{Cat}(F_i) \leq i$. Obviously we have a homeomorphism $F_1 = \Sigma CP^3$. Since the cell $e^6$ is attached to $F_1$, we obtain that $F_2 \simeq F_1 \cup C(S^5 \cup A)$ using Lemma 3.1.

Since we have $e^9 \cup e^{11} \cup e^{13} = e^6(e^3 \cup e^5 \cup e^7)$, the composition map

$(CS^8, S^5) \times (CP^3, CP^3) \rightarrow (CS^8, S^5) \times (\Sigma CP^3, *) \rightarrow (F_2 \cup e^9 \cup e^{11} \cup e^{13}, F_2)$

is a relative homeomorphism. Then we obtain $F_2 \cup e^9 \cup e^{11} \cup e^{13} = F_2 \cup C(S^5 \ast CP^3)$ using Lemma 4.1. The cell $e^{15}$ is the highest dimensional cell of $SU(4)$ and is attached to $F_2$. Then we obtain $F_3 \simeq F_2 \cup C(S^{14} \cup (S^5 \ast CP^3))$. Now we consider the following composition map:

$(C(S^5 \ast A), S^5 \ast A) = (CS^8, S^5) \times (CA, A) \rightarrow (CS^8, S^5) \times (F_2, F_1) \rightarrow (F_4, F_3)$. 

Since we have $e^{14} \cup e^{16} \cup e^{18} = e^6(e^8 \cup e^{10} \cup e^{12})$, the right map is a relative homeomorphism. The left map induces an isomorphism of homologies of pairs so that the map $H_\ast(F_3 \cup C(S^5 \ast A), F_3; \mathbb{Z}) \to H_\ast(F_4, F_3; \mathbb{Z})$ is an isomorphism. Thus we obtain $F_4 \simeq F_3 \cup C(S^5 \ast A)$. Obviously we have a homeomorphism $F_3 = F_4 \cup CS^{20}$. □

References
