Lusternik-Schnirelmann category of a sphere-bundle over a sphere

Dedicated to Professor J. R. Hubbuck on his 60th birthday

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Abstract

We determine the L-S category of a total space of a sphere-bundle over a sphere in terms of primary homotopy invariants of its characteristic map, and thus providing a complete answer to Ganea’s Problem 4. As a result, we obtain a necessary and sufficient condition for a total space $N$ to have the same L-S category as its ‘once punctured submanifold’ $N \setminus \{ P \}$, $P \in N$. Also necessary and sufficient conditions for a total space $M$ to satisfy Ganea’s conjecture are described.

Key words: Lusternik-Schnirelmann category, higher Hopf invariant, sphere bundle, manifold, Ganea conjecture.


1 Introduction

The (normalised) L-S category $\text{cat}(X)$ of $X$ is the least number $m$ such that there is a covering of $X$ by $m + 1$ open subsets each of which is contractible in $X$, which equals to the least number $m$ such that the diagonal map $\Delta_{m+1} : X \to \prod^{m+1} X$ can be compressed into the ‘fat wedge’ $T^{m+1}(X)$ (see James [8] and Whitehead [21]). By definition, we have $\text{cat}(*)) = 0$.

A simple definition, however, does not always suggest a simple way of calculation. In fact, to determine the L-S category of a sphere-bundle over a sphere in terms of homotopy invariants of its characteristic map is listed as Problem 4 of...
Ganea [2] in 1971. Ganea’s Problem 2 is also a basic problem on $\text{cat}(X \times S^n)$, where we easily see that $\text{cat}(X \times S^n) = \text{cat}(X)$ or $\text{cat}(X) + 1$: Can the latter case only occur on any $X$ and $n \geq 1$? The affirmative answer had become known as “the Ganea conjecture” (see James [9]), particularly for manifolds.

Although a tight connection between L-S category and the Bar resolution ($A_\infty$-structure) has been pointed out by Ginsburg [3] in 1963, a homological approach could not succeed to solve Ganea’s problems on L-S category. By Singhof [18] followed by Montejano [11], Gómez-Larrañaga and González-Acuña [4], Rudyak [16,17] and Oprea and Rudyak [15], the conjecture is validated for a large class of manifolds. The first closed manifold counter-example to the conjecture was given by the author [7] as a total space of a sphere-bundle over a sphere, using the $A_\infty$-method with concrete computations of Toda brackets depending on results by Toda [20] and Oka [14]. Also, Lambrechts, Stanley and Vandembroucq [10] and the author [7] provided manifolds each of which has the same L-S category as its once punctured submanifold.

The purpose of this paper is to determine the L-S category of a sphere-bundle over a sphere in terms of a primary homotopy invariant of the characteristic map of a bundle, providing simpler proofs of manifold examples in [7]. Using it, we could obtain many closed manifolds each of which has the same L-S category as its once punctured submanifold and many closed manifold counter-examples to Ganea’s conjecture on L-S category.

Throughout this paper, we follow the notations in [6,7]: In particular for a map $f : S^k \to X$, a homotopy set of higher Hopf invariants $H^S_m(f) = \{ [H^S_m(f)] | \sigma$ is a structure map of $\text{cat}(X) \leq m \}$ (or its stabilisation $H^S_m(f) = \Sigma^\infty H^S_m(f)$) is referred simply as a (stabilised) higher Hopf invariant of $f$, which plays a crucial role in this paper. For a sphere map $f : S^k \to S^\ell$ with $k, \ell > 1$, we identify $H^S_1(f)$ and $H^S_1(f)$ with their unique elements, $H_1(f)$ and $H_1(f) = \Sigma^\infty H_1(f)$, since a sphere $S^n$ has the unique structure $\sigma(S^n) : S^n \to \Sigma \Omega S^n$ for $\text{cat}(S^n) = 1$, $n > 1$.

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2 L-S category of a sphere-bundle over a sphere

Let $r \geq 1, t \geq 0$ and $E$ be a fibre bundle over $S^{t+1}$ with fibre $S^r$. Then $E$ can be described as $S^r \cup_{\Psi} S^r \times D^{t+1}$, with $\Psi : S^r \times S^t \to S^r$ (see Whitehead [21]). Hence $E$ has a CW decomposition $S^r \cup_{\alpha} e^{t+1} \cup_{\Psi} e^{r+t+1}$ with $\alpha : S^t \to S^r$ and
\[ \psi : S^{r+t} \to Q = S^r \cup_{\alpha} e^{t+1} \] given by the following formulae:

\[
\alpha = \Psi|((*) \times S^t)\text{, } \psi|\Sigma_{r-1}^{*} \times D^{t+1} = \chi_{\alpha} \circ \text{pr}_2, \quad \psi|D^r \times S^t = \Psi_{\omega_r \times 1_S} (\cdot),
\]

where we denote by \( \chi_f : (C(A), A) \to (C_f, B) \) the characteristic map for \( f : A \to B \) and let \( \omega_r = \chi_{(*) \times S^{r-1} \to (* \times 1)} \). When \( r = 1 \), the L-S categories of \( E \) and \( Q \) are studied by several authors; especially by Singhof [18] and Oprea-Rudyak [15] in the case when \( r = t = 1 \). We summarise known results.

**Fact 2.1** Let \( r = 1 \). Then we have the following.

\[
\begin{align*}
(t = 0) & \quad \text{cat}(Q \times S^n) = 2, \text{cat}(Q) = 1, \text{cat}(E) = 2, \text{cat}(E \times S^n) = 3. \\
(t = 1, \alpha = \pm 1) & \quad \text{cat}(Q \times S^n) = 1, \text{cat}(Q) = 0, \text{cat}(E) = 1, \text{cat}(E \times S^n) = 2. \\
(t = 1, \alpha = 0) & \quad \text{cat}(Q \times S^n) = 2, \text{cat}(Q) = 1, \text{cat}(E) = 2, \text{cat}(E \times S^n) = 3. \\
(t = 1, \alpha \neq 0, \pm 1) & \quad \text{cat}(Q \times S^n) = 3, \text{cat}(Q) = 2, \text{cat}(E) = 3, \text{cat}(E \times S^n) = 4. \\
(t > 1) & \quad \text{cat}(Q \times S^n) = 2, \text{cat}(Q) = 1, \text{cat}(E) = 2, \text{cat}(E \times S^n) = 3.
\end{align*}
\]

When \( r > 1 \), we identify \( H^S_1(\alpha) \) with its unique element \( H_1(\alpha) \). We summarise the known results (due to Berstein-Hilton [1]) from [7, Facts 7.1, 7.2].

**Fact 2.2** Let \( r > 1 \). Then we have the following.

\[
\begin{align*}
(t < r) & \quad \text{cat}(Q \times S^n) = 2, \text{cat}(Q) = 1, \text{cat}(E) = 2, \text{cat}(E \times S^n) = 3. \\
(t = r, \alpha = \pm 1_{S^r}) & \quad \text{cat}(Q \times S^n) = 1, \text{cat}(Q) = 0, \text{cat}(E) = 1, \text{cat}(E \times S^n) = 2. \\
(t = r, \alpha \neq \pm 1_{S^r}) & \quad \text{cat}(Q \times S^n) = 2, \text{cat}(Q) = 1, \text{cat}(E) = 2, \text{cat}(E \times S^n) = 3. \\
(t > r, H_1(\alpha) = 0) & \quad \text{cat}(Q \times S^n) = 2, \text{cat}(Q) = 1, \text{cat}(E) = 2, \text{cat}(E \times S^n) = 3. \\
(t > r, H_1(\alpha) \neq 0) & \quad \text{cat}(Q \times S^n) = 3 \text{ or } 2, \text{cat}(Q) = 2, \text{cat}(E) = 2 \text{ or } 3, \text{cat}(E \times S^n) = 3 \text{ or } 4.
\end{align*}
\]

By [6] and [7, Theorem 5.2, 5.3, 7.3], the following is also known.

**Fact 2.3** When \( r > 1 \), \( t \geq r \) and \( \alpha \neq \pm 1 \), we also have the following.

\[
\begin{align*}
(1) & \quad \Sigma^n H_1(\alpha) = 0 \implies \text{cat}(Q \times S^n) = 2, \text{and } \Sigma^{n+1} H_1(\alpha) \neq 0 \implies \text{cat}(Q \times S^n) = 3. \\
(2) & \quad \text{cat}(E) = 2 \text{ if and only if } H^S_2(\psi) \ni 0, \text{and } \text{cat}(E) = 2 \implies \text{cat}(E \times S^n) = 3 \text{ for all } n. \\
(3) & \quad \Sigma^n H^S_2(\psi) \ni 0 \implies \text{cat}(E \times S^n) = 3, \text{and } \Sigma^{n+r+1} h_2(\alpha) \neq 0 \implies \text{cat}(E \times S^n) = 4.
\end{align*}
\]

**Remark 2.4** When \( \alpha \) is in meta-stable range, \( H_1(\alpha) : S^t \to \Omega S^r \ast \Omega S^r \) is given by the second James-Hopf invariant \( h_2(\alpha) : S^t \to \Sigma S^{r-1} \wedge S^{r-1} \) composed with an appropriate inclusion to a wedge-summand. Thus we may regard
h_2(\alpha) = H_1(\alpha) \text{ when } \alpha \text{ is in meta-stable range.}

Our main result is as follows:

**Theorem 2.5** Let cat(Q) = 2 with t > r > 1, Then $H^S_2(\psi)$ contains 0 if and only if $\Sigma^r H_1(\alpha) = 0$. More generally for a co-H-map $\beta : S^v \to S^{r+t}$ with $v < t + 2r - 1$, $H^S_2(\psi \circ \beta) = \beta^* H^S_2(\psi)$ contains 0 if and only if $\Sigma^r H_1(\alpha) \circ \beta = 0$.

The main result is obtained by the following lemma for $Q$ of cat $(Q) = 2$ with $t > r > 1$.

**Lemma 2.6** $H^S_2(\psi) \ni \pm [(\hat{i}^* \Omega_Q, \Omega_Q) \circ \Sigma^r H_1(\alpha)]$, where the bottom-cell inclusion $\hat{i} : S^{r-1} \to \Omega Q$ denotes the adjoint of the inclusion $i : S^r \to Q$.

By combining above facts with Theorem 2.5, we obtain an answer to Ganea’s Problem 4:

**Theorem 2.7 (Table of L-S categories)** For an $S^r$-bundle $E$ over $S^{t+1}$ and its once-punctured submanifold $E \setminus \{P\} \simeq Q$, we have the following table:

<table>
<thead>
<tr>
<th>Conditions</th>
<th>L-S categories</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = 1$</td>
<td></td>
</tr>
<tr>
<td>$t = 0$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$t = 1$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$\alpha = \pm 1$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$\alpha = 0$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$\alpha \neq 0, \pm 1$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$t &gt; 1$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$r &gt; 1$</td>
<td></td>
</tr>
<tr>
<td>$t &lt; r$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$t = r$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$\alpha = \pm 1$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$\alpha \neq \pm 1$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$H_1(\alpha) = 0$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$H_1(\alpha) \neq 0$ &amp; $\Sigma^r H_1(\alpha) = 0$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$\Sigma^r H_1(\alpha) \neq 0$</td>
<td>$Q \times S^n$, $Q$, $E$, $E \times S^n$</td>
</tr>
<tr>
<td>$3$ or $2$ (1)</td>
<td>$2$, $3$</td>
</tr>
<tr>
<td>$2$, $3$ (2)</td>
<td>$2$, $3$</td>
</tr>
</tbody>
</table>

(1): $\Sigma^n H_1(\alpha) = 0$ implies $\text{cat}(Q \times S^n) = 2$ and $\Sigma^{n+1} H_1(\alpha) \neq 0$ implies $\text{cat}(Q \times S^n) = 3$.
(2): $\Sigma^{n+1} H_1(\alpha) = 0$ implies $\text{cat}(E \times S^n) = 3$ and $\Sigma^{n+2} H_2(\alpha) \neq 0$ implies $\text{cat}(E \times S^n) = 4$.  

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3 Applications and examples

Firstly, Theorem 2.7 yields the following result.

**Theorem 3.1** Let a manifold \( N \) be the total space of a \( S^r \)-bundle over \( S^{t+1} \) with a characteristic map \( \Psi : S^r \times S^t \to S^r, t > r > 1 \), and let \( \alpha = \Psi|_{S^t} \). Then \( \text{cat}(N \setminus \{ P \}) = \text{cat}(N) \) if and only if \( H_1(\alpha) \neq 0 \) and \( \Sigma^r H_1(\alpha) = 0 \).

This theorem provides the following examples.

**Example 3.2** Let \( p \) be an odd prime and \( \alpha = \eta_2 \circ \alpha_1(3) \circ \alpha_1(2p) \). Then we have that \( H_1(\alpha) = \alpha_1(3) \circ \alpha_1(2p) \neq 0 \) and \( \Sigma^2 H_1(\alpha) = 0 \) by [20]. Let \( N_p \to S^{4p-2} \) be the bundle with fibre \( S^2 \) induced by \( \Sigma(\alpha_1(3) \circ \alpha_1(2p)) : S^{4p-2} \to S^4 \) from the bundle \( CP^2 \to HP^1 = S^4 \) with fibre \( Sp(1)/U(1) = S^2 \). By the arguments given in [7], we obtain that \( N_p \) has a CW-decomposition as \( N_p \approx S^2 \cup e^{4p-2} \cup e^{4p} \). Then Theorem 3.1 implies that \( \text{cat}(N_p) = \text{cat}(N_p \setminus \{ P \}) = 2 \).

**Example 3.3 ([7])** Let \( p \) be a prime \( \geq 5 \) and \( \alpha = \eta_2 \circ \alpha_1(3) \circ \alpha_2(2p) \) as in [7]. Then we have that \( H_1(\alpha) = \alpha_1(3) \circ \alpha_2(2p) \neq 0 \) and \( \Sigma^2 H_1(\alpha) = 0 \) by [20]. Let \( L_p \to S^{6p-4} \) be the bundle with fibre \( S^2 \) induced by \( \Sigma(\alpha_1(3) \circ \alpha_2(2p)) : S^{6p-4} \to S^4 \) from the bundle \( CP^3 \to HP^1 = S^4 \) with fibre \( Sp(1)/U(1) = S^2 \). By the arguments given in [7], we obtain that \( L_p \) has a CW-decomposition as \( L_p \approx S^2 \cup e^{6p-4} \cup e^{6p-2} \). Then Theorem 3.1 implies that \( \text{cat}(L_p) = \text{cat}(L_p \setminus \{ P \}) = 2 \).

Secondly, Theorem 2.7 also yields the following result.

**Theorem 3.4** Let a manifold \( M \) be the total space of a \( S^r \)-bundle over \( S^{t+1} \) with a characteristic map \( \Psi : S^r \times S^t \to S^r, t > r > 1 \), and let \( \alpha = \Psi|_{S^t} \). If \( \Sigma^r H_1(\alpha) \neq 0 \) and \( \mathcal{H}_t(\alpha) = 0 \), then \( M \) is a counter-example to the Ganea’s conjecture on \( L \)-\( S \) category; more precisely, \( \text{cat}(M) = \text{cat}(M \times S^n) = 3 \) if \( \Sigma^r H_1(\alpha) \neq 0 \) and \( \Sigma^{r+t} H_1(\alpha) = 0 \).

This theorem provides the following manifold counter examples to Ganea’s conjecture on \( L \)-\( S \) category.

**Example 3.5** Let \( p = 2 \) and \( \alpha = \eta_2 \circ \eta_3 \circ \epsilon_5 \). Then we have that \( H_1(\alpha) = \eta_3 \circ \epsilon_5 \neq 0 \), \( \Sigma^2 H_1(\alpha) \neq 0 \) and \( \Sigma^6 H_1(\alpha) = 0 \) by [20]. Let \( M_2 \to S^{14} \) be the bundle with fibre \( S^2 \) induced by \( \Sigma(\eta_3 \circ \epsilon_5) : S^{14} \to S^4 \) from the bundle \( CP^3 \to HP^1 = S^4 \) with fibre \( Sp(1)/U(1) = S^2 \). By the arguments given in [7] we obtain that \( M_2 \) has a CW-decomposition as \( M_2 \approx S^2 \cup e^{14} \cup e^{16} \). Then Theorem 3.4 implies that \( \text{cat}(M_2 \times S^n) = \text{cat}(M_2) = 3 \) for \( n \geq 4 \).

**Example 3.6 ([7])** Let \( p = 3 \) and \( \alpha = \eta_2 \circ \alpha_1(3) \circ \alpha_2(6) \) as in [7]. Then we have that \( H_1(\alpha) = \alpha_1(3) \circ \alpha_2(6) \neq 0 \), \( \Sigma^2 H_1(\alpha) \neq 0 \) and \( \Sigma^4 H_1(\alpha) = 0 \) by [20].
Let $M_3 \to S^{14}$ be the bundle with fibre $S^2$ induced by $\Sigma(\alpha_1(3)\circ\alpha_2(6)) : S^{14} \to S^4$ from the bundle $\mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$ with fibre $S^2$. By the arguments given in [7] we obtain that $M_3$ has a CW-decomposition as $M_3 \approx S^2 \cup \alpha e^{14} \cup \psi e^{16}$. Then Theorem 3.4 implies that $\text{cat}(M_3 \times S^n) = \text{cat}(M_3) = 3$ for $n \geq 2$.

Finally, Theorem 2.5 and [7, Theorem 5.2] imply the following result.

**Theorem 3.7** Let a manifold $X$ be the total space of a $S^r$-bundle over $S^{t+1}$ with a characteristic map $\Psi : S^r \times S^t \to S^r$, $t > r > 1$, and let $\alpha = \Psi|_{S^r}$. When $H_1(\alpha) \neq 0$ and $\beta$ is a co-H-map, we obtain that $X(\beta) = S^r \cup_\alpha e^{t+1} \cup_{\psi_0\beta} e^{r+1}$ is of $\text{cat}(X(\beta)) = 3$ if and only if $\Sigma^r H_1(\alpha) \circ \beta \neq 0$.

**Remark 3.8** All examples obtained here still support the conjecture in [6].

4 Proof of Lemma 2.6

Let $\text{cat}(Q) = 2$ with $t > r > 1$. In the remainder of this paper, we distinguish a map from its homotopy class to make the arguments clear.

Here, let us recall the definition of a relative Whitehead product: For maps $f : \Sigma X \to M$ and $g : (C(Y), Y) \to (K, L)$, we denote by $[f, g]_{\text{rel}} : X \ast Y = C(X) \times Y \cup X \times C(Y) \to M \times L \cup \{\ast\} \times K$ the relative Whitehead product, which is given by

$$[f, g]_{\text{rel}}|_{C(X) \times Y}(t \wedge x, y) = (f(t \wedge x), g(y)) \quad \text{and} \quad [f, g]_{\text{rel}}|_{X \times C(Y)}(x, t \wedge y) = (\ast, g(t \wedge y)).$$

Also a pairing $F : M \times L \to M$ with axes $1_M$ and $h : L \to M$ (see Oda [13]) determines a map

$$(F \cup \chi_h) : (M \times L \cup \{\ast\} \times K) \to (M \cup_h K, M)$$

by $(F \cup \chi_h)|_{M \times L} = F$ and $(F \cup \chi_h)|_{\{\ast\} \times K} = \chi_h$, where $\chi_h : (K, L) \to (M \cup_h K, M)$ is a relative homeomorphism given by the restriction of the identification map $M \cup K \to M \cup_h K$. Then we can easily see that $\psi : S^{r+t} \to Q$ is given as

$$\psi = (\Psi \cup \chi_h) \circ [t_r, C(t_k)], \quad (4.1)$$

where $t_k : S^k \to S^k$ and $C(t_k) : C(S^k) \to C(S^k)$ denote the identity maps.

We denote by $j_i^Q : P^i(\Omega Q) \hookrightarrow P^\infty(\Omega Q)$ the classifying map of the fibration $\tilde{p}_i^Q : E^i+1(\Omega Q) \to P^i(\Omega Q)$ and $e_i^Q = e_\Sigma^Q j_i^Q$, where $e_i^Q : P^\infty(\Omega Q) \to Q$ is a homotopy equivalence extending the evaluation map $e_i^Q = e_v : \Sigma \Omega Q \to Q$. 

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Let $\sigma_\infty$ be the homotopy inverse of $e_\infty^Q$. Then we may assume that $\sigma_\infty|_{S^r} = j_1^Q \circ \sigma(S^r)$ for dimensional reasons.

**Proposition 4.1** The following without the dotted arrows is a commutative diagram where the lower squares are pull-back diagrams.

\[
\begin{array}{cccc}
E^3(\Omega Q) & \longrightarrow & \Omega Q \ast E^2(\Omega Q) & \longrightarrow & \Omega Q \ast \Omega Q \ast \Omega Q \\
\sigma_0 \circ j_1^Q \circ e_1^Q \circ \chi_{[\ast, \ast]} \rel & \downarrow & \downarrow \rel \sigma_1^Q \circ (e_1^Q \circ e_1^Q) \circ \chi_{[\ast, \ast]} \rel & & \downarrow \\
P^2(\Omega Q) & \longrightarrow & \Delta Q \ast \Omega Q \ast \Omega Q & \longrightarrow & T^3 Q \\
\Delta Q & \rightarrow & Q \ast Q & \rightarrow & Q \ast Q \ast Q \\
\end{array}
\]

(4.2)

Therefore, there is a lifting $\sigma'_0$ of $\Delta Q$ and hence a lifting $\sigma_0$ of the identity $1_Q$.

**Remark 4.2** The homotopy fibre $\Omega Q \ast \Omega Q \ast \Omega Q \to T^3 Q$ of the inclusion

\[T^3 Q = Q \times (Q \ast Q) \cup \{\ast\} \times (Q \times Q) \to Q \times (Q \times Q)\]

is given by a relative Whitehead product $[e_1^Q, (e_1^Q \circ e_1^Q) \circ \chi_{[\ast, \ast]}] \rel$, where $\iota$ denotes the identity $1_{\Sigma \Omega Q}$ and

\[\chi_{[\ast, \ast]} : \mathcal{C}(\Omega Q \ast \Omega Q), \Omega Q \ast \Omega Q) \to (\Sigma \Omega Q \ast \Sigma \Omega Q, \Sigma \Omega Q \ast \Sigma \Omega Q)\]

denotes a relative homeomorphism.

A lifting $\sigma'_0$ of $\Delta Q$ in diagram (4.2) is given by the following data:

\[\sigma'_0|_{S^r}(y) = ((j_1^Q \circ \sigma(S^r))(y), \sigma(S^r)(y)) \quad \text{for } y \in S^r,\]

and for $u \land x \in (0, 1] \times S^t \setminus \{1\} \times S^t = Q \setminus S^r$ with $\mu_t(x) = (x_1, x_2)$,

\[\sigma'_0|_{Q \setminus S^r}(u \land x) = \begin{cases} 
(j_1^Q \circ \sigma(S^r)) \circ \alpha \times \sigma(S^r) \circ \alpha) \circ H_1(2u \land x), & \text{if } u \leq \frac{1}{2} \\
(\hat{\chi}_\alpha(2u - 1, x_1), \hat{\chi}_\alpha(2u - 1, x_2)), & \text{if } u \geq \frac{1}{2},
\end{cases}\]

where $H_1$ is a homotopy $\Delta_{S^t} \sim \mu_t$ in $S^t \setminus S^t$, $\mu_k = \Sigma^{k-1}_{i=1} \mu_1 : S^k \to S^k \setminus S^k$ denotes the unique co-H-structure of $S^k$ and $\hat{\chi}_\alpha$ is a null-homotopy $\sigma_\infty \circ \chi_\alpha : (C(S^t), S^t) \to (Q, S^r) \to (P^\infty(\Omega Q), \text{im}(j_1^Q \circ \sigma(S^r)))$ of $j_1^Q \circ \sigma(S^r) \circ \alpha \sim \ast$.

Since the lower left square of diagram (4.2) is a homotopy pullback diagram, $\sigma'_0$ and the identity $1_Q$ defines a lifting $\sigma_0 : Q \to P^2(\Omega Q)$ of $1_Q$. 

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Proof of Proposition 4.1. By [6, Lemma 2.1] with \((X, A) = \(P^\infty(\Omega Q), \{\ast\}\), 
\((Y, B) = \(P^\infty(\Omega Q), \Sigma\Omega Q\), \(Z = P^\infty(\Omega Q)\) and \(f = g = 1_{P^\infty(\Omega Q)}\), we have the following homotopy pushout-pullback diagram:

\[
\begin{array}{ccc}
E^2(\Omega Q) & \longrightarrow & \{\ast\} \\
\downarrow \rho_1^Q & & \\
\Sigma\Omega Q & \longrightarrow & P^2(\Omega Q) \\
\downarrow \Delta Q_{\text{HPO}} & & \downarrow \Delta Q_{\Sigma\Omega Q} \\
Q & \longrightarrow & Q \times Q
\end{array}
\]

where we replaced \(P^\infty(\Omega Q)\) by \(Q\) in the bottom, since \(P^\infty(\Omega Q)\) is the homotopy equivalent with \(Q\) by \(e_{\infty}^Q : P^\infty(\Omega Q) \rightarrow Q\) and \(\sigma_{\infty} : Q \rightarrow P^\infty(\Omega Q)\).

By [6, Lemma 2.1] with \((X, A) = \(P^\infty(\Omega Q), \{\ast\}\), 
\((Y, B) = \(P^\infty(\Omega Q), \Sigma\Omega Q\), \(Z = \{\ast\}\) and \(f = g = \ast\), we have the following homotopy pushout-pullback diagram:

\[
\begin{array}{ccc}
\Omega Q \times E^2(\Omega Q) & \longrightarrow & \Omega Q \\
\downarrow \text{pr}_1 & & \\
E^2(\Omega Q) & \longrightarrow & \Omega Q \ast E^2(\Omega Q) \\
\downarrow \text{pr}_2 & & \downarrow \text{pr}_2 \big[Q \ast \delta_{\Omega Q}^E\big]_{\text{rel}} \\
\{\ast\} & \longrightarrow & Q \times Q
\end{array}
\]

where \(\chi_{\rho_1^Q} : (C(E^2(\Omega Q)), E^2(\Omega Q)) \rightarrow (P^2(\Omega Q), \Sigma\Omega Q)\) is a relative homeomorphism.

The above constructions give a standard \(\Omega Q\)-projective plane \(P^2(\Omega Q)\) and a standard projection \(p_2^{\Omega Q} : E^3(\Omega Q) \rightarrow P^2(\Omega Q)\). In fact, the diagonal map \(\Delta^3_Q : Q \rightarrow Q \times Q \times Q\) is the composition \((1_Q \times \Delta_Q) \circ \Delta_Q\) and there is the following homotopy pushout-pullback diagram by [6, Lemma 2.1] with \((X, A) = (Q, \{\ast\})\),

\[
\begin{array}{ccc}
\Omega Q \times E^2(\Omega Q) & \longrightarrow & \Omega Q \\
\downarrow \text{pr}_1 & & \\
E^2(\Omega Q) & \longrightarrow & \Omega Q \ast E^2(\Omega Q) \\
\downarrow \text{pr}_2 & & \downarrow \text{pr}_2 \big[Q \ast \delta_{\Omega Q}^E\big]_{\text{rel}} \\
\{\ast\} & \longrightarrow & Q \times Q
\end{array}
\]
(Y, B) = (Q × Q, Q ∨ Q), Z = Q × Q, f = pr₁ and g = Δ_{Q} \circ \text{pr}_2:

\begin{align*}
\begin{array}{c}
\{\ast\} \times \Sigma Q \\
HPO
\end{array}
\begin{array}{c}
\rightarrow \{\ast\} \times Q
\end{array}
\begin{array}{c}
\rightarrow P^\infty(\Omega Q) \times \Sigma Q \cup \{\ast\} \times P^\infty(\Omega Q)
\end{array}
\begin{array}{c}
\rightarrow T^3 Q
\end{array}
\begin{array}{c}
HPO
\end{array}
\begin{array}{c}
\begin{array}{c}
\varepsilon_0^Q \times \varepsilon_1^Q \cup \varepsilon_2^Q \\
\end{array}
\begin{array}{c}
\rightarrow Q \times Q
\end{array}
\begin{array}{c}
i_Q \times \Delta Q \\
\rightarrow Q \times Q \times Q.
\end{array}
\end{align*}

By combining this diagram with diagrams (4.3) and (4.4), we obtain the desired diagram.

\[QED.\]

Since there is a right action of \(S^1 \times S^1\) on \(S^r \times S^r\) by \(\Psi^2 = (\Psi \times \Psi) \circ (1 \times T \times 1) : S^r \times S^r \times S^1 \times S^1 \to S^r \times S^r\), we obtain the following.

**Proposition 4.3** The map \(\sigma'_0 \circ \psi : S^t \to P^\infty(\Omega Q) \times \Sigma Q \cup \{\ast\} \times P^\infty(\Omega Q)\) satisfies

\[
\sigma'_0 \circ \psi \sim \left(\left( (j^t_1 \circ \sigma(S^r) \right) \times \sigma(S^r) \right) \circ \Psi^2 (\mu_t, C(\mu_t))^{\text{rel}},
\]

where \(\Psi^2_0 = \Psi^2|_{(S^r \times S^r) \times (S^t \times S^t)} : (S^r \times S^r) \times (S^t \times S^t) \to S^r \times S^r\).

**Proof.** By (4.1), we know \(\sigma'_0 \circ \psi = \sigma'_0 (\Psi \cup \chi_a) \circ \mu_t, C(\mu_t))^{\text{rel}} = \sigma'_0 (\Psi \cup \chi_a) = (\sigma'_0 |_{\text{im} \sigma(S^r)} \circ \Psi \cup \sigma'_0 \circ \chi_a) \circ \mu_t, C(\mu_t))^{\text{rel}},\) where we have

\[
\sigma'_0 |_{\text{im} \sigma(S^r)} = j^t_1 \circ \sigma(S^r) \circ \Delta_{S^r} \circ \Psi = j^t_1 \circ \sigma(S^r) \circ \Psi^2 (\Delta_{S^r} \times \Delta_{S^r}) \quad \text{and} \quad \\
\sigma'_0 \circ \chi_a = (j^t_1 \circ \sigma(S^r)) \circ \alpha \times (j^t_1 \circ \sigma(S^r)) \circ \alpha) \circ H_t + (\hat{\chi_a} \vee \hat{\chi_a}) \circ C(\mu_t),
\]

where the addition denotes the composition of homotopies. Using the same homotopy \(H_t : \Delta_{S^r} \sim \mu_t,\) we obtain homotopies

\[
\sigma'_0 |_{\text{im} \sigma(S^r)} = j^t_1 \circ \sigma(S^r) \circ \Psi^2 (\mu_t, \Delta_{S^r} \times \mu_t) \quad \text{and} \quad \\
\sigma'_0 \circ \chi_a = (\hat{\chi_a} \vee \hat{\chi_a}) \circ C(\mu_t)
\]

which fit together into a homotopy

\[
\sigma'_0 (\Psi \cup \chi_a) \sim \left(\left( (j^t_1 \circ \sigma(S^r)) \times \sigma(S^r) \right) \circ \Psi^2 (\mu_t, \Delta_{S^r} \times \mu_t) \cup (\hat{\chi_a} \vee \hat{\chi_a}) \circ C(\mu_t))^{\text{rel}}.
\]

Then the homotopy \(H_r : \Delta_{S^r} \sim \mu_t\) gives the homotopy relation

\[
\sigma'_0 \circ \psi \sim \left(\left( (j^t_1 \circ \sigma(S^r)) \times \sigma(S^r) \right) \circ \Psi^2 (\mu_t, \Delta_{S^r} \times \mu_t) \cup (\hat{\chi_a} \vee \hat{\chi_a}) \circ C(\mu_t))^{\text{rel}},
\]
which yields \( \sigma'_0 \psi \sim ((j_1^Q \circ \sigma(S')) \times \sigma(S')) \circ \Psi_0^2 \cup (\hat{x}_\alpha \lor \hat{x}_\alpha) \circ [\mu_r, C(\mu_t)]^{\text{rel}}. \) QED.

Hence by the definition of \( \sigma_0 \) and \( \psi \), we obtain the following proposition.

**Proposition 4.4** We have \( \hat{\Delta}_Q \circ \hat{p}_2^Q \circ H_2^{\sigma_0}(\psi) \sim [j_1^Q \circ \sigma(S'), \chi_0]^{\text{rel}}. \)

**Proof.** By the definition of \( \sigma_0 \), we obtain

\[
\hat{\Delta}_Q \circ \sigma_0 \circ \psi \sim \sigma'_0 \psi \sim ((j_1^Q \circ \sigma(S')) \times \sigma(S')) \circ \Psi_0^2 \cup (\hat{x}_\alpha \lor \hat{x}_\alpha) \circ [\mu_r, C(\mu_t)]^{\text{rel}}.
\]

Let \( \ln_i : Z \to Z \lor Z \) be the inclusion to the \( i \)-th factor. Then \( [\mu_r, C(\mu_t)]^{\text{rel}} : S^{r+t} \to (S^r \lor S^t) \times (S^r \lor S^t) \) can be deformed as

\[
[\mu_r, C(\mu_t)]^{\text{rel}} \sim \left[ \ln_1 \circ \ln_r + \ln_2 \circ \ln_r, \ln_1 \circ C(t_r) + \ln_2 \circ C(t_r) \right]^{\text{rel}}
\]

\[
\sim \left[ \ln_1 \circ \ln_r, \ln_1 \circ C(t_r) \right]^{\text{rel}} + \left[ \ln_2 \circ \ln_r, \ln_2 \circ C(t_r) \right]^{\text{rel}}
\]

\[
\sim \left[ \ln_1 \circ \ln_r, \ln_1 \circ C(t_r) \right]^{\text{rel}} + \left[ \ln_2 \circ \ln_r, \ln_2 \circ C(t_r) \right]^{\text{rel}}
\]

\[
\sim \ln_1 \circ \psi \circ (\hat{x}_\alpha \lor \hat{x}_\alpha) \circ [t_r, C(t_r)]^{\text{rel}}
\]

\[
+ \ln_2 \circ (j_1^Q \circ \sigma(S')) \circ \psi \cup \hat{x}_\alpha \circ [t_r, C(t_r)]^{\text{rel}}
\]

\[
+ [\hat{x}_\alpha, j_1^Q \circ \sigma(S')]^{\text{rel}} \lor \hat{T} + [j_1^Q \circ \sigma(S'), \chi_0]^{\text{rel}},
\]

where \( \hat{T} : S^{r+t} = S^{-1} \ast S^t \to S^r \ast S^{-1} = S^{r+t} \) is a switching map. Since \([\hat{x}_\alpha, j_1^Q \circ \sigma(S')]^{\text{rel}} \sim * \) in \( P^\infty(\Omega Q) \times \Sigma \Omega Q \cup \{\} \times P^\infty(\Omega Q) \), we obtain

\[
\hat{\Delta}_Q \circ \sigma_0 \circ \psi \sim \ln_1 \circ \sigma_{\infty} \circ \psi + \ln_2 \circ \sigma_{\infty} \circ \psi + [j_1^Q \circ \sigma(S'), \chi_0]^{\text{rel}}.
\]

On the other hand, we have

\[
\hat{\Delta}_Q \circ \Sigma \Omega \psi \circ \sigma(S^{r+t}) = (j_1^Q \times j_1^Q) \circ \Delta \Sigma \Omega Q \circ \Sigma \Omega \psi \circ \sigma(S^{r+t})
\]

\[
= (j_1^Q \times j_1^Q) \circ (\Sigma \Omega \psi \circ \sigma(S^{r+t}) \times \Sigma \Omega \psi \circ \sigma(S^{r+t})) \circ \Delta \Sigma \Omega Q
\]

\[
\sim (j_1^Q \circ \Sigma \Omega \psi \circ \sigma(S^{r+t}) \lor j_1^Q \circ \Sigma \Omega \psi \circ \sigma(S^{r+t})) \circ [\mu_r, C(t_r)]^{\text{rel}}
\]

\[
\sim \ln_1 \circ \sigma_{\infty} \circ \psi + \ln_2 \circ \sigma_{\infty} \circ \psi.
\]

Since \( p_2^{\Sigma \Omega Q} \circ H_2^{\sigma_0}(\psi) \) is the difference between \( \sigma_0 \psi \) and \( j_1^Q \circ \Sigma \Omega \psi \circ \sigma(S^{r+t}) \), we have the desired homotopy relation. QED.
Next we show the following description of $\hat{\chi}_a$ up to homotopy.

**Proposition 4.5** For some $\delta_0 : S^{t+1} \to \Sigma \Omega Q$, there is a homotopy relation

$$\hat{\chi}_a \simeq j_2^Q \circ \chi_{p_1^Q} \circ C(H_1(\alpha)) + j_1^Q \circ \delta_0 : (C(S^t), S^t) \to \left(P^\infty(\Omega Q), \text{im}(j_1^Q \circ \sigma(S^r))\right),$$

where the addition is given by the coaction $(C(S^t), S^t) \to (C(S^t) \vee S^{t+1}, S^t)$.

**Proof.** Let $\chi'_a : (C(S^t), S^t) \to (P^2(\Omega Q), \Sigma \Omega Q)$ be the map given by the deformation of $\alpha$ to $p_1^Q \circ H_1(\alpha)$ in $\Sigma \Omega Q$ and by $\chi_{p_1^Q} \circ C(H_1(\alpha)) : (C(S^t), S^t) \to (P^2(\Omega Q), \Sigma \Omega Q)$ as in [6, Lemma 5.4, Remark 5.5], where we denote by $C$ the functor taking cones. Then by definition, we have $\chi_a \sim \chi_{p_1^Q} \circ C(H_1(\alpha))$ in $(P^2(\Omega Q), \Sigma \Omega Q)$ and $j_1^Q \circ \chi'_a|_{S^t} = j_1^Q \circ \sigma(S^r) \circ \alpha = \hat{\chi}_a|_{S^t}$. Hence the difference between $\hat{\chi}_a$ and $j_1^Q \circ \chi'_a$ is given by a map $\delta : S^{t+1} \to P^\infty(\Omega Q) \simeq Q$, which can be pulled back to $\delta_0 : S^{t+1} \to \Sigma \Omega Q \subset P^2(\Omega Q)$ (see the proof of [6, Theorem 5.6]). Thus we have $\hat{\chi}_a \simeq j_2^Q \circ \chi'_a + j_1^Q \circ \delta_0 \sim j_2^Q \circ \chi_{p_1^Q} \circ C(H_1(\alpha)) + j_1^Q \circ \delta_0$. QED.

Now we prove Lemma 2.6 using Propositions 4.1, 4.4 and 4.5:

$$[j_1^Q, j_2^Q \circ \chi_{p_1^Q}]_{\text{rel}} \circ H_2^\sigma(\psi) \sim \tilde{\Delta}_Q p_2^Q \circ H_2^\sigma(\psi) \sim [j_1^Q \circ \sigma(S^r), \hat{\chi}_a]_{\text{rel}}$$

$$\sim [j_1^Q \circ \sigma(S^r), j_2^Q \circ \chi_{p_1^Q} \circ C(H_1(\alpha))]_{\text{rel}} + [j_1^Q \circ \sigma(S^r), j_1^Q \circ \delta_0]$$

$$= \pm [j_1^Q, j_2^Q \circ \chi_{p_1^Q}]_{\text{rel}} \circ (\tilde{i}*1_{\Omega Q \circ \Omega Q}) \circ (1_{S^r-1}*H_1(\alpha)) + (j_1^Q \vee j_1^Q) \circ \sigma(S^r), \delta_0].$$

Since $[\sigma(S^r), \delta_0] \simeq 0$ in $\Sigma \Omega Q \times \Sigma \Omega Q$, we proceed as

$$[j_1^Q, j_2^Q \circ \chi_{p_1^Q}]_{\text{rel}} \circ H_2^\sigma(\psi) \sim \pm [j_1^Q, j_2^Q \circ \chi_{p_1^Q}]_{\text{rel}} \circ (\tilde{i}*1_{\Omega Q \circ \Omega Q}) \circ \Sigma^r H_1(\alpha).$$

Since the relative Whitehead product $[j_1^Q, j_2^Q \circ \chi_{p_1^Q}]_{\text{rel}}$ induces a split monomorphism in homotopy groups, we have $H_2^\sigma(\psi) \sim \pm (\tilde{i}*1_{\Omega Q \circ \Omega Q}) \circ \Sigma^r H_1(\alpha)$. Thus we obtain $H_2^\sigma(\psi) \ni [H_2^\sigma(\psi)] = \pm [(\tilde{i}*1_{\Omega Q \circ \Omega Q}) \circ \Sigma^r H_1(\alpha)]$. This completes the proof of Lemma 2.6.

**5 Proof of Theorem 2.5**

In this section, we always assume that $\beta : S^v \to S^{r+t}$ is a co-H-map and $v < t + 2r - 1$. If $[\Sigma^r H_1(\alpha) \circ \beta] = 0$, then we have $H_2^\sigma(\psi \circ \beta) \ni [H_2^\sigma(\psi \circ \beta)] = \pm [(\tilde{i}*1_{\Omega Q \circ \Omega Q}) \circ \Sigma^r H_1(\alpha) \circ \beta] = 0$ by Lemma 2.6. Hence we show the converse. There are cofibre sequences as follows:

$$S^t \xleftarrow{\alpha} S^r \xrightarrow{i} Q \xrightarrow{j} S^{t+1}, \quad S^{r+t} \xleftarrow{\psi} Q \xrightarrow{j} E \xrightarrow{\delta} S^{r+t+1}.$$
By the arguments given in Section 4, we know there are ‘standard’ structures \( \sigma(S^r) : S^r \to P^1(\Omega S^r) \) and \( \sigma_0 : Q \to P^2(\Omega Q) \) for \( \text{cat}(S^r) = 1 \) and \( \text{cat}(Q) = 2 \), respectively, where \( \sigma_0|_{S^r} = \sigma(S^r) \) in \( P^2(\Omega Q) \).

Let \( \sigma \) be a structure for \( \text{cat}(Q) = 2 \) with \( H_2^\sigma(\psi) : \beta \sim 0 \) in \( E^3(\Omega Q) \). For dimensional reasons, \( \sigma|_{S^r} \) is homotopic to \( \sigma(S^r) \) which is given by the bottom-cell inclusion. We regard \( e_2^Q : P^2(\Omega Q) \to Q \) as a fibration with fibre \( E^3(\Omega Q) \xrightarrow{\beta} P^2(\Omega Q) \) and \( \sigma_0 \) as a cross-section of \( e_2^Q \). Then by the definition of a structure, we have \( e_2^Q \circ \sigma \sim 1_Q \). Thus we obtain the following homotopy relations:

\[
\sigma|_{S^r} \sim \sigma(S^r) = \sigma_0|_{S^r} \quad \text{in} \quad P^2(\Omega Q), \quad e_2^Q \circ \sigma \sim e_2^Q \circ \sigma_0 = 1_Q.
\]

Thus the difference between \( \sigma \) and \( \sigma_0 \) is given by a map \( \gamma_0 : S^{r+1} \to P^2(\Omega Q) \) which can be lift to \( E^3(\Omega Q) \):

\[
\sigma \sim \sigma_0 + \gamma_0 \quad \text{in} \quad P^2(\Omega Q),
\]

where the addition is taken by the coaction \( \mu : Q \to Q \vee S^{t+1} \) along the collapsing \( q : Q \to S^{t+1} \). Thus we obtain that \( \sigma \circ \psi \sim \{ \sigma_0, \gamma_0 \}_Q \circ \mu \psi \) in \( P^2(\Omega Q) \), where \( \{ \sigma_0, \gamma_0 \} : Q \vee S^{t+1} \to P^2(\Omega Q) \) is a map given by \( \{ \sigma_0, \gamma_0 \}|_Q = \sigma_0 \) and \( \{ \sigma_0, \gamma_0 \}|_{S^{t+1}} = \gamma_0 \).

By the definition of \( \psi \), we have \( \text{pr}_1 \circ \mu \circ \psi \sim \psi \) and \( \text{pr}_2 \circ \mu \circ \psi \sim q \circ \psi \sim * \), and hence we obtain

\[
\mu \circ \psi \sim (\psi \vee *) \circ \mu + a[t'_r, t''_{t+1}] \quad \text{in} \quad Q \vee S^{t+1} \quad \text{for some} \quad a \in \mathbb{Z},
\]

where \( t'_r : S^r \hookrightarrow Q \hookrightarrow Q \vee S^{t+1} \) and \( t''_{t+1} : S^{t+1} \hookrightarrow Q \vee S^{t+1} \) are inclusions. Hence by putting \( \gamma = a \gamma_0 \), we obtain

\[
\sigma \circ \psi \sim \sigma_0 \circ \psi + [\sigma(S^r), \gamma] \quad \text{in} \quad P^2(\Omega Q),
\]

which yields the following homotopy relation in \( P^2(\Omega Q) \) for a co-H-map \( \beta \):

\[
\begin{align*}
& p_2^{\Omega Q} \circ H_2^\beta(\psi) \circ \beta \sim P^2(\Omega \psi) \circ \sigma(S^{r+1}) \circ \beta - \sigma \circ \psi \circ \beta \\
& \sim P^2(\Omega \psi) \circ \sigma(S^{r+1}) \circ \beta - (\sigma_0 \circ \psi \circ \beta + [\sigma(S^r), \gamma] \circ \beta) \\
& \sim (P^2(\Omega \psi) \circ \sigma(S^{r+1}) - \sigma_0 \circ \psi \circ \beta - [\sigma(S^r), \gamma] \circ \beta) \\
& \sim p_2^{\Omega Q} \circ H_2^\beta(\psi) \circ \beta - \sigma(S^r, \gamma) \circ \beta \\
& \sim \pm p_2^{\Omega S^r} \circ \Sigma^r H_1(\alpha) \circ \beta - [\sigma(S^r), \gamma] \circ \beta
\end{align*}
\]

To proceed, we consider the following commutative ladder of fibre sequences.

\[
\begin{array}{c}
\Omega S^r \xrightarrow{e_2^S} E^3(\Omega S^r) \xrightarrow{p_2^{\Omega S^r}} P^2(\Omega S^r) \xrightarrow{e_2^S} S^r \\
\Omega Q \xrightarrow{e_2^Q} E^3(\Omega Q) \xrightarrow{p_2^{\Omega Q}} P^2(\Omega Q) \xrightarrow{e_2^Q} Q.
\end{array}
\]
Since the pair \((E^3(\Omega Q), E^3(\Omega S^r))\) is \((t + 2r - 1)\)-connected and \(t + 1 < r + t < t + 2r - 1, r > 1\), we have \(\pi_{t+1}(E^3(\Omega Q)) \cong \pi_{t+1}(E^3(\Omega S^r))\) and \(\pi_{r+t}(E^3(\Omega Q)) \cong \pi_{r+t}(E^3(\Omega S^r))\). Since \(\gamma\) can be lift to \(E^3(\Omega Q)\) and we know \(\pi_{t+1}(E^3(\Omega Q)) \cong \pi_{t+1}(E^3(\Omega S^r))\), we may regard that the image of \(\gamma\) is contained in \(P^2(\Omega S^r)\). Hence \(\gamma\) vanishes in \(P^\infty(\Omega S^r)\), and so is \([\sigma(S^r), \gamma]\). Thus \([\sigma(S^r), \gamma]\) can be lift to \(\hat{\gamma}: S^{r+1} \to E^3(\Omega S^r)\) as \([\sigma(S^r), \gamma] \sim p^2_2 S^r \cdot \hat{\gamma}\) in \(P^2(\Omega S^r)\).

Therefore, the hypothesis \(H^2_\psi(\psi)\beta \sim \ast\) together with the homotopy equation (5.1) implies the homotopy relation

\[
p_2^{S^r} |_{S^{r+1} \to E^2(\Omega S^r)} \cdot \Sigma^r H_1(\alpha) \cdot \beta \sim \pm p_2^{S^r} \cdot \hat{\gamma} \cdot \beta \quad \text{in} \quad P^2(\Omega Q).
\]

(5.2)

Since \(p^Q_2\) induces a split monomorphism in homotopy groups and \(\pi_v(E^3(\Omega Q)) \cong \pi_v(E^3(\Omega S^r))\) for \(v < t + 2r - 1\), (5.2) implies a homotopy relation

\[
p_2^{S^r} |_{S^{r+1} \to E^2(\Omega S^r)} \cdot \Sigma^r H_1(\alpha) \cdot \beta \sim \pm [\sigma(S^r), \gamma] \cdot \beta \quad \text{in} \quad P^2(\Omega S^r).
\]

To show \(\Sigma^r H_1(\alpha) \cdot \beta\) is trivial, we use the following proposition obtained by a straightforward calculation (see Mac Lane [12], Stasheff [19] or [5], for example) of Bar resolution:

**Proposition 5.1** The composition map \(\partial: E^{m+1}(\Omega S^r) \xrightarrow{\partial} P^m(\Omega S^r) \to P^m(\Omega S^r)/\Sigma E^m(\Omega S^r) \cong \Sigma E^m(\Omega S^r)\) induces a homomorphism

\[
\partial_*: \tilde{H}_*(\wedge^{m+1} \Omega S^r; \mathbb{Z}) \to \tilde{H}_*(\wedge^m \Omega S^r; \mathbb{Z}),
\]

which is given by

\[
\partial_* (x^{a_0} \otimes x^{a_1} \otimes \cdots \otimes x^{a_m}) = \sum_{i=1}^{m} (-1)^i x^{a_0} \otimes \cdots \otimes x^{a_i-1} \otimes a_i \otimes \cdots \otimes x^{a_m},
\]

where \(a_0, \ldots, a_m \geq 1\) and \(x \in H_{r-1}(\Omega S^r; \mathbb{Z})\) is the generator of the Pontryagin ring \(H_*(\Omega S^r; \mathbb{Z})\).

**Corollary 5.1.1** The composition map \(\partial': S^{r-1} \ast E^2(\Omega S^r) \subset E^3(\Omega S^r) \xrightarrow{\partial'} \Sigma E^2(\Omega S^r) \to \Sigma E^2(\Omega S^r)/\Sigma(S^{r-1} \ast \Omega S^r)\) induces an isomorphism

\[
\partial_*: \tilde{H}_*(S^{r-1} \wedge \Omega S^r \wedge \Omega S^r; \mathbb{Z}) \to \tilde{H}_*((\Omega S^r/S^{r-1}) \wedge \Omega S^r; \mathbb{Z}),
\]

which is given by \(\partial'_* (x \otimes x^j \otimes x^k) = -x^{j+1} \otimes x^k\) for \(j, k \geq 1\).

Thus we obtain a left homotopy inverse of \(p_2^{S^r} |_{S^{r+1} \to E^2(\Omega S^r)}\) as a composition map \(P^2(\Omega S^r) \to P^2(\Omega S^r)/\Sigma \Omega S^r \cong \Sigma E^2(\Omega S^r) \to \Sigma E^2(\Omega S^r)/\Sigma(S^{r-1} \ast \Omega S^r) \cong S^{r-1} \ast E^2(\Omega S^r)\), where the image of \(\Sigma^r H_1(\alpha)\) lies in \(S^{r-1} \ast E^2(\Omega S^r)\). On the other hand by the fact that \(\text{im} \sigma(S^r) \subset \Sigma \Omega S^r\), we also know that the Whitehead product \([\sigma(S^r), \gamma]\) vanishes in the quotient space.
$P^2(\Omega S^r)/\Sigma \Omega S^r$, and hence never appears non-trivially in $S^{r-1}E^2(\Omega S^r)$. Thus we conclude that $\Sigma H_1(\alpha)\circ \beta$ is trivial.

References


