HIGHER HOMOTOPY ASSOCIATIVITY

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§1. Introduction

Throughout the paper we work in the category of spaces having the homotopy type of 1-connected CW-complexes with base point. Let us recall some notions introduced by Stasheff [S].

Definition. An A_n-structure on a space X consists of a sequence of quasi-fibrations $p_i : E^i \to P^{i-1}$;

$$\begin{array}{ccccccccc} X = E^1 & \subset & E^2 & \subset & \cdots & \subset & E^n & & \\ \downarrow p_1 & & \downarrow p_2 & & & & \downarrow p_n & & \text{(commutative)} \\ * = P^0 & \subset & P^1 & \subset & \cdots & \subset & P^{n-1} & \subset & P^n \end{array}$$

with E^i contractible in E^{i+1} and $P^1 = SX$.

Stasheff then defines a special complex $K(i)$ such that

(1) $K(2) = *$, $K(i) \cong I^{i-2}$ (homeomorphic),

(2) the boundary $\partial K(i)$ of $K(i)$ is the union of $i(i-1)/2-1$ faces $K_k(r,s)$ for $2 \leqq r$, s; $1 \leqq k \leqq r$, $r+s = i+1$, where each face $K_k(r,s)$ is affine homeomorphic to $K(r)\times K(s)$ by the map $\partial_k(r,s) : K(r)\times K(s) \to K_k(r,s)$, a face operator,

(3) it has degeneracy operators $s_j : K(i) \to K(i-1)$ for $1 \leqq j \leqq i$.

Definition. An A_n-form on X consists of a family of maps

$$m_i : K(i)\times X^i \to X \quad \text{for} \quad 1 \leqq j \leqq i$$

such that

(1) m_2 is a multiplication with unit, $m_2(*,e,x) = m_2(x,*,e) = x$,

(2) $m_i(\partial_k(r,s)(\rho,\sigma);\ x_1,\ldots,x_i)$

$= m_r(\rho;x_1,\ldots,x_{k-1},m_s(\sigma;x_k,\ldots,x_{k+s-1}),x_{k+s},\ldots,x_i)$,

(3) $m_i(\tau;x_1,\ldots,x_{j-1},*,x_{j+1},\ldots,x_i)$

$= m_i(s_j(\tau);x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_i)$.

Definition. The pair $(X,\{m_i\})$ is called an A_n-space.

Definition. If there exist m_i for any i, we call $\{m_i\}$ an A_∞-form and $(X,\{m_i\})$ an A_∞-space.

Remark. 1) A space is an A_2-space iff it is an H-space.

2) For $i = 3$, $K_3 = I$ the unit interval. The second condition (2) says that $m_2\circ(\mathrm{Id}\times m_2) \simeq m_2\circ(m_2\times\mathrm{Id})$, i.e. $x(yz) \sim (xy)z$. Thus m_3 is an associating homotopy (so that m_2 is a homotopy associative multiplication).

3) Any associative H-space admits n-forms for any n

$$m_n(\tau;x_1,\ldots,x_n) = x_1\ldots x_n$$

so that it is an A_∞-space.

4) In the complex K_i, the symmetries are lost, because we do not assume strict associativity for the H-space. The faces are in the one to one correspondence with the variation of the (non-commutative) product and there are no inessential faces.

One of the main results by Stasheff in [S] is

Theorem. A space X has an A_n-form iff it admits an A_n-structure.

In the process of the proof he defined a space D^i such that there exists a relative homeomorphism $\sigma_{k+1} : (D^k,E^k) \to (P^k,P^{k-1})$

satisfying

(1) $$\begin{array}{ccccc} D^{k-1} & \subset & E^{k} & \subset & D^{k} \\ \downarrow{\scriptstyle \sigma_{k-1}} & & \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle \sigma_k} \\ P^{k-1} & = & P^{k-1} & \subset & P^{k} \end{array} \qquad \text{(commutative)}$$

(2) $$(CE^n, E^n) \simeq (D^k, E^k) \qquad \text{(homotopy equivalence).}$$

Notation (Convention). When we want to express the original space X explicitly (or to avoid ambiguity) we write:

$$E^n(X),\ P^{n-1}(X),\ D^n(X),\ p_n^X,\ m_i^X, \text{ etc.}$$

It is quite natural to ask a functorial definition of an A_n-structure and A_n-form i.e. a definition of a map $f : X \to Y$ to be an A_n-map between A_n-spaces X and Y, which preserve, up to homotopy, A_n-structures of X and Y.

Before we give an explicit definition of an A_n-map, we state its fundamental properties:

P1) A map homotopic to an A_n-map is an A_n-map.

P2) A composition of A_n-maps is an A_n-map.

P3) An A_n-homomorphism is an A_n-map.

P4) The localization map is an A_n-map. The localization of an A_n-map is an A_n-map.

P5) If a homotopy equivalence is an A_n-map, so is its homotopy inverse.

P6) The homotopy fibre of an A_n-map admits an A_n-structure.

P7) The pull-back of two A_n-maps admits an A_n-structure.

P8) A map is an A_2-map, A_3-map or A_∞-map iff it is an H-map, an

H-map preserving homotopy associativity or a loop map, respectively.

P9) Let X be an A_n-space and G a monoid. A map $f : X \to G$ is an A_n-map iff its adjoint $ad(f) : SX \to BG$ is extendable over $P^n(X)$.

P10) Suppose that an A_n-space X dominates an A_{n-1}-space Y, namely, there are maps $f : X \to Y$, $g : Y \to X$ such that $f \circ g \simeq 1_Y$. If one of them is an A_{n-1}-map, Y has an A_n-form.

Historical remarks. 1) Stasheff [S] defined an A_n-homomorphism between A_n-spaces X and Y iff f commutes strictly with A_n-forms $(X,\{m_i\})$ and $(Y,\{n_i\})$: $f \circ m_i = n_i \circ (1_K \times f \times \ldots \times f)$ (a map homotopic to an A_n-homomorphism is not necessarily an A_n-homomorphism).

2) When the target space admits an A_∞-structure, he defined an A_n-form.

3) He also described a parametric complex for $n = 4$ giving an A_4-form of a map but did not give a unified construction of such complexes for all n.

4) Zabrodsky [Z1] defined an A_n-map for $n \leq 3$ and mentioned the possibility for general n (but did not give an explicit definition).

The paper is organized as follows: In Section 2 we define a parameter complex $J(i)$. Then the notions, A_n-form and A_n-structure of a map, are defined by making use of the complex in Section 3. In Section 4 we give an outline of the proofs of the fundamental properties P1) $\sim$ P10). In Section 5 we give some applications. In the last section, Section 6, we generalize the Zabrodsky theorem [Z1] and [Z2].

The details will appear somewhere.

§2. The construction of complex J(i)

In this section we define a complex J(i) which will be needed to define an A_n-map in the later section.

The definition of J(i)

The definition of J(i) is somewhat 'flexible'. Let n be any positive integer. Note that the n-1 cell K(n+1) is homeomorphic with a complex $\overline{K}(n+1)$ whose boundary is a PL-manifold as follows:

$$\overline{K}(n+1) \subset \prod_{j=0}^{n}[0,j],$$
$$\overline{K}(n+1) \ni (u_0,\dots,u_n) \text{ if } \sum_{i=0}^{j}u_i \leqq j \text{ and } \sum_{i=0}^{n}u_i = n,$$
$$\overline{K}_{k+1}(r,s) \ni (u_0,\dots,u_n) \text{ if } (u_k,\dots,u_{k+s-1}) \in \overline{K}(s).$$

From now on we identify $\overline{K}(n+1)$ with K(n+1) if there is no misunderstanding.

The boundary of K(n+1) is the union of $K_{k+1}(r,s)$'s. The face operators are described as

$$\partial_{k+1}(r+1,s+1)(\rho,\sigma) = (u_0,\dots,u_{k-1},v_0,\dots,v_s,u_k,\dots,u_r)$$

for $\rho = (u_0,\dots,u_r)$ in K(r+1) and $\sigma = (v_0,\dots,v_s)$ in K(s+1).

Now we define a complex J(n+1) as follows:

$$J(n+1) \subset \prod_{j=0}^{n}[0,2j+1],$$

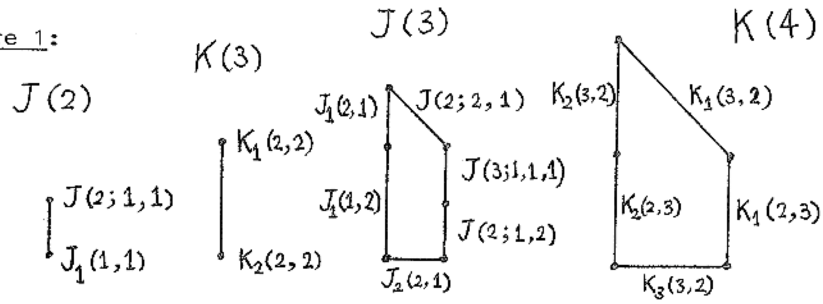
$$J(n+1) \ni (u_0,\dots,u_n) \text{ if } \sum_{i=0}^{j}u_i \leqq 2j+1 \text{ and } \sum_{i=0}^{n}u_i = 2n+1$$

and face operators are given by

$\delta_{k+1}(r+1,s+1)(\rho,\sigma) = (u_0,\ldots,u_{k-1},2v_0,\ldots,2v_s,u_k,\ldots,u_r)$
where $\rho = (u_0,\ldots,u_r)$ in $\bar{J}(r)$, $r+s = n$, and $\sigma = (v_0,\ldots,v_s)$ in $K(s+1)$ and

$\delta(t+1;r_0,\ldots,r_t)(\tau;\rho_0,\ldots,\rho_t)$
$= (\rho_0',\ldots,\rho_t')$, $\rho_i' = (v_{i,0},\ldots,v_{i,r_i}+u_i)$
where $\rho_i = (v_{i,0},\ldots,v_{i,r_i})$ in $J(r_i+1)$, $t+r_0+\ldots+r_t = n$ and $\tau = (u_0,\ldots,u_t)$ in $K(t+1)$ (see Figure 1).

Figure 1:



Next we will show the relationship of $J(n)$ with $K(n)$. For any given real number $a \geq 0$, we can define a complex $\bar{J}(a)$ as follows:

$\bar{J}(a+1) \subset \prod_{j=0}^{n}[0,j+1]$ with $n = [a]$,
$\bar{J}(a+1) \ni (u_0,\ldots,u_n)$ if $\sum_{i=0}^{j} u_i \leq a-n+j$ and $\sum_{i=0}^{n} u_i = a$;

and its face operators are given by

$\bar{\delta}_{k+1}(b+1,s+1)(\rho,\sigma) = (u_0,\ldots,u_{k-1},v_0,\ldots,v_s,u_k,\ldots,u_r)$
where $\rho = (u_0,\ldots,u_r)$ in $\bar{J}(b)$, $b+s = a$, $[b] = r$, $\sigma = (v_0,\ldots,v_s)$ in $K(s+1)$ and

$$\bar{\delta}(t+1;a_0+1,\dots,a_t+1)(\tau;\rho_0,\dots,\rho_t)$$
$$= (\rho_0',\dots,\rho_t'),\ \rho_i' = (v_{i,0},\dots,v_{i,r_i}+u_i') \quad \text{and} \quad u_i' = u_i(n+1-a),$$

where $\rho_i = (v_{i,0},\dots,v_{i,r_i})$ is in $\bar{J}(b_i+1)$, $[b_i] = r_i$, $b_i-[b_i] = a-[a]$, $t(n+1-a)+a_0+\dots+a_t = a$ and $\sigma = (u_0,\dots,u_t)$ is in $K(t+1)$.

We denote the faces by $\bar{J}_k(r,s)$ which is the image of $\bar{\delta}_k(r,s)$ and by $\bar{J}(t+1;b_0+1,\dots,b_t+1)$ which is the image of $\bar{\delta}(t+1;b_0+1,\dots,b_t+1)$. We call the last ones upper faces and denote by $\partial_+\bar{J}(a+1)$, the union of all upper faces of $\bar{J}(a+1)$.

<u>Remark</u>. 1) The faces are all homeomorphisms if a is not an integer.

2) $J(n)$ is naturally equivalent to $\bar{J}(n+1/2)$ as complexes.

Under these notations, we have

Proposition 2.1. 1) $K(n+1) \supseteq \bar{J}(a) \supseteq K(n)$ <u>for</u> $[a] = n \geqq 1$ <u>and further</u> $\bar{J}(a) = K(n)$ <u>if</u> a <u>is an integer</u>.

2) <u>Let</u> a <u>be a non-integral real number and</u> $[a] = [b] = n \geqq 1$. <u>Then there is a map</u> $f_{a,b}$ <u>from</u> $\bar{J}(a)$ <u>to</u> $\bar{J}(b)$ <u>such that</u>

$$f_{a,b}\circ\bar{\delta}(t;a_1,\dots,a_t) = \bar{\delta}(t;b_1,\dots,b_t),$$
$$f_{a,b}\circ\bar{\delta}_k(a',s) = \bar{\delta}_k(b',s)\circ(f_{a',b'}\times \mathrm{Id}),$$

<u>where</u> b' <u>and</u> b_i <u>are given by</u> $a'-b' = a_i-b_i = a-b$. <u>Moreover the boundaries of</u> $\bar{J}(a)$ <u>and</u> $\bar{J}(b)$ <u>are equivalent as complexes, if</u> a <u>and</u> b <u>are not integers and</u> $[a] = [b] \geqq 1$.

3) <u>Let</u> a <u>be a non-integral real number and</u> $[a] = n \geqq 1$. <u>Then there is a projection</u> $\pi_n : J(n) \to K(n)$ <u>such that</u>

$$\pi_n\circ\delta(n;1,\dots,1) = \mathrm{Id},$$
$$\pi_n\circ\delta_k(r,s) = \delta_k(r,s)\circ(\pi_r\times \mathrm{Id}).$$

Proof. 1) is clear by the above definitions of $K(n)$'s and $\bar{J}(n)$'s. We show 2). We have to show that there is a cellular map $f_{a,b} : \bar{J}(a) \to \bar{J}(b)$. Since $\bar{J}(a)$ is the union of the upper faces $\partial_+\bar{J}(a')$ for $n \leqq a' \leqq a$, it suffices to define the map on the upper faces of $\bar{J}(a)$. We define the map by induction on n. In the case $n = 1$, $\bar{J}(a) = \bar{J}(b)$ = one point set and the map is trivial. In general, we define $f_{a,b}$ by $f_{a,b}\circ\bar{\delta}(t;a_1,\ldots,a_t) = \bar{\delta}(t;b_1,\ldots,b_t)$ on the upper faces and by the hypothesis. The latter identity obtained by the relation of the face operators. 3) is obvious, because $\pi_n = f_{n+1/2,n}$ satisfies the required properties by 1). This implies the proposition. Q.E.D.

Next, we define the degeneracies d_j on $J(n)$. It suffices to define degeneracies on $\partial_+\bar{J}(a)$. So we can define degeneracies by the relation with the upper faces (d-3) below and obtain

Proposition 2.2. $\pi_{n-1}\circ d_j = s_j\circ\pi_n$ for $1 \leqq j \leqq n$.

The properties of $J(i)$

The complexes $J(i)$ satisfy the following properties (2-a) $\sim$ (2-e):

(2-a) $J(1) = \{*\}$, and $J(i)$ for $i \geqq 2$ is affine homeomorphic with I^{i-1}.

(2-b) The boundary $\partial J(i)$ of the complex $J(i)$ is the union of $i(i-1)/2+2^{i-1}-1$ faces

$J_k(r,s)$ for $1 \leqq k \leqq r$, $1 \leqq r \leqq i-1$, $r+s = i+1$,

$J(t;r_1,\ldots,r_t)$ for $2 \leqq t \leqq i$, $r_j \geqq 1$, $r_1+\ldots+r_t = i$.

(2-c) Let $r+s-1 = r_1+\ldots+r_t = i$. The faces $J_k(r,s)$ and

$J(t;r_1,\ldots,r_t)$ of $J(i)$ are affine and piecewise affine homeomorphic with $J(r)\times K(s)$ and $K(t)\times J(r_1)\times\ldots\times J(r_t)$ respectively through affine and piecewise affine homeomorphisms:

$$\delta_k(r,s) : J(r)\times K(s) \to J_k(r,s),$$

$$\delta(t;r_1,\ldots,r_t) : K(t)\times J(r_1)\times\ldots\times J(r_t) \to J(t;r_1,\ldots,r_t),$$

which are called face operators. The second face operators are called upper face operators by virtue of this property. The face operators satisfy the following four relations:

(c-1)
$$\begin{aligned}&\delta_k(r,s+t-1)\ (\mathrm{Id}\times\partial_j(s,t))\\ &= \delta_{k+j-1}(r+s-1,t)\circ(\delta_k(r,s)\times\mathrm{Id})\\ &: J(r)\times K(s)\times K(t) \to J(r+s+t-2),\end{aligned}$$

(c-2)
$$\begin{aligned}&\delta_k(r+s-1,t)\circ(\delta_j(r,s)\times\mathrm{Id})\\ &= \begin{cases}\delta_{j+s-1}(r+t-1)\circ(\delta_k(r,t)\times\mathrm{Id})\circ(\mathrm{Id}\times T) & \text{for } k < j\\ \delta_j(r,s+t-1)\circ(\mathrm{Id}\times\partial_{k-j+1}(s,t)) & \text{for } j \leqq k < j+s\\ \delta_j(r+t-1,s)\circ(\delta_{k-s+1}(r,t)\times\mathrm{Id})\circ(\mathrm{Id}\times T) & \text{for } j+s \leqq k\end{cases}\\ &: J(r)\times K(s)\times K(t) \to J(r+s+t-2),\end{aligned}$$

(c-3) for given $(r_1,\ldots,r_t)$ with $\sum_j r_j = i$ and k, let j be the index such that $r_1+\ldots+r_{j-1} < k \leqq r_1+\ldots+r_j$. Then

$$\begin{aligned}&\delta_k(i,s)\circ(\delta(t;r_1,\ldots,r_t)\times\mathrm{Id})\\ &= \delta(t;r_1,\ldots,r_{j-1},r_j+s-1,r_{j+1},\ldots,r_t)\circ(1_A\times\delta_{k-i'}(r_j,s)\times 1_B)\circ T'\\ &: K(t)\times J(r_1)\times\ldots\times J(r_t)\times K(s) \to J(i+s-1),\end{aligned}$$

where 1_A and 1_B are the identity maps of $A = K(t)\times J(r_1)\times\ldots\times J(r_{j-1})$ and $B = J(r_{j+1})\times\ldots\times J(r_t)$ respectively, $i' = r_1+\ldots+r_{j-1}$ and $T' : A\times J(r_j)\times B\times K(s) \to A\times J(r_j)\times K(s)\times B$ is the map switching factors B and $K(s)$,

(c-4)
$$\begin{aligned}&\delta(t+u-1;r_1,\ldots,r_{t+u-1})\circ(\partial_k(t,u)\times\mathrm{Id})\\ &= \delta(t;r_1,\ldots,r_{k-1},i'',r_{k+u},\ldots,r_{t+u-1})\circ(1_C\times\end{aligned}$$

$$\delta(u;r_k,\dots,r_{t+u-1})\times 1_D)\circ T''$$

$$: K(t)\times K(u)\times J(r_1)\times\dots\times J(r_{t+u-1}) \to J(i),$$

where $i = r_1+\dots+r_{t+u-1}$, $i'' = r_k+\dots+r_{t+u-1}$, $C = K(t)\times C'$ with $C' = J(r_1)\times\dots\times J(r_{k-1})$, $D = J(r_{k+u})\times\dots\times J(r_{t+u-1})$ and T'' : $K(t)\times K(u)\times C'\times C''\times D \to K(t)\times C'\times K(u)\times C''\times D$ with $C'' = J(r_k)\times\dots\times J(r_{k+u-1})$ is the map switching factors $K(u)$ and C'.

(2-d) The complexes $J(i)$ have degeneracy operators d_j : $J_i \to J_{i-1}$, $1 \leqq j \leqq i$, satisfying the following three relations:

(d-1) $$d_k\circ d_j = \begin{cases} d_{j-1}\circ d_k & \text{for } k < j,\\ d_j\circ d_k & \text{for } k \geqq j,\end{cases}$$

(d-2) $$d_k\circ \delta_j(r,s) : J(r)\times K(s) \to J_i(r,s)$$
$$= \begin{cases} \delta_{j-1}(r-1,s)\circ(d_k\times Id) & \text{for } k < j,\\ \delta_j(r,s-1)\circ(Id\times s_{k-j+1}) & \text{for } j \leqq k < j+s,\ s > 2,\\ pr_1 & \text{for } j \leqq k < j+s,\ s = 2,\\ \delta_j(r-1,s)\circ(d_{k-s+1}\times Id) & \text{for } j+s \leqq k,\end{cases}$$

(d-3) for given k and $(r_1,\dots,r_t)$ with $\sum_a r_a = i$ let j be the index such that $r_1+\dots+r_{j-1} < k \leqq r_1+\dots+r_j$ and put $i' = r_1+\dots+r_{j-1}$. Then

$$d_k\circ \delta(t;r_1,\dots,r_t) : K(t)\times J(r_1)\times\dots\times J(r_t) \to J(t;r_1,\dots,r_t)$$
$$= \begin{cases} \delta(t;r_1,\dots,r_{j-1},r_j-1,r_{j+1},r_{j+1},\dots,r_t)\circ(1_E\times d_{k-i'}\times 1_F) & \\ \qquad\qquad \text{for } r_j \geqq 2, & \\ \delta(t-1;r_1,\dots,r_{j-1},r_{j+1},\dots,r_t)\circ(s_j\times 1_G)\circ\pi' & \text{for } r_j = 1,\ t \geqq 3,\\ pr_2 & \text{for } t = 2,\ k = r_1+1 = i,\ j = 2,\\ pr_3 & \text{for } t = 2,\ k = r_1 = 1,\ j = 1,\end{cases}$$

where $E = K(t)\times E'$ with $E' = J(r_1)\times\dots\times J(r_{j-1})$, $F = J(r_{j+1})\times\dots\times J(r_t)$, $G = E'\times F$, pr_t is the projection to the t-th factor and $\pi' : E\times J(r_j)\times F \to E\times F$ is the natural projection.

(2-e) There is the map $\omega_i : K(i) \to J(i)$ satisfying the following:

$$\omega_i \circ \partial_k(r,s) = \delta_k(r,s) \circ (\omega_r \times Id),$$

$$\begin{array}{ccc}
K(r) \times K(s) & \xrightarrow{\omega_r \times Id} & J(r) \times K(s) \\
\partial_k(r,s) \big\downarrow & & \big\downarrow \delta_k(r,s) \\
K_k(r,s) & & J_k(r,s) \\
\cap & & \cap \\
K(i) & \xrightarrow{\omega_i} & J(i)
\end{array}$$

$$d_j \circ \omega_i = \omega_{i-1} \circ s_j \quad \text{for} \quad 2 \leqq j \leqq i-1,$$

$$\text{Image } \omega_i = \bigcup J(t; r_1, \ldots, r_t),$$

where the union runs over all the upper faces of $J(i)$.

§3. An A_n-map, an A_n-action and an A_n-equivariant map

In this section we introduce the notion of an A_n-map, an A_n-action (see [N]) and an A_n-equivariant map.

An A_n-map

Let X and Y be A_n-spaces.

Definition. An A_n-structure of $f : X \to Y$ consists of a sequence of maps $\{f_k^D\}$, $\{f_k^P\}$, $1 \leqq k \leqq n$ such that

1) $\sigma_{k+1}^Y \circ f_k^D = f_k^D \circ \sigma_{k+1}^X$

$$\begin{array}{ccc}
(D^k X,\ E^k X) & \xrightarrow{f_k^D} & (D^k Y,\ E^k Y) \\
\sigma_{k+1}^X \big\downarrow & & \big\downarrow \sigma_{k+1}^Y \\
(P^k X,\ P^{k-1} X) & \xrightarrow{f_k^P} & (P^k Y,\ P^{k-1} Y)
\end{array}$$

2) $f = f_1^D|_X$, $f_k^D = f_n^D|_{D^kX}$, $f_k^P = f_n^P|_{P^kX}$.

We have already introduced parameter complexes $J(i)$ for $i \geqq 1$ in the earlier section in order to define the notion "A_n-form" of a map.

Let X and Y be A_n-spaces with A_n-forms $\{M_i^X, i \leqq n\}$, $\{M_i^Y, i \leqq n\}$ respectively.

Definition. An A_n-form of a based map $f : X \to Y$ consists of a family of maps $\{F_i : J(i)\times X^n \to Y;\ i \leqq n\}$ satisfying

(1) $F_1 = f$

(2) $F_i(\delta_k(r,s)(\rho,\sigma);x_1,\dots,x_i)$
$= F_r(\rho;x_1,\dots,x_{k-1},m_s^X(\sigma;x_k,\dots,x_{k+s-1}),x_{k+s},\dots,x_i)$

(3) $F_i(\delta(t;r_1,\dots,r_t)(\tau;\rho_1\dots,\rho_t);x_1,\dots,x_i)$
$= m_t^Y(\tau;F_{r_1}(\rho_1;x_1,\dots,x_{r_1}),\dots,F_{r_t}(\rho_t;x_{r_1+\dots+r_{t-1}+1},\dots,x_i))$

(4) $F_i(\gamma;x_1,\dots,x_{j-1},*,x_{j+1},\dots,x_i)$
$= F_{i-1}(d_j(\gamma);x_1,\dots,x_{j-1},x_{j+1},\dots,x_i)$

Definition. We call the pair $(f,\{F_i, i \leqq n\})$ an A_n-map. The pair $(f,\{F_i\})$ is called an A_∞-map if F_i exists for every i.

Theorem 3.1. A map admits an A_n-structure iff it has an A_n-form.

(Outline of the proof) Firstly we remark that $K(i)$ and the upper faces in $J(i+1)$ are homeomorphic and we can use the upper faces of $J(i+1)$ and d_j instead of $K(i)$ and s_j. Let f have an A_n-form. We define an A_n-structure as follows:

Definition. For σ in $K(i+1)$, $\omega_{i+1}(\sigma) = \delta(t;r_1,\dots,r_t)(\tau;\rho_1,\dots,\rho_t)$, we define $f_i^E : E^i(X) \to E^i(Y)$, $f_i^P : P^i(X) \to P^i(Y)$ and $f_i^D : D^i(X) \to D^i(Y)$ as follows:

$$(3.2.1)\quad f_{i-1}^E(\alpha_{i-1}^X(\sigma;x_1,\dots,x_{i-1})) = \alpha_{t-1}^Y(\tau;F^f(r_1)(\rho_1;x_1,\dots,x_{r_1}),\dots,F^f(r_{t-1})(\rho_{t-1};x_{r_1+\dots+r_{t-2}+1},\dots,x_{r_1+\dots+r_{t-1}}))$$

$$(3.2.2)\quad f_{i-1}^P(\beta_i^X(\sigma;x_2,\dots,x_i)) = \beta_{t-1}^Y(\tau;F^f(r_2)(\rho_2;x_{r_1+1},\dots,x_{r_1+r_2}),\dots,F^f(r_{t-1})(\rho_{t-1};x_{r_1+\dots+r_{t-2}+1},\dots,x_{r_1+\dots+r_{t-1}}))$$

$$(3.2.3)\quad f_{i-1}^D(\gamma_i^X(\sigma;x_2,\dots,x_i)) = \begin{cases} \alpha_{t-1}^Y(\tau;F^f(r_1-1)(d_1(\rho_1);x_2,\dots,x_{r_1}),\dots,F^f(r_{t-1})(\rho_{t-1}; \\ \qquad x_{r_1+\dots+r_{t-2}+1},\dots,x_{r_1+\dots+r_{t-1}})), & r_1 > 1, \\ \gamma_{t-1}^Y(\tau;F^f(r_2)(\rho_2;x_{r_1+1},\dots,x_{r_1+r_2}),\dots,F^f(r_{t-1})(\rho_{t-1}; \\ \qquad x_{r_1+\dots+r_{t-2}+1},\dots,x_{r_1+\dots+r_{t-1}})), & r_1 = 1. \end{cases}$$

We leave the reader to check the well-definedness of this definition. The converse is similar to the proof of Theorem 5 of [S,I]. Q.E.D.

<u>An</u> A_n-<u>action</u> ([N])

We introduce here the idea of higher homotopy (<u>right</u>) action. A higher homotopy left action is defined similarly, so we omit it. Let G be an A_n-space with A_n-structure $\{p_i^G : E^i(G) \to P^{i-1}(G)$ for $i \leq n\}$ and h a based map from G to a space W, where $P^i(G)$ is the G-projective i-space.

Definition. An A_n-structure of a (right) action along the map $h : G \to W$ consists of a sequence of quasi-fibrations $p_i^h : E^i(h) \to P^{i-1}(G)$ with fibre W and relative homeomorphisms $\sigma_i^h : (D^i(h), E^i(h)) \simeq (P^i(G), P^{i-1}(G))$, where $E^1(h) = W$, and $E^i(h)$ and $D^i(h)$ are defined similarly to the definition of $E^i(G)$ and $D^i(G)$ respectively by the relative homeomorphisms

$$\alpha_{i+1}^h : (K(i+2)\times W\times G^i, \partial K(i+2)\times W\times G^i \cup K(i+2)\times W\times G^{[i]}) \to (E^{i+1}(h), E^i(h)),$$
$$\gamma_{i+1}^h : (K(i+2)\times G^i, \partial K(i+2)\times G^i \cup K(i+2)\times G^{[i]}) \to (D^i(G), E^i(G)),$$

and σ_i^h is obtained by the projection.

Remark. 1) $D^i(h)$ is not contractible in general.

2) The fibre of p_i^h is W.

3) Let $h : H \to G$ be a map between A_∞-spaces. If h is an A_∞-map, $E^\infty(h)$ can be identified with the homotopy fibre of the map $Bh : BH \to BG$. We may write $E^\infty(h)$ by G/H, if further h is an injective A_∞-homomorphism.

Let $\{m_i^G, i \leqq n\}$ be the A_n-form of G.

Definition. An A_n-form of a (right) action along $h : G \to W$ consists of a family of maps $\{N_i^f : K(i)\times W\times G^{i-1} \to W;\ i \leqq n\}$ satisfying

(1) $N_1 = Id$,

(2) $N_i(\partial_j(r,s)(\rho,\sigma);w;g_2,\dots,g_i)$
$= N_r(\rho;w;g_2,\dots,g_{j-1},m_s^G(\sigma;g_j,\dots,g_{j+s-1}),g_{j+s},\dots,g_i)$,

(2)' $N_i(\partial_1(r,s)(\rho,\sigma);w;g_2,\dots,g_i)$
$= N_r(\rho;N_s(\sigma;w;g_2,\dots,g_s),g_{s+1},\dots,g_i)$,

(3) $N_i(\tau;w;g_2,\dots,g_{j-1},*,g_{j+1},\dots,g_i)$
$= N_{i-1}(s_j(\tau);w;g_2,\dots,g_{j-1},g_{j+1},\dots,g_i)$,

(3)' $N_i(\tau;*;g_2,\dots,g_i) = h\circ m_{i-1}^G(s_1(\tau);g_2,\dots,g_i)$.

Definition. We call the pair $(h, \{N_i, i \leqq n\})$ a (right) A_n-action along h. The pair $(h, \{N_i\})$ is called a (right) A_∞-action along h if N_i exists for all i.

From a similar argument to [S,I], it follows

Theorem 3.3. _An_ A_n_-structure of an action along_ h _is equivalent to an_ A_n_-action along_ h.

_An_ A_n_-equivariant map_

Let H and K be A_n-spaces and $f : H \to K$ be an A_n-map with the A_n-structure $\{f_i^P, f_i^D; i \leqq n\}$. We assume that there are A_n-structures of actions $\{p_i^h, \sigma_i^h\}$ on V along h and $\{p_i^k, \sigma_i^k\}$ on W along k. Let $F : V \to W$ be a based map with $F \circ h = k \circ f$.

Definition. An A_n-structure of $F : V \to W$ along f consists of a sequence of maps $\{F_i^D\}$, $1 \leqq i \leqq n$ such that

1) $f_i^P \circ \sigma_{i+1}^h = \sigma_{i+1}^k \circ F_i^D$

$$\begin{array}{ccc} D^i(h) & \xrightarrow{F_i^D} & D^i(k) \\ \sigma_{i+1}^h \big\downarrow & & \big\downarrow \sigma_{i+1}^k \\ P^i(H) & \xrightarrow{f_i^P} & P^i(K) \end{array}$$

2) $F = F_1^D|_V$, $F_i^D = F_n^D|_{D^i(h)}$.

Let $\{N_i^h, i \leqq n\}$ (and $\{N_i^k, i \leqq n\}$) be the A_n-action along h (and along k, resp.) and $\{F_i^f\}$ the A_n-form of f.

Definition. An A_n-form of the map $F : V \to W$ consists of a family of maps $\{R_i : J(i)\times V\times H^{i-1} \to W;\ i \leqq n\}$ satisfying

(1) $R_1 = F$

(2) $R_i(\delta_j(r,s)(\rho,\sigma);v;g_2,\dots,g_i)$
$= R_r(\rho;v;g_2,\dots,g_{j-1},m_s^H(\sigma;g_j,\dots,g_{j+s-1}),g_{j+s},\dots,g_i)$

(2)' $R_i(\delta_1(r,s)(\rho,\sigma);v;g_2,\dots,g_i)$
$= R_r(\rho;N_s^h(\sigma;v;g_2,\dots,g_s),g_{s+1},\dots,g_i)$

(3) $R_i(\delta(t;r_1,\dots,r_t)(\tau;\rho_1\dots,\rho_t);v;g_2,\dots,g_i)$
$= N_t^k(\tau;R_{r_1}(\rho_1;v;g_2,\dots,g_{r_1}),F_{r_2}^f(\rho_2;g_{r_1+1},\dots,g_{r_1+r_2}),\dots$
$F_{r_t}^f(\rho_t;g_{r_1+\dots+r_{t-1}+1},\dots,g_i))$

(4) $R_i(\gamma;v;g_2,\dots,g_{j-1},*,g_{j+1},\dots,g_i)$
$= R_{i-1}(d_j(\gamma);v;g_2,\dots,g_{j-1},g_{j+1},\dots,g_i)$

(4)' $R_i(\gamma;*;g_2,\dots,g_i) = k\circ F_{i-1}^f(d_1(\gamma);g_2,\dots,g_i)$

Definition. We call the triple $(F,f,\{R_i,\ i \leqq n\})$ an A_n-equivariant map along f. The triple $(F,f,\{R_i\})$ is called an A_∞-equivariant map along f if R_i exists for all i.

By a same argument as above, we have

Theorem 3.4. <u>An</u> A_n-<u>structure of a map along</u> f <u>is equivalent to an</u> A_n-<u>equivariant map along it</u>.

§4. Fundamental properties

We indicate an outline of the proof of the properties.

P1) We deform the A_n-forms by the homotopy of maps.

P2) The composition of the A_n-structures gives the A_n-structure of the composition of the A_n-maps.

P3) By using π_n, we can define A_n-forms for an A_n-homomorphism.

P4) As the localization functor is continuous (by [I3]), the localization map is an A_n-homomorphism and the localization of an A_n-map is again an A_n-map.

P5) Let (f,g) be the homotopy equivalence pair with f an A_n-map, given homotopies $f\circ g \simeq Id$ and $g\circ f \simeq Id$. Through these homotopies, we can define an A_n-form for g.

P7) The homotopy fibre of the A_n-structure of an A_n-map gives the A_n-structure of the homotopy fibre of the A_n-map.

P6) This is a corollary of P7)

P8) We obtain this directly by the definition.

P9) It suffices to show the existence of the A_n-structure for a given map, if its adjoint has an extension to the X-projective n-space. We can construct it by the homotopy extension property of $(D^i(X),E^i(X))$ and by the homotopy lifting property of the principal fibration $p_i^X : E^i(X) \to P^{i-1}(X)$.

P10) Using the A_n-form of f (or g), we can define an n-form for Y similarly to the case when $n = 2$.

§5. Some applications

<u>An A_n-primitive space</u>

Let X be an A_n-space of finite type such that

$$H^*(X;\mathbf{Q}) \cong E(x_1,\dots,x_r)\otimes P[y_1,\dots,y_s]$$

with $\deg(x_i) = 2n_i-1$, $n_1 \leq \dots \leq n_r$ and $\deg(y_j) = 2m_j$, $m_1 \leq \dots \leq m_s$. Then as is well known,

$$X \simeq_0 \prod S^{2n_i-1} \times \prod K(\mathbf{Z},2m_j).$$

Recall from [I2, Theorem B]

(5.1) Let X be a finite CW-complex having an A_n-form. There is a homotopy equivalence $h : X_{(0)} \simeq \prod S^{2n_i-1}_{(0)}$ which is an A_{n-1}-map.

Definition. An A_n-space X is A_n-primitive if h is an A_n-map.

Definition. An element x in $H^*(X;Q)$ is A_n-primitive if there is an element y in $H^*(P^nX;Q)$ such that $s^*x = \iota_2^* \cdots \iota_n^* y$, where $\iota_i : P^{i-1} \hookrightarrow P^i$ is a natural inclusion and s^* is the suspension isomorphism.

Proposition 5.2. X <u>is an</u> A_n-<u>primitive iff</u> $H^*(X;Q)$ <u>is generated by</u> A_n-<u>primitive elements</u>.

<u>Notation</u>. $\mathfrak{A}_n$ = {finite A_n-spaces}

$\mathfrak{A}'_n$ = {finite A_n-primitive spaces}

By the definitions and by the above proposition we have the inclusions

(5.3) $\mathfrak{A}_2 \supseteq \mathfrak{A}'_2 \supseteq \mathfrak{A}_3 \supseteq \cdots \supseteq \mathfrak{A}'_{n-1} \supseteq \mathfrak{A}_n \supseteq \mathfrak{A}'_n \supseteq \cdots$

<u>Example</u>. The seven sphere S^7 is an example of $\mathfrak{A}'_2 \supsetneqq \mathfrak{A}_3$.

Recall that whether or not $\mathfrak{A}_2 \supsetneqq \mathfrak{A}'_2$ is still open problem.

Let $(X;\{m_i, i\leqq n\})$ be a finite A_n-space. For the dimensional reasons, we have

Proposition 5.4. *If, for any* $i \leq r$ *and any* r-*tuple* $\{\alpha_j ; \emptyset \neq \alpha_j \subseteq \{1,\ldots,r\}, 1 \leq j \leq r\}$, *we have* $2n_i \neq (n-1) + \sum_{j=1}^{r} |\alpha_j|$ *with* $|\alpha_j| = \sum(2n_t-1)$ *where* t *ranges over* α_j, *then* $(X,\{m_i, i \leq n\})$ *is in* $\mathfrak{A}'_n$.

Corollary 5.5. *Let* $G(m) = SU(m)$ *for* $d = 2$, $Sp(m)$ *for* $d = 4$. *If* $d(m-1) \leq 4n$, *then* $G'(m) = (G(m), m_n)$ *is in* $\mathfrak{A}'_n$ *for any* n-*form* m_n. *So,* $P^nG'(m)$ *is rationally equivalent to* $P^nG(m)$.

Proposition 5.6 (Counter examples).

1) *If* $d(m-1) > 4n$, *then there is an* n-*form* m'_n *of* $G(m)$ *such that* $(G(m), m'_n)$ *is not in* $\mathfrak{A}'_n$

2) *If* $m \geq 2n+1$, *then there is an* n-*form* m'_n *of* $S^3 \times S^3 \times SU(m)$ *such that* $(S^3 \times S^3 \times SU(m), m'_n)$ *is not in* $\mathfrak{A}'_n$.

In the proof of these proposition we need the following facts.

Let $(Y,\{m_i, i \leq n\})$ be an A_n-space $(n \geq 3)$.

Notation. $\mathbf{A}_n(X;\{m_i\}) = \{m'_n : K_n \times X^n \to X ; \{m_i, i < n$ and $m'_n\}$ is an A_n-form of $X\}$

Definition. For any n-forms m'_n and m''_n in $\mathbf{A}_n(X;\{m_i\})$, $m'_n \sim_{A_n} m''_n$ iff there is an n-form $F(n) : J_n \times X^n \to X$ of the identity map 1_X such that $\{m_i \circ (\pi_i \times 1_X \times \ldots \times 1_X), i < n$ and $F(n)\}$ is an A_n-form of the identity 1_X.

By the fundamental property P5), we have

Proposition 5.7. *The relation* $\sim_{A_n}$ *is an equivalence relation*.

Lemma 5.8. $A_n(X,\{m_i\})/\sim_{A_n}$ <u>is in the one to one correspondence with</u> $\pi_0 A_n(X,\{m_i\})$.

<u>(Outline of the proof)</u> Since 1_X is an A_{n-1}-homomorphism, the obstruction for the existence of an n-form of 1_X is deformed to be the obstruction for the existence of the homotopy between two n-forms m_n and m'_n.

The latter obstruction is classified by the set $[S^{n-2}\wedge X\wedge\ldots\wedge X,X]$. So we have

Theorem 5.9. $A_n(X,\{m_i\})/\sim_{A_n} \cong [S^{n-2}\wedge X\wedge\ldots\wedge X,X]$.

Localization and Zabrodsky's theorem

Zabrodsky constructs in [Z2] an example of a finite A_{p-1}-space which does not have an A_p-form for each prime $p \geqq 3$. It seems, however, that his construction needs more precise arguments.

Let Π be the set of all primes and $\mathbf{P}$ be a subset of Π.

Definition. A space X or a map f admits a mod $\mathbf{P}$ A_n-form (or A_n-structure) iff $X_{\mathbf{P}}$ or $f_{\mathbf{P}}$ admits an A_n-form (or A_n-structure).

Let $\Pi = \coprod_i \mathbf{P}_i$ (a finite partition).

Proposition 5.10. 1) X <u>is an</u> A_n-<u>primitive iff</u> X <u>is a</u> mod $\mathbf{P}_i$ A_n-<u>primitive for all</u> i,

2) f <u>is an</u> A_n-map <u>iff</u> f <u>is a</u> mod $\mathbf{P}_i$ A_n-<u>map for all</u> i.

Proposition 5.11 (Mixing homotopy types) (see [MNT]). <u>If</u> X_i <u>are</u> mod $\mathbf{P}_i$ A_n-<u>space</u> $(i \geqq 1)$ <u>such that</u> $(X_i)_{(0)}$ <u>is</u> A_n-<u>equivalent to</u> $2n_i - 1$

$\prod S_{(0)}$. *Then there exists an* A_n*-space* X *such that* X *is* mod $\mathbf{P}_i$ A_n*-homotopy equivalent to* X_i.

Proposition 5.12 (see [MNT]). *In the category of connected complexes,*

1) X *is an* A_n*-space iff* X *is an* A_n*-space* mod $\mathbf{P}_i$ *for each* i *and the rationalization* $X_{\mathbf{P}_i} \to X_{(0)}$ *induces an equivalent* A_n*-structure on* $X_{(0)}$.

2) *A map* f *between* A_n*-spaces is an* A_n*-map iff* f *is an* A_n*-map* mod $\mathbf{P}_i$ *for each* i *and the rationalization of* $f_{\mathbf{P}_i}$ *induces an equivalent* A_n*-structure on* $f_{(0)}$.

Using these propositions, we have

Theorem 5.13 (Zabrodsky). *For every prime* $p \geq 3$ *there exists a finite* CW-*complex which admits an* A_{p-1}*-structure but no* A_p*-structure*.

§6. The sphere extension of a complex Lie group

A Lie group G often acts transitively on a sphere and is regarded as a total space of the fibre bundle over the sphere [B]. Then the Lie group G is called a sphere extension of the isotropy subgroup G_0. Let us consider a new sphere extension $\overline{G}$ of G_0 induced by a map f on spheres. Then $\overline{G}$ is no longer a Lie group, in general. Zabrodsky [Z] shows, however, that $\overline{G}$ is an H-space, provided that the map degree deg(f) of f is prime to 2. It is natural to ask when $\overline{G}$ is a group-like space, in other words, when it is an A_3-space. If the bundle projection $e : G \to S^{2n-1}$ were an A_3-map, it would be trivial. But it is not true in general. Y. Hemmi [H] shows the following

Theorem. *Let* n *be a positive integer not dividing* $2\cdot 3^*$ *and let* G *be* $U(n)$, $SU(n)$ *or* $Sp(m)$ $(n = 2m)$ *acts on the odd sphere* S^{2n-1}. *Then* $\deg(f)$ *is prime to* 6, *provided that* $\overline{G}$ *is an* A_3*-space*.

We discuss the existence problem of a higher homotopy associativity of $\overline{G}$ in this section. Firstly by a simple computation of the action of the mod p Steenrod algebra $\mathbf{A}(p)$ on the mod p cohomology algebra of the projective spaces, we have a generalization of the above theorem.

Theorem 6.1. *Let* p *be a prime number and* n *a positive integer not dividing* $(p-1)\cdot p^*$ *and let* G *be* $U(n)$, $SU(n)$ *or* $Sp(m)$ $(n = 2m)$ *acting on the odd sphere* S^{2n-1}. *Then* $\deg(f)$ *is prime to* p, *provided that* $\overline{G}$ *is a* mod p A_p*-space and* $\overline{f}$ *is a* mod p A_p*-map*.

Corollary 6.2. *Let* n *be prime to* $p!$. *Then* $\deg(f)$ *is prime to* $p!$, *provided that* $\overline{G}$ *is an* A_p*-space and* $\overline{f}$ *is an* A_p*-map*.

(Outline of the proof)

Assuming that $(\deg(f), p) \neq 1$, we deduce a contradiction. By the hypothesis on $\overline{G}$ and $\overline{f}$, the map $\overline{f}$ induces a homomorphism F between the spectral sequences $E_k(\overline{G})$ and $E_k(G)$ of the Stasheff type. We remark that $\overline{G}$ and G have torsion free integral cohomologies and $E_k(\overline{G})$ has also a torsion free integral cohomology for $k \leq p$. By comparing the spectral sequences, we obtain

Proposition 6.3. $\overline{G}$ *is* A_p*-primitive*.

Then the mod p cohomology of $P^k(\bar{G})$ is the direct sum of a polynomial ring and a nilpotent ideal S_k (see [I2]). Let R be the quotient algebra of the mod p cohomology of $P^k(\bar{G})$ by the ideal generated by the image of $\bar{f}^*$ and the nilpotent ideal S_k. By the hypothesis, $R = \mathbf{Z}/p\mathbf{Z}[v]/(v^{p+1})$ with $\deg(v) = 2n$ must be an $\mathbf{A}(p)$-algebra. On the other hand, by the Adem relation, we have

Lemma 6.4. Let $R = \mathbf{Z}/p\mathbf{Z}[v]/(v^{p+1})$ with $\deg(v) = 2n$. Then R can not be an $\mathbf{A}(p)$-algebra unless n divides $(p-1)\cdot p^*$.

It is a contradiction and the proof of Theorem 6.1 is completed.

Q.E.D.

Our main goal of this section is the following

Theorem 6.5. Let G be a compact complex Lie group complex-linearly and transitively acting on the odd sphere S^{2n-1}. Then $\bar{G}$ is an A_k-space A_{k-1}-acting on the sphere and the map $\bar{f}$ covering f is an A_k-map, if the degree deg(f) of f is prime to k!. Moreover $\bar{f}$ preserves A_{k-1}-action in a homotopical sense (see Section 3 and also [N] for the definition of A_{k-1}-action).

Corollary 6.6. Let n be a positive integer not dividing $2\cdot 3^*$. Then $\bar{G}$ is a homotopy associative H-space iff deg(f) is prime to 6.

Remark that the conclusion is equivalent to that $\bar{G}$ is an A_4-space and $\bar{f}$ is an A_4-map.

We use the following method: For the unitary group U(n-1), taking the equivariant localization of the base space S^{2n-1} and the map f, we get an equivariant A_k-space X and an equivariant A_k-map F. By the obstruction theory for an A_k-space and an A_k-map, we can show that this space has a higher homotopy associative equivariant structure for the total space G and the sphere map has also a higher homotopy associative equivariant structure. We often call the higher homotopy associative equivariant structure the A_k-equivariant structure for some k.

Decomposition of the equivariant A_n-action of $U(n)$

We work in the category of (strictly) $U(n-1)$-equivariant spaces and maps. Let p be an odd prime and $\mathbf{P}$ the set of primes $\geqq p$. Recall that the unitary group $U(n)$ acts complex linearly and transitively on the odd sphere $S^{2n-1} \subset \mathbf{C}^n$. Therefore S^{2n-1} is an equivariant based space whose fixed point set by any subgroup H is always a sphere of odd dimension $2n(H)-1$; $2n-1 \geqq 2n(H)-1 > 0$. So we can consider the equivariant localization X (and F) of S^{2n-1} (and a map $f : S^{2n-1} \to S^{2n-1}$ continuously, resp.) (see [I3]). Let $C_2(X)$ be the double (associative) loop space of the double reduced suspension of X. Then $C_2(X)$ is an equivariant A_∞-space by the first loop structure of the double loop.

Recall that the unitary group $U(n)$ is a left equivariant group by the conjugate action of $U(n-1)$ and is also a right equivariant space by the product from the right. We denote by $U(n) \times_{U(n-1)} S^{2n-1}$ the equivariant product of the right equivariant space $U(n)$ and the left equivariant space S^{2n-1}. Then S^{2n-1} admits an equivariant (strict) action of $U(n)$ by the equivariant map from $U(n) \times_{U(n-1)} S^{2n-1}$ to S^{2n-1} along the projection $e : U(n) \to S^{2n-1}$. On the other hand, X is equivariantly mod $\mathbf{P}$ equivalent to S^{2n-1}. Therefore $U(n) \times_{U(n-1)} X$ is mod $\mathbf{P}$

equivariantly equivalent to $U(n)\times_{U(n-1)}S^{2n-1}$ and X admits an A_∞-action of U(n) (see [I1]) by the equivariant A_∞-form, a tuple of k-forms from $K_k \times U(n)\times_{U(n-1)}\cdots\times_{U(n-1)}U(n)\times_{U(n-1)}X$ to X.

Proposition 6.7. <u>There is an equivariant homotopy action</u> T <u>of</u> U(n) <u>on</u> X <u>satisfying the following two conditions</u>:

(1) <u>The</u> A_∞-<u>action of</u> U(n) <u>on</u> X <u>is</u> U(n-1)-<u>equivariantly equivalent to</u> $e'(g) \oplus T(g,x)$ <u>in</u> $C_2(X)$ <u>for</u> $x \in X$, $g \in U(n)$, <u>where</u> $\oplus$ <u>means the associative loop product</u>,

(2) $T(h,x) = hx$, $T(gh,x) = T(g,hx)$ <u>and</u> $T(hg,x) = hT(g,x)$, <u>for</u> $x \in X$, $g \in U(n)$, $h \in U(n-1)$,

<u>where</u> $e' = j\circ e$, j <u>is the inclusion of</u> X <u>into</u> $C_2(X)$.

We inductively construct the k-form of the homotopy action of U(n) on $C_2(X)$ decomposed by the A_{k-1}-form of the homotopy action T. We may assume that the first loop structure of the double loop space $C_2(X)$ is equivariantly associative. We deform the A_k-form of the action in $C_2(X)$ to be decomposed by the A_k-form of T' given by $T'(g,w)(s) = T(g,w(s))$.

The action on X of U(n) is homotopy equivalent to $e'(g) \oplus T'(g)(w)$ for all w in $C_2(X)$. The key lemma of the main theorem is described as follows:

Lemma 6.8. T <u>is an</u> A_{p-1}-<u>action with a</u> k-<u>form</u> N_k^T : $K_k\times U(n)\times_{U(n-1)}\cdots\times_{U(n-1)}U(n)\times_{U(n-1)}X \to X$ <u>of</u> U(n) <u>on</u> X <u>for</u> $k < p$ <u>and there is an</u> A_∞-<u>action</u> $N_k^{e'}$: $K_k\times U(n)\times_{U(n-1)}\cdots\times_{U(n-1)} U(n)\times_{U(n-1)}C_2(X) \to C_2(X)$ <u>of</u> U(n) <u>on</u> $C_2(X)$ <u>equivariantly equivalent to the usual action of</u> U(n) <u>on</u> X <u>in</u> $C_2(X)$; <u>for given</u> $(\sigma;g_1,\ldots,g_{k-1};w)$ <u>in</u> $K_k\times\prod^{k-1}_{U(n-1)}U(n)\times_{U(n-1)}C_2(X)$, <u>the</u> A_∞-<u>action</u> $N_k^{e'}$ <u>has the form</u>:

(6.9) $N_k^{e'}(\sigma;g_1,\dots,g_{k-1};w)$
$= \bigoplus_{i=1}^{k-1} N_i^{T'}(s_{i+1}\cdots s_{k+1}(\sigma);g_1,\dots,g_{i-1};e'(g_i))$
$\oplus N_k^{T'}(\sigma;g_1,\dots,g_{k-1};w).$

Proof. We construct inductively the A_k-forms R_k^j and $N_k^{T'}$ of the map j along the identity and the action along T' respectively by the following formulae:

(1) $R_1^j(*;x) = j(x)$,

(2) $R_k^j(\delta_j(r,s)(\rho,\sigma);g_1,\dots,g_{k-1};x)$
$= R_r^j(\rho;g_1,\dots,g_{j-1},g_j\cdots g_{j+s-1},g_{j+s},\dots,g_{k-1};x)$,

(2)' $R_k^j(\delta_r(r,s)(\rho,\sigma);g_1,\dots,g_{k-1};x)$
$= R_r^j(\rho;g_1,\dots,g_{r-1};g_r\cdots g_{k-1}x)$,

(3) $R_k^j(\delta(t;r_1,\dots,r_t)(\tau;\rho_1,\dots,\rho_t);g_1,\dots,g_{k-1};x)$
$= \bigoplus_{i=1}^{t-1} N_i^{T'}(s_{i+1}\cdots s_t(\tau);g_1\cdots g_{r_1},\dots,g_{n_{i-2}+1}\cdots g_{n_{i-1}};$
$R_{r_i}^j(\rho_i;g_{n_{i-1}+1},\dots,g_{n_i}))$
$\oplus N_t^{T'}(\tau;g_1\cdots g_{r_1},\dots,g_{n_{i-2}+1}\cdots g_{n_{i-1}};$
$R_{r_t}^j(\rho_t;g_{n_{t-1}+1},\dots,g_{k-1};x))$, where $n_i = r_1 + \dots + r_i$,

(4) $R_k^j(\gamma;g_1,\dots,g_{j-1},*,g_{j+1},\dots,g_{k-1};x)$
$= R_{k-1}^j(d_j(\gamma);g_1,\dots,g_{j-1},g_{j+1},\dots,g_{k-1};x)$,

(4)' $R_k^j(\gamma;g_1,\dots,g_{k-1};*) = e'(g_1\cdots g_{k-1})$.

(Outline of the proof of Theorem 6.5)

Since X is mod **P** equivariantly equivalent to S^{2n-1}, we may identify X with S^{2n-1}. Recall that $e : G \to X$ is a fibration and $e' = j\circ e : G \to C_2(X)$. Let $\bar{e}$ be the induced fibration of e by f and let $f' = j\circ f$. Then $e\circ\bar{f} = f\circ\bar{e}$ and $e'\circ\bar{f} = f'\circ\bar{e}$.

We prove Theorem 6.5 through the inverse process of Lemma 6.8. Let $Y = C_2(X)\times U(n)$ and $\bar{E} = \langle f'\circ\bar{e},\bar{f}\rangle : \bar{G} \to Y$ which we may assume to be an inclusion. Then Y is the A_∞-space with the following A_k-form m_k^Y:

(6.10) $pr_1 \circ m_k^Y(\tau; y_1, \ldots, y_k) = \bigoplus_{i=1}^{k} N_i^{T'}(s_{i+1} \cdots s_k(\tau); g_1, \ldots, g_{i-1}); x_i)$,

$pr_2 \circ m_k^Y(\tau; y_1, \ldots, y_k) = g_1 \cdots g_k$ for $y_j = (x_j, g_j)$.

By using the equivariant obstruction theory, we obtain that $\overline{G}$ is an A_k-space A_k-acting on X and the map f along $\overline{f}$ preserves A_k-actions by induction on $k < p$, namely,

Proposition 6.11. <u>There are A_{p-1}-forms $m_k^{\overline{G}}$, $F_k^{\overline{E}}$, $N_k^{\overline{e}}$ and $R_k^{f'}$ of the space $\overline{G}$, the map $\overline{E}$, the action along $\overline{e}$ and the map f' preserving action along the homomorphism $\overline{f}$ respectively satisfying the following formulae</u>:

(1) $R_1^{f'}(*; x) = j(x)$,

(2) $R_k^{f'}(\delta_j(r,s)(\rho,\sigma); \overline{g}_1, \ldots, \overline{g}_{k-1}; x)$
$= R_r^{f'}(\rho; \overline{g}_1, \ldots, \overline{g}_{j-1}, m_s^{\overline{G}}(\sigma; \overline{g}_j, \ldots, \overline{g}_{j+s-1}), \overline{g}_{j+s}, \ldots, \overline{g}_{k-1}; x)$,

(2)' $R_k^{f'}(\delta_r(r,s)(\rho,\sigma); \overline{g}_1, \ldots, \overline{g}_{k-1}; x)$
$= R_r^{f'}(\rho; \overline{g}_1, \ldots, \overline{g}_{r-1}; N_s^{\overline{e}}(\sigma; \overline{g}_r, \ldots, \overline{g}_{k-1}; x))$,

(3) $R_k^{f'}(\delta(t; r_1, \ldots, r_t)(\tau; \rho_1, \ldots, \rho_t); \overline{g}_1, \ldots, \overline{g}_{k-1}; x)$

$= \bigoplus_{i=1}^{t-1} N_i^{T'}(s_{i+1} \cdots s_t(\tau); m_{r_1}^{\overline{G}}(\rho_1; \overline{g}_1, \ldots, \overline{g}_{r_1}), \ldots;$

$R_{r_i}^{f'}(\rho_i; \overline{g}_{n_{i-1}+1}, \ldots, \overline{g}_{n_i}))$

$\oplus\ N_t^{T'}(\tau; m_{r_1}^{\overline{G}}(\rho_1; \overline{g}_1, \ldots, \overline{g}_{r_1}), \ldots; R_{r_t}^{f'}(\rho_t; \overline{g}_{n_{t-1}+1}, \ldots, \overline{g}_{k-1}; x))$,

where $n_i = r_1 + \cdots + r_i$,

(4) $R_k^{f'}(\gamma; \overline{g}_1, \ldots, \overline{g}_{j-1}, *, \overline{g}_{j+1}, \ldots, \overline{g}_{k-1}; x)$
$= R_{k-1}^{f'}(d_j(\gamma); \overline{g}_1, \ldots, \overline{g}_{j-1}, \overline{g}_{j+1}, \ldots, \overline{g}_{k-1}; x)$,

(4)' $R_k^{f'}(\gamma; \overline{g}_1, \ldots, \overline{g}_{k-1}; *) = e' \circ F_{k-1}^{\overline{f}}(\gamma; \overline{g}_1, \ldots, \overline{g}_{k-1})$,

(5) $pr_1 \circ F_k^{\overline{E}}(\gamma; \overline{g}_1, \ldots, \overline{g}_k) = R_k^{f'}(\gamma; \overline{g}_1, \ldots, \overline{g}_{k-1}; x_k)$

(6) $pr_2 \circ F_k^{\overline{E}}(\gamma; \overline{g}_1, \ldots, \overline{g}_k) = g_1 \cdots g_k$ for $\overline{g}_j = (x_j, g_j)$.

Remark. 1) The A_k-form $R_k^{f'}$ gives the A_k-form R_k^f of A_k-equivariant map f (along $\bar{f}$) by using the equivariant compression theory.

2) These homotopy actions are left homotopy actions.

This completes the proof of Theorem 6.5.

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