The Milnor–Stasheff Filtration on Spaces and Generalized Cyclic Maps

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Abstract. The concept of $C_k$-spaces is introduced, situated at an intermediate stage between $H$-spaces and $T$-spaces. The $C_k$-space corresponds to the $k$-th Milnor–Stasheff filtration on spaces. It is proved that a space $X$ is a $C_k$-space if and only if the Gottlieb set $G(Z, X) = [Z, X]$ for any space $Z$ with $\text{cat}Z \leq k$, which generalizes the fact that $X$ is a $T$-space if and only if $G(\Sigma B, X) = [\Sigma B, X]$ for any space $B$. Some results on the $C_k$-space are generalized to the $C_k$-space for a map $f: A \to X$. Projective spaces, lens spaces and spaces with a few cells are studied as examples of $C_k$-spaces, and non-$C_k$-spaces.

1 Introduction

A 0-connected space $X$ is called a $T$-space if the fibration $\Omega X \to X^{S^1} \to X$ is fiber homotopically trivial [1], and it is known that any 0-connected $H$-space is a $T$-space. To investigate intermediate stages between $H$-spaces and $T$-spaces, Aguadé [1] defined $T_k$-spaces for any integer $k \geq 1$ and $k = \infty$, making use of the Milnor–Stasheff filtration on spaces, so that the $T_\infty$-space is an $H$-space and the $T_1$-space is a $T$-space. It seems that relations between $T_k$-spaces and the L-S category of spaces were not investigated clearly after his work. In this paper we define the concept of the $C_k$-space for $k \geq 1$ so that the $C_1$-space is the same as the $T$-space and the $C_\infty$-space is an $H$-space. We also employ the Milnor–Stasheff filtration on spaces to define $C_k$-spaces. However, the definition of the $C_k$-space is directly connected with the L-S category; it enables us to prove, for example, that a space $X$ is a $C_k$-space if and only if the Gottlieb set $G(Z, X) = [Z, X]$ for any space $Z$ with $\text{cat}Z \leq k$ (Theorem 2.3), which is a generalization of the fact that $X$ is a $T$-space if and only if the Gottlieb group $G(\Sigma B, X) = [\Sigma B, X]$ for any space $B$ [26, Theorem 2.2].

For each $k$, let $j^k_1: \Sigma Omega X \to P^1(\Omega X) \to P^k(\Omega X)$ and $c^k_1: P^k(\Omega X) \to P^{k\infty}(\Omega X) \simeq X$ be the natural inclusions for the spaces $P^k(\Omega X)$ [16, 21] (see [22]). Let $f: A \to X$ be any map. A 0-connected space $X$ is called a $C^f_k$-space if $c^k_1: p^k(\Omega X) \to X$ is $f$-cyclic (Definition 3.1). A $C^k_0$-space $X$ is called a $C^k_0$-space (Definition 2.1).

We show that a space $X$ is a $C^f_k$-space if and only if $G^f(Z, X) = [Z, X]$ for any space $Z$ with $\text{cat}Z \leq k$ (Theorem 3.2). Let $f: A \to X$ and $g: B \to Y$ be any maps. The product space $X \times Y$ is a $C^f_{k\times g}$-space if and only if $X$ is a $C^f_k$-space and $Y$ is a $C^g_k$-space (Theorem 4.7). It follows that the product space $X \times Y$ is a $C^f_k$-space if and only if both $X$ and $Y$ are $C^g_k$-spaces (Theorem 4.8).
Let $\tilde{X}$ be a covering space of a space $X$ with the covering map $p: \tilde{X} \to X$ and $1 \leq k \leq \infty$. Let $f: A \to X$, $\tilde{f}: B \to \tilde{X}$, and $q: B \to A$ be maps such that the following diagram is homotopy commutative,

$$
\begin{array}{ccc}
B & \xrightarrow{\tilde{f}} & \tilde{X} \\
\downarrow{q} & & \downarrow{p} \\
A & \xrightarrow{f} & X
\end{array}
$$

In Theorem 4.9 we show that if $X$ is a $C^f_k$-space, then the covering space $\tilde{X}$ is a $C^{\tilde{f}}_k$-space. A relation between two “multiplications” that are induced by a pairing and a copairing [18, Proposition 3.4] will be used to prove Theorem 4.9. A similar result holds for the $T^f_k$-space, which is a generalization of Aguadé’s $T_k$-space (see Definition 3.3). If we put $f = 1_X$, $\tilde{f} = 1_{\tilde{X}}$, $q = p$, then we see that any covering space of a $C_k$-space (resp. Aguadé’s $T_k$-space) is a $C_k$-space (resp. $T_k$-space) for any $1 \leq k \leq \infty$ (Theorem 4.10).

In the last section we study projective spaces, lens spaces and spaces with a few cells.

### 2 $C_k$-Spaces

We work in the category of topological spaces with base point. The symbol $f \sim g: X \to Y$ means the based homotopy relation and the symbol $X \simeq Y$ the based homotopy equivalence. The set of based homotopy classes of maps $[f]: X \to Y$ is denoted by $[X, Y]$. Let $f: A \to X$ be a map. A based map $g: B \to X$ is said to be $f$-cyclic [17] if there exists a map $\phi: A \times B \to X$ such that the diagram

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\phi} & X \\
\downarrow{j} & & \downarrow{\nabla} \\
A \vee B & \xrightarrow{f \vee g} & X \vee X
\end{array}
$$

is homotopy commutative, where $j: A \vee B \to A \times B$ is the inclusion and $\nabla: X \vee X \to X$ is the folding map. We call such a map $\phi$ an associated map of an $f$-cyclic map $g$.

Clearly, $g$ is $f$-cyclic if and only if $f \perp g$ is $g$-cyclic. We write $f \perp g$ if $g$ is $f$-cyclic. If $f \perp g$ for maps $f: A \to X$ and $g: B \to X$, then $(w \circ f \circ f') \perp (w \circ g \circ g')$ for any maps $w: X \to W$, $f': A' \to A$, and $g': B' \to B$ by [17, Theorems 1.4 and 1.5]. This formula is used repeatedly in the following arguments without further reference. A based map $g: B \to X$ is said to be cyclic [23] if $1_X \perp g$, that is, $g$ is $1_X$-cyclic. The Gottlieb set denoted by $G(B, X)$ is the set of all homotopy classes of cyclic maps from $B$ to $X$. 
The Milnor–Stasheff Filtration on Spaces and Generalized Cyclic Maps

The loop space $\Omega X$ of any space $X$ has a homotopy type of an associative $H$-space. A 0-connected space $X$ is filtered by the projective spaces of $\Omega X$ [16, 21]:

$$* = P^0(\Omega X) \hookrightarrow \Sigma \Omega X = P^1(\Omega X) \hookrightarrow \cdots \hookrightarrow P^k(\Omega X) \hookrightarrow \cdots \hookrightarrow P^\infty(\Omega X) \simeq X.$$  

For each $k$, let $j_k^X: \Sigma \Omega X = P^1(\Omega X) \to P^k(\Omega X)$ and $e_k^X: P^k(\Omega X) \to P^\infty(\Omega X) \simeq X$ be the natural inclusions. We write $e_k^X = e_k: \Sigma \Omega X \to P^k(\Omega X) \to X$. We see that $j_k^X \sim e_k^X: \Sigma \Omega X \to X$ and $e_k^X \sim 1_X: X \to X$.

A 0-connected space $X$ is called a $T_k$-space [1] if $1_X \perp e_k^X$ for some extension $\eta_k: P^k(\Omega X) \to X$ of $e_k^X: \Sigma \Omega X \to X$, that is, there exists a map $\phi_k: X \times P^k(\Omega X) \to X$ such that $\phi_k \circ j \circ (1_X \vee j_k^X) \sim \nabla \circ (1_X \vee e_k^X): X \vee \Sigma \Omega X \to X$. Aguadé showed that $X$ is a $T$-space if and only if $X$ is a $T_k$-space [1, Proposition 4.1]. If $X$ is a $T_k$-space, then it is a $T_i$-space for any $1 \leq i \leq k$. By [1, Proposition 4.1(i)(ii)], a 0-connected space is an $H$-space if and only if it is a $T_\infty$-space; we remark that $T_\infty \sim 1_X$ when $X$ is a 0-connected CW complex. The concepts of the $T$-space and the Gottlieb set are closely connected by the fact that $X$ is a $T$-space if and only if $G(\Sigma B, X) = [\Sigma B, X]$ for any space $B$ [26, Theorem 2.2].

**Definition 2.1** Let $k \geq 1$ be an integer or $k = \infty$. A 0-connected space $X$ is called a $C_k$-space if $1_X \perp e_k^X$, that is, the inclusion $e_k^X: P^k(\Omega X) \to X$ is cyclic. A 0-connected space $X$ is called an NC-space if $X$ is not a $C_k$-space for any $k \geq 1$.

Clearly any $C_k$-space is a $T_k$-space for any $k \geq 1$. We use the L-S category $\text{cat} X$ for a 0-connected space $X$ in the sense that $\text{cat} X = n$ if $n$ is the minimum number of categorical open coverings $U_0, U_1, \ldots, U_n$ of $X$, so that $\text{cat} X = 0$ if and only if $X$ is contractible and $\text{cat} X \leq 1$ if $X$ is a suspension. Throughout this paper, we follow Iwase for the notations for the L-S category; his list of references covers much of the widely-known literature [11].

We now recall Ganea’s theorem [10, 11].

**Theorem 2.2** (Ganea [3, 10]) Let $k \geq 1$ be an integer or $k = \infty$ and assume that $X$ is a 0-connected space. The category $\text{cat} X \leq k$ if and only if $e_k^X: P^k(\Omega X) \to X$ has a right homotopy inverse.

In the rest of this section, we mention some results on the $C_k$-space that are obtained as special cases of the results on the $C_k^f$-spaces for a map $f: A \to X$ in the following sections, since the $C_k$-space is the $C_k^f$-space for the identity map $f = 1_X: X \to X$.

The property of the $T$-spaces in [26, Theorem 2.2] is extended to the $C_k$-spaces using the L-S category in the sense that the L-S category of any suspension space $\Sigma B$ satisfies $\text{cat} \Sigma B \leq 1$.

**Theorem 2.3** Let $k \geq 1$ be an integer. A space $X$ is a $C_k$-space if and only if $G(Z, X) = [Z, X]$ for any space $Z$ with $\text{cat} Z \leq k$.

Theorem 2.3 is a special case of Theorem 3.2 which is proved in the next section. The following proposition is a direct consequence of the definition.
Proposition 2.4  
(i) A space $X$ is a $T$-space if and only if $X$ is a $C_1$-space.  
(ii) Any $C_m$-space is a $C_n$-space for $\infty \geq m \geq n \geq 1$.  
(iii) A space $X$ is an $H$-space if and only if $X$ is a $C_\infty$-space.

As a direct consequence of Proposition 2.4(ii),(v) and Theorem 4.3, the following theorem is obtained.

Theorem 2.5 Assume that $\text{cat}(X) = k \geq 1$. Then $X$ is an $H$-space if and only if $X$ is a $C_k$-space for some $n \geq k$.

It is known [14] that $\text{cat}(X) \leq \dim X$ for any finite CW complex $X$. Thus, we obtain the following corollary.

Corollary 2.6 If a $T$-space $X$ is a $1$-dimensional finite CW complex, then $X = S^1$.

Example 2.7 By [1, Proposition 4.2] Agadé obtained a space $X$ such that $X$ is a $T_{p-1}$-space but not a $T_p$-space. This space $X$ is not a $C_p$-space, but it is not known whether $X$ is a $C_{p-1}$-space or not.

3 $C^f_k$-Spaces for a Map $f: A \to X$

We denote the set of all homotopy classes of $f$-cyclic maps from $B$ to $X$ by

$$G(B;A,f,X) = G^f(B,X) = f^\perp(B,X) \subset [B,X].$$

This is called the Gottlieb set for a map $f: A \to X$. If $f = 1_X: X \to X$, then we recover the set $G(B,X)$ defined by Varadaran [23]:

$$G(B,X) = G(B;X,1_X,X) = G^{1_X}(B,X) = (1_X)^\perp(B,X).$$

In general, $G(B,X) \subset G^f(B,X) \subset [B,X]$ for any spaces $A,B,X$ and any map $f: A \to X$. An example is shown in [27] such that $G(B,X) \neq G(B,A,f,X) \neq [B,X]$:

$$G_S(S^5 \times S^5) \cong 2\mathbb{Z} \oplus 2\mathbb{Z} \neq G_S(S^5, i_1, S^5 \times S^5) \cong 2\mathbb{Z} \oplus \mathbb{Z} \neq \pi(S^5 \times S^5) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Definition 3.1 Let $k \geq 1$ be an integer or $k = \infty$. Let $f: A \to X$ be any map. A 0-connected space $X$ is called a $C^f_k$-space if $f \perp e^X_k$ (or $e^X_k: P^k(\Omega X) \to X$ is $f$-cyclic). A 0-connected space $X$ is called an $NC^f_k$-space if $X$ is not a $C^f_k$-space for any $k \geq 1$.

We see that a $C^f_k$-space $X$ is a $C_k$-space.

Theorem 3.2 Let $f: A \to X$ be any map. A space $X$ is a $C^f_k$-space if and only if $G^f(Z,X) = [Z,X]$ for any space $Z$ with $\text{cat}(Z) \leq k$.

Proof Suppose that $X$ is a $C^f_k$-space, namely, $f \perp e^X_k$. Let $Z$ be a space with $\text{cat}(Z) \leq k$ and $g: Z \to X$ any map. Since $\text{cat}(Z) \leq k$, there exists a map $s^Z_k: Z \to P^k(\Omega Z)$ such
that $e_k^x \circ s_k^x \sim 1_Z$. We see that $e_k^x \circ P^k(\Omega g) \sim g \circ e_k^x$ by the naturality of the construction of $P^k(\Omega Z)$, as is shown in the following homotopy commutative diagram:

$$
\begin{array}{ccc}
P^k(\Omega Z) & \xrightarrow{P^k(\Omega g)} & P^k(\Omega X) \\
\zeta_k & \downarrow g & \downarrow \zeta_k \\
Z & \rightarrow & X
\end{array}
$$

Hence the relation $f \perp e_k^X$ implies $f \perp (e_k^X \circ P^k(\Omega g) \circ s_k^X)$ or $f \perp g$. It follows that $G^l(Z, X) = \{Z, X\}$.

Conversely, assume that $G^l(Z, X) = \{Z, X\}$ for any space $Z$ with $\text{cat } Z \leq k$. It is known that $\text{cat } C_0 \leq \text{cat } Y + 1$ for any map $\theta: X \rightarrow Y$ [24, (1.6) Theorem, p. 459], where $C_0$ is the mapping cone of $\theta$. Thus $\text{cat } P^k(\Omega X) = \text{cat } C_0 \leq \text{cat } P^{k-1}(\Omega X) + 1$, where $\theta: (\Omega X) * \cdots * (\Omega X)(k-times) \rightarrow P^{k-1}(\Omega X)$ is the map in [21, Part I, Theorem 12]. By induction, we have $\text{cat } P^k(\Omega X) \leq k$. Thus we know that $e_k^X: P^k(\Omega X) \rightarrow X$ is $f$-cyclic by our assumption, and hence $X$ is a $C^l_k$-space. \hfill \blacksquare

A space $X$ is called an $H^l$-space for a map $f: A \rightarrow X$ if $1_X$ is $f$-cyclic (namely $f \perp 1_X$), and a $T^l$-space for a map $f: A \rightarrow X$ if $e^X: \Sigma \Omega X \rightarrow X$ is $f$-cyclic (namely $f \perp e^X$)[28,29]. Any $H$-space $X$ is an $H^l$-space and any $H^l$-space $X$ is a $T^l$-space for any map $f: A \rightarrow X$. We remark that the 2-dimensional sphere $S^2$ is not an $H$-space nor a $T$-space, but it is an $H^n_\infty$-space and a $T^n_\infty$-space for the Hopf map $\eta_2: S^3 \rightarrow S^5$ [29, Example 2.10], [26, Corollary 2.8].

**Definition 3.3** Let $f: A \rightarrow X$ be any map. A space $X$ is called a $T^l_k$-space if $f \perp \zeta_k$ for some extension $\zeta_k: P^k(\Omega X) \rightarrow X$ of $e^X: \Sigma \Omega X \rightarrow X$, that is, there exists a map $\phi_k: A \times P^l(\Omega X) \rightarrow X$ such that $\phi_k \circ f \circ (1_X \cup j_k^X) \sim \nabla \circ (f \circ e^X): A \cup P^l(\Omega X) \rightarrow X$.

An $H^{l+s}$-space $X$ is an $H$-space and a $T^{l+s}_k$-space $X$ is a $T_k$-space.

**Proposition 3.4** Let $f: A \rightarrow X$ be any map.

(i) $X$ is a $C^l_1$-space $\iff$ $X$ is a $T^l_1$-space $\iff$ $X$ is a $T^l$-space.

(ii) Any $C^m_\infty$-space is a $C^m_l$-space for $\infty \geq m \geq n \geq 1$.

(iii) Any $T^m_\infty$-space is a $T^m_l$-space for $\infty \geq m \geq n \geq 1$.

(iv) If $X$ is a $C^l_\infty$-space, then $X$ is a $T^l_\infty$-space for $\infty \geq k \geq 1$.

(v) If $X$ has the homotopy type of a CW complex, then the following equivalences hold:

$$
X \text{ is an } H^l \text{-space } \iff X \text{ is a } C^l_\infty \text{-space } \iff X \text{ is a } T^l_\infty \text{-space}.
$$

**Proof** These results are direct consequences of the definitions except the following part of (v): "$X$ is a $T^l_\infty$-space $\Rightarrow$ $X$ is an $H^l$-space", which is proved by a method similar to the proof of [1, Proposition 4.1 (ii)] as follows.

Suppose that $X$ is a $T^l_\infty$-space. Then $f \perp \tau$ for some extension $\tau: P^\infty(\Omega X)(\simeq X) \rightarrow X$ of $e^1_\tau: \Sigma \Omega X \rightarrow X$, and there exists a map $m: A \times P^\infty(\Omega X) \rightarrow X$ with axes $f$ and $\tau$. \hfill \blacksquare
making the following diagram commutative up to homotopy:

\[
\begin{array}{ccc}
A \times X & \xrightarrow{1 \times e_\infty^X} & A \times P^\infty(\Omega X) \\
\cong & & \rightarrow \\
\cup & & m \\
1 \times e_\Sigma \Omega X & \xleftarrow{1 \times e_\Sigma \Omega X} & A \times \Sigma \Omega X
\end{array}
\]

Let \( g: X \to X \) be a map given by \( g(x) = m \circ (1 \times e_\infty^X)^{-1}(\ast, x) \) for any \( x \in X \). Then, \( g \sim \tilde{e} \circ (e_\infty^X)^{-1} \) and we have \( g \circ e_1^X \sim e_1^X \), and hence \( \Omega g \sim 1_{\Omega X} \) by taking adjoints. Then it follows that \( g: X \to X \) is a weak homotopy equivalence and hence is a homotopy equivalence if \( X \) has the homotopy type of a CW complex, by a theorem of J. H. C. Whitehead, and there exists a map \( h: X \to X \) such that \( g \circ h \sim 1_X \). Hence we have \( f \perp g \), which implies that \( f \perp (g \circ h) \) or \( f \perp 1_X \) by the composition formula we discussed at the start of Section 2.

4 More about \( T^f_k \)-Spaces and \( C^f_k \)-Spaces

**Proposition 4.1** Let \( f: A \to X \) and \( g: B \to A \) be any maps.

(i) If \( X \) is an \( H^f \)-space, then \( X \) is an \( H^f \circ g \)-space.

(ii) If \( X \) is a \( T^f_k \)-space, then \( X \) is a \( T^f_k \circ g \)-space.

(iii) If \( X \) is a \( C^f_k \)-space, then \( X \) is a \( C^f_k \circ g \)-space.

**Proof** The relations (i) \( f \perp 1_X \), (ii) \( f \perp \tilde{e}_k \), and (iii) \( f \perp e_k^X \) imply (i) \( f \perp (g \circ h) \perp 1_X \), (ii) \( (f \circ g) \perp e_k^X \), and (iii) \( (f \circ g) \perp e_k^X \), respectively, and we have the results.

**Proposition 4.2** Assume that \( f: A \to X \) has a right inverse \( s: X \to A \), i.e., \( f \circ s \sim 1_X \). Then the following results hold.

(i) An \( H^f \)-space \( X \) is an \( H \)-space.

(ii) A \( T^f_k \)-space \( X \) is a \( T_k \)-space.

(iii) A \( C^f_k \)-space \( X \) is a \( C_k \)-space.

**Proof** These are immediate by Proposition 4.1.

If \( X \) is an \( H^f \)-space, then \( X \) is a \( C^f_k \)-space for any \( k \geq 1 \) by Proposition 3.4 (ii), (v).

The following theorem shows that the converse holds if \( \text{cat } X \leq k \).

**Theorem 4.3** Let \( f: A \to X \) be any map.

(i) If \( X \) is a \( C^f_k \)-space and \( \text{cat } X \leq k \), then \( X \) is an \( H^f \)-space.

(ii) If \( X \) is a \( C_k \)-space and \( \text{cat } X \leq k \), then \( X \) is an \( H \)-space.

**Proof** (i) Since \( \text{cat } X \leq k \), we see that \( G^f(X, X) = [X, X] \) by Theorem 3.2. It follows that \( f \perp 1_X \). (ii) is the case where \( f = 1_X \), and hence \( 1_X \perp 1_X \).

**Theorem 4.4** Assume that \( Y \) is a homotopy retract of \( X \) with the maps \( r: X \to Y \) and \( s: Y \to X \) such that \( r \circ s \sim 1_Y \).
The Milnor–Stasheff Filtration on Spaces and Generalized Cyclic Maps

(i) If $X$ is a $C_k^{pf}$-space, then $Y$ is a $C_k^{pf}$-space for any map $f: A \to X$.

(ii) If $X$ is a $C_k$-space, then $Y$ is a $C_k$-space.

**Proof** Let $\tau_k = P^k(\Omega r) : P^k(\Omega X) \to P^k(\Omega Y)$ and $\bar{\tau}_k = P^k(\Omega s) : P^k(\Omega Y) \to P^k(\Omega X)$ be the maps induced by $r$ and $s$, respectively. Then we see that

$$e_k^X = r \circ s \circ e_k^Y = e_k^X \circ \tau_k \circ \bar{\tau}_k = r \circ e_k^Y \circ \bar{\tau}_k : P^k(\Omega Y) \to Y.$$ 

Then (i) the relation $f \perp e_k^X$ implies $(r \circ f) \perp (r \circ e_k^Y \circ \bar{\tau}_k)$, or $(r \circ f) \perp e_k^Y$ and (ii) the relation $1_X \perp e_k^X$ implies $(r \circ 1_X \circ s) \perp (r \circ e_k^Y \circ \bar{\tau}_k)$, or $1_Y \perp e_k^X$ [17, Theorems 1.4, 1.5].

The following result is a generalization of Woo and Kim [25, Theorem 3.6].

**Proposition 4.5** Let $f: A \to X$ and $g: B \to Y$ be any maps. The relation

$$G^{f \times g}(Z, X \times Y) \cong G^f(Z, X) \times G^g(Z, Y)$$

holds for any space $Z$ (under the identification $[Z, X \times Y] \cong [Z, X] \times [Z, Y]$).

**Proof** Let $\alpha: Z \to X$ and $\beta: Z \to Y$ be maps. We define a map $\langle \alpha, \beta \rangle: Z \to X \times Y$ by $\langle \alpha, \beta \rangle = (\alpha \times \beta) \circ \Delta_Z$ for the diagonal map $\Delta_Z: Z \to Z \times Z$. Suppose that $\langle \alpha, \beta \rangle \in G^f(Z, X) \times G^g(Z, Y)$, which is identified with a map $\langle f, g \rangle: Z \times X \to Y$. Since $f \perp \alpha$ and $g \perp \beta$, we have $(f \times g) \perp \langle \alpha, \beta \rangle$ [17, Proposition 1.7]. It follows that $(f \times g) \perp \langle \alpha, \beta \rangle$, or $(f \times g) \perp \langle f \times g, \alpha, \beta \rangle$, and hence $\langle \alpha, \beta \rangle \in G^{f \times g}(Z, X \times Y)$.

Conversely, suppose that $\langle \alpha, \beta \rangle \in G^{f \times g}(Z, X \times Y)$. Let $p_1: X \times Y \to X$ and $p_2: X \times Y \to Y$ be the projections and $i_1: X \to X \times Y$ and $i_2: Y \to X \times Y$ be the inclusions defined by $i_1(x) = (x, y_0)$ and $i_2(y) = (x_0, y)$ for any $x \in X$ and $y \in Y$, where $x_0 \in X$ and $y_0 \in Y$ are base points. It follows that

$$\{ p_1 \circ (f \times g) \circ i_1 \} \perp \{ p_1 \circ (\alpha, \beta) \} \quad \text{and} \quad \{ p_2 \circ (f \times g) \circ i_2 \} \perp \{ p_2 \circ (\alpha, \beta) \}$$

and we have $f \perp \alpha$ and $g \perp \beta$. It follows that $\alpha \in G^f(Z, X)$ and $\beta \in G^g(Z, Y)$.

**Remark 4.6** The converse of Proposition 1.7 of [17] holds by an argument similar to the proof of Proposition 4.5. Let $f_1: X_1 \to Z_1$, $f_2: X_2 \to Z_2$, $g_1: Y_1 \to Z_1$, $g_2: Y_2 \to Z_2$ be any maps. Then the following statements are equivalent.

(i) $f_1 \perp g_1$ and $f_2 \perp g_2$.

(ii) $(f_1 \times f_2) \perp (g_1 \times g_2)$

**Theorem 4.7** Let $f: A \to X$ and $g: B \to Y$ be any maps. The product space $X \times Y$ is a $C_k^{f \times g}$-space if and only if $X$ is a $C_k^{f}$-space and $Y$ is a $C_k^{g}$-space.

**Proof** If $X \times Y$ is a $C_k^{f \times g}$-space, then for any space $Z$ with cat $Z \leq k$ we see

$$G^f(Z, X) \times G^g(Z, Y) \cong G^{f \times g}(Z, X \times Y) = [Z, X \times Y] \cong [Z, X] \times [Z, Y]$$

by Theorem 4.5 and Proposition 4.5 and hence $G^f(Z, X) = [Z, X]$ and $G^g(Z, Y) = [Z, Y]$ for any space $Z$ with cat $Z \leq k$. Conversely, suppose that $X$ is a $C_k^{f}$-space and $Y$ is a $C_k^{g}$-space. Then $G^f(Z, X) = [Z, X]$ and $G^g(Z, Y) = [Z, Y]$ for any space $Z$ with cat $Z \leq k$ by Theorem 4.5. It follows that $G^{f \times g}(Z, X \times Y) \cong G^f(Z, X) \times G^g(Z, Y) = [Z, X \times Y]$ for any space $Z$ with cat $Z \leq k$. 


**Theorem 4.8** The product space $X \times Y$ is a $C_k$-space if and only if both $X$ and $Y$ are $C_k$-spaces.

**Proof** Set $f = 1_X$ and $g = 1_Y$ in Theorem 4.7. Then we have the result. 

We now consider covering spaces of $C_k^f$-spaces and $T_k^f$-spaces.

**Theorem 4.9** Let $\tilde{X}$ be a covering space of a space $X$ with the covering map $p: \tilde{X} \to X$ and $1 \leq k \leq \infty$. Let $f: A \to X$, $\tilde{f}: B \to \tilde{X}$, and $q: B \to A$ be maps such that the following diagram is homotopy commutative:

$$
\begin{array}{ccc}
B & \xrightarrow{\tilde{f}} & \tilde{X} \\
\downarrow q & & \downarrow p \\
A & \xrightarrow{f} & X
\end{array}
$$

(i) If $X$ is a $C_k^f$-space, then the covering space $\tilde{X}$ is a $C_k^\tilde{f}$-space.

(ii) If $X$ is a $T_k^f$-space, then the covering space $\tilde{X}$ is a $T_k^\tilde{f}$-space.

**Proof** (i) Since $X$ is a $C_k^f$-space, there exists a map $m_k$ for $f \perp e_X^k$. Consider the following diagram.

$$
\begin{array}{ccc}
B \times P^k(\Omega \tilde{X}) & \xrightarrow{\tilde{m}_k} & \tilde{X} \\
\downarrow q \times P^k(\Omega p) & & \downarrow p \\
A \times P^k(\Omega X) & \xrightarrow{m_k} & X
\end{array}
$$

We must show that

$$(m_k \circ (q \times P^k(\Omega p)))_*(\pi_1(B \times P^k(\Omega \tilde{X}))) \subset p_*\pi_1(\tilde{X})$$

To obtain a map $\tilde{m}_k: B \times P^k(\Omega \tilde{X}) \to \tilde{X}$ for $f \perp e_X^k$. Let $(\alpha, \beta) \in \pi_1(B \times P^k(\Omega \tilde{X}))$ be any element. We see that

$$(m_k \circ (q \times P^k(\Omega p))_*((\alpha, \beta)) = (p \circ \tilde{f})_*((\alpha, \beta))$$

by [18, Proposition 3.4 (1)], since $f \circ q \sim p \circ \tilde{f}$ by assumption and the following
The Milnor–Stasheff Filtration on Spaces and Generalized Cyclic Maps

Diagram is homotopy commutative:

\[
\begin{array}{ccc}
P^k(\Omega \widetilde{X}) & \xrightarrow{\delta_k^X} & \widetilde{X} \\
\downarrow p^k(\Omega) & & \downarrow p \\
P^k(\Omega X) & \xrightarrow{\delta_k^X} & X
\end{array}
\]

(ii) is proved by an argument similar to (i); the proof is omitted.

The following theorem is obtained by setting \(A = X\), \(B = \widetilde{X}\), \(q = p: \widetilde{X} \to X\), \(f = 1_X\), and \(\bar{f} = 1_{\widetilde{X}}\) in Theorem 4.9.

**Theorem 4.10** Any covering space of a \(C_k\)-space (resp. \(T_k\)-space) is a \(C_k\)-space (resp. \(T_k\)-space) for any \(1 \leq k \leq \infty\).

5 Applications and Examples

We have the following result by Theorem 2.5.

**Proposition 5.1** If \(X\) is a \(C_m\)-space with \(\text{cat} X \leq m\) for some \(m \geq 1\), then \(X\) is an \(H\)-space.

**Proposition 5.2** (i) If \(\text{cat} X = 1\) (for example, \(X = \Sigma A\), or a general co-\(H\)-space) and \(X\) is not an \(H\)-space, then \(X\) is an \(NC\)-space.

(ii) If \(\Sigma X\) is a \(C_1\)-space, then \(\Sigma X = S^1, S^3,\) or \(S^7\).

**Proof** (i) and (ii) are obtained by Proposition 5.1.

Let \(X\) be a 0-connected space. A space \(X\) is called a Gottlieb space or a \(G\)-space if the Gottlieb group \(G_m(X) = \pi_m(X)\) for any \(m \geq 1\) \([4, 5]\). A space \(X\) is called a Whitehead space or a \(W\)-space if every Whitehead product \([\alpha, \beta] = 0\) in \([S^m + n + 1, X]\) for any \(\alpha \in [S^m + 1, X], \beta \in [S^m + 1, X]\), and any \(n, m \geq 0\). A space \(X\) is called a generalized Whitehead space or a \(GW\)-space if every generalized Whitehead product on \(X\) is trivial, that is, \([\alpha, \beta] = 0\) in \([\Sigma(A \wedge B), X]\) for any \(\alpha \in [\Sigma A, X], \beta \in [\Sigma B, X]\), and any spaces \(A, B\).

**Remark 5.3** The following implications hold:

(i) \(X\) is a \(C_1\)-space \(\Rightarrow X\) is a \(G\)-space \(\Rightarrow X\) is a \(W\)-space.

(ii) \(X\) is a \(C_1\)-space \(\Rightarrow X\) is a \(GW\)-space \(\Rightarrow X\) is a \(W\)-space.

(See \([26,\) Theorem 2.2] and \([20,\) Theorem 1.9] for (i); \([12,\) Remark (4), p. 616] for (ii).)

The complex projective space \(CP^3\) is a \(GW\)-space \([12,\) Theorem 1\] such that \(\text{cat}(CP^3) = 3\), but it is not a \(C_k\)-space for any \(k\) (Example 5.7). We note that \(CP^3\) is not a \(G\)-space \([20,\) Remark 3.4].

If \(p > 2\), then \(L^3(p)\) is a \(G\)-space, but it is not a \(C_k\)-space for any \(k \geq 2\) (see Example 5.10 and Theorem 5.13).
Proposition 5.4 Assume that $X$ is a 1-connected space.

(i) $X$ is a $G$-space $\implies X$ is a rational $H$-space.

(ii) If $k \geq 1$, then the rationalization of any $T_k$-space (and hence any $C_k$-space) is an $H$-space.

Proof (i) is obtained by Haslam [7] (see also [13, Theorem 3.4]). (ii) is a direct consequence of (i).

Example 5.5 It is known that $H$-spaces, $T$-spaces, and $GW$-spaces are equivalent in the class of spaces of L-S category $\leq 1$ (see Propositions 5.4 and the definition of the $GW$-space). Then the following results hold by Proposition 5.4(v) and Theorem 4.3(ii).

(i) $S^1, S^3$, and $S^7$ are $H$-spaces and hence $C_k$-spaces for any $k \geq 1$.

(ii) If $1 \leq n < \infty$ and $n \neq 1, 3, 7$, then $S^n$ is not an $H$-space and hence an $NC$-space, since cat $S^n = 1$.

In the following argument we consider projective spaces $RP^n$, $CP^n$, and lens spaces $L^n(p)$ ($p \geq 2$); however, the cases $RP^\infty$, $CP^\infty$, and $L^\infty(p)$ are not referred to, since they are $H$-spaces and hence $C_k$-spaces for any $1 \leq k \leq \infty$.

Example 5.6 If $1 \leq n < \infty$ and $n \neq 1, 3, 7$, then the real projective space $RP^n$ is an $NC$-space by Example 5.5(ii) and Theorem 4.10. However, $RP^1, RP^3$, and $RP^7$ are $H$-spaces and hence $C_k$-spaces for any $1 \leq k \leq \infty$.

Example 5.7 If a 1-connected space $X$ is not a rational $H$-space, then $X$ is an $NC$-space by Proposition 5.4. For $1 \leq n < \infty$, the complex projective space $CP^n$ is not a rational $H$-space, and hence it is an $NC$-space.

Let $S^{2n+1}$ be the unit sphere in the $(n+1)$-dimensional complex vector space $C^{n+1}$ ($n \geq 1$). Let $\omega$ be the $p$-th root of unity ($p \geq 2$). Then the group $\Gamma$ generated by $\omega$ acts on $S^{2n+1}$ by $\omega \cdot (z_0, z_1, \ldots, z_n) = (\omega z_0, \omega z_1, \ldots, \omega z_n)$. Let the lens space be $L^{2n+1}(p) = S^{2n+1}/\Gamma$, the quotient space of $S^{2n+1}$ by $\Gamma$. See [24, Example 3, p. 91].

Proposition 5.8 ([24, Theorem (7.9), Chapter II]) Let $p$ be an odd prime.

$$H^*(L^{2n+1}(p); Z/p) = \bigwedge_{Z/p} (x_1) \otimes \{Z/p [x_2]/(x_2^{2n+1})\},$$

where $x_1 \in H^1(L^{2n+1}(p); Z/p)$ and $x_2 = \beta_p x_1 \in H^2(L^{2n+1}(p); Z/p)$.

Proposition 5.9 Let $p$ be a prime.

(i) If $2n + 1 \neq 3, 7$, then $L^{2n+1}(p)$ is not a $G$-space.

(ii) If $2n + 1 \neq 3, 7$, then $L^{2n+1}(p)$ is a $NC$-space.

Proof (i) If $L^{2n+1}(p)$ is a $G$-space, then $S^{2n+1}$ is a $G$-space [6, Theorem 2.2].

(ii) If $L^{2n+1}(p)$ is a $C_k$-space, then $S^{2n+1}$ is a $C_k$-space by Theorem 4.10.\]
Let us recall that $L^3(p)$ is a $G$-space by [15, Corollary II.10], since $S^3 = \text{Sp}(1)$ is a Lie group. For general $L^{2n+1}(p)$, we only know that $\pi_1(L^{2n+1}(p)) = G_1(L^{2n+1}(p))$ by [2, Theorem] or [19, Theorem A]. See also [4, Theorems II.4, II.5] and [5, Theorem 6.2]. However, for $L^3(p)$, we obtain the result using an argument similar to [15], including a proof for the fundamental group that is simpler than [2, 19] in this particular case.

**Example 5.10** $L^3(p)$ is a $G$-space for any $p \geq 2$.

Actually, we can show the result in this way. Assume that $\pi_1(L^3(p)) = \mathbb{Z}/p$ is generated by the inclusion map $\alpha : S^1 \hookrightarrow L^3(p)$, which has a lift $\tilde{\alpha} : [0, 1] \to S^3$ such that $\tilde{\alpha}(0) = 1$, $\tilde{\alpha}(1) = \xi$ and $\pi \circ \tilde{\alpha} = \alpha \circ \omega$, where $\pi : S^3 \to L^3(p)$ is the canonical projection taking the orbit space by the action of $(\xi, \xi^r)$ to $\mathbb{Z}/p$ a subgroup of a Lie group $S^3$, and where $\omega : [0, 1] \to S^1$ is the standard identification map. Since $S^3$ is a Lie group, there is an associative unital multiplication $\mu : S^3 \times S^3 \to S^3$ that defines a map $\tilde{f} : [0, 1] \times S^3 \to S^3$ by $\tilde{f} = \mu \circ (\tilde{\alpha} \times 1)$. Then $\tilde{f}$ induces a map $f$ of orbit spaces by the action of $\mathbb{Z}/p$, since $\tilde{f}(1, \xi^r \cdot x) = \alpha(1) \cdot \xi^r \cdot x = \xi \cdot \xi^r \cdot x = \xi^{r+1} \cdot x = \xi^r \cdot \tilde{f}(0, x)$:

$$
\begin{array}{c}
[0, 1] \times S^3 & \xrightarrow{\tilde{f}} & S^3 & \xleftarrow{\alpha} & [0, 1] \\
\downarrow{\omega \times \pi} & & \downarrow{\pi} & & \downarrow{\omega} \\
S^1 \times L^3(p) & \xrightarrow{f} & L^3(p) & \leftarrow & S^1 \\
\cup & & & & \langle \alpha, L^3(p) \rangle \\
S^1 \lor L^3(p),
\end{array}
$$

Thus $\alpha \in G_1(L^3(p))$ and hence $G_1(L^3(p)) = \pi_1(L^3(p))$. Since the universal cover of $L^3(p)$ is $S^3$, which is a Lie group, we see that the projection $\pi : S^3 \to L^3(p)$ is a cyclic map, and hence $G_n(L^3(p)) = \pi_n(L^3(p))$ for $n \geq 2$. It follows that $L^3(p)$ is a $G$-space.

To examine the existence of a $C_1$-structure on $L^3(p)$, we need the following lemma for a space $X$ using observations on $\Sigma \Omega X$.

**Lemma 5.11** Let $X$ be a 0-connected CW-complex whose universal cover $\tilde{X}$ satisfies that $\Sigma \Omega \tilde{X}$ has the homotopy type of a wedge sum of spheres. Then $X$ is a $C_1$-space if and only if $X$ is a $G$-space.

**Proof** Since $\Omega X \cong \pi_1(X) \times \Omega \tilde{X}$, we have

$$
\Sigma \Omega X \simeq \left( \bigvee_{0 \neq \lambda \in \pi_1(X)} S^1_{\lambda} \right) \lor \Sigma \Omega \tilde{X} \lor \left( \bigvee_{0 \neq \lambda \in \pi_1(X)} S^1_{\lambda} \land \Omega \tilde{X} \right),
$$

which has the homotopy type of a wedge of spheres. Thus we have the lemma. □

**Proposition 5.12** $L^3(p)$ is a $C_1$-space for any $p \geq 2$. 
Remark When \( L^1_2 \) is not a \( C^1 \)-space, then there is a map
\[
m: P^2(\Omega L^3(p)) \times L^1(p) \to L^3(p)
\]
whose axes are \( e_2^{L^1_2} \) and the identity of \( L^3(p) \).

Let \( L^3(p)^{(2)} = S^1 \cup e_2 \) be the 2-skeleton of \( L^3(p) = S^1 \cup e_2 \cup e_3 \). Then there is a map
\[
s_2: L^3(p)^{(2)} \to P^2(\Omega L^3(p)^{(2)}) \subset P^2(\Omega L^3(p)) \text{ such that } e_2^{L^1_2} \circ s_2 \sim i_2: L^3(p)^{(2)} \hookrightarrow L^3(p)
\]
is the canonical inclusion. On the other hand, we have
\[
H^*(L^3(p); \mathbb{Z}/p) \cong \bigwedge_{\mathbb{Z}/p} (x_1) \otimes \{\mathbb{Z}/p[x_2]/(x_2^2)\}
\]
\[
\cong H^*(L^3(p); \mathbb{Z}/p) \oplus \mathbb{Z}/p[x_1x_2], \quad \ker i_2^* = \mathbb{Z}/p[x_1x_2],
\]
where \( x_i \) is in \( H^i(L^3(p); \mathbb{Z}/p) \subset H^i(L^3(p); \mathbb{Z}/p) \) with a Bockstein relation \( \beta p x_i = x_i \). Thus \( e_2^{L^1_2} x_i \neq 0 \) for \( i = 1, 2 \), since \( e_2^{L^1_2} \circ s_2 \sim i_2 \).

Now let \( h: \Sigma P^2(\Omega L^3(p)) \wedge L^3(p) \to \Sigma L^3(p) \) be the Hopf construction of the map
\[
m: P^2(\Omega L^3(p)) \times L^1(p) \to L^3(p),
\]
and let \( C_h \) be the mapping cone of \( h \). Then the connecting homomorphism
\[
\delta: H^5(\Sigma L^3(p); \mathbb{Z}/p) \to H^6(C_h; \mathbb{Z}/p)
\]
is an isomorphism, since \( H^q(\Sigma L^3(p); \mathbb{Z}/p) = 0 \) for \( q \geq 5 \). Thus we have
\[
H^6(C_h; \mathbb{Z}/p) \cong H^4(\Sigma P^2(\Omega L^3(p)) \wedge L^3(p); \mathbb{Z}/p) \cong H^2(L^3(p)^{(2)}; \mathbb{Z}/p) \oplus H^2(L^3(p); \mathbb{Z}/p).
\]
Let \( s^*: H^n(\Sigma X) \to H^{n-1}(X) \) be the suspension homomorphism \( (n \geq 1) \). For dimensional reasons, we know that \( x_1 \) and \( x_2 \) are primitive with respect to \( m \), and hence \( s^{-1}x_1, x_2 \) lies in the image of the restriction \( H^{i+1}(C_h; \mathbb{Z}/p) \to H^{i+1}(\Sigma L^3(p); \mathbb{Z}/p) \), say
\[
y_i+1|_{\Sigma L^3(p)} = s^{-1}x_i \text{ for } i = 1, 2.
\]
Then by [22, Corollary 1.4(a)], we know
\[
y_2^3 = \pm \delta(s^{-1}(x_2 \otimes x_2)) \neq 0,
\]
while we know that \( y_2^3 = -y_2^3 \) and hence \( 2y_2^3 = 0 \). Thus we have \( p = 2 \).

Making use of the classification of \( GW \)-spaces of type \( (q, n, m) \) in [12, Theorem 1], the following result is proved.
Theorem 5.14  Let \( X \) be a \( C_k \)-space for some \( k \geq 1 \) with at most three cells (other than the base point 0-cell). Then \( X \) has the homotopy type of one of the spaces in the following list.

(i) \( X = S^1, S^3, S^7 \) or their products; otherwise;
(ii) If \( \pi_1(X) \) is a non-zero finite group, then \( X = L^1(p, \ell) \) for an integer \( p \geq 2 \), where \( \ell \) is a unit of the quotient ring \( \mathbb{Z}[\tau]/(1 + \tau + \cdots + \tau^p - 1) \) of the group ring \( \mathbb{Z}[\pi] \) for the group \( \pi = \langle \tau \mid \tau^p = 1 \rangle \cong \mathbb{Z}/p \);
(iii) If \( \pi_1(X) = 0 \), then \( X = SU(3) \) or \( E_k(\omega) \) (\( k \not \equiv 2 \mod 4 \)); in the latter case \( E_k(\omega) \) is an \( H \)-space.

Proof  Since a \( C_k \)-space for some \( k \geq 1 \) is a \( T \)-space and hence a \( GW \)-space, we can examine the \( GW \)-spaces with up to 3 cells listed in Theorem 1 of [12]. However, \( CP^3 \) in the theorem is an \( NC \)-space by Example 5.7, and hence the result follows.

Remark 5.15  In view of Theorem 5.14 we see that any real, complex or quaternionic Stiefel manifold of 2-frames is an \( NC \)-space unless it is an \( H \)-space. We note that a Stiefel manifold is an \( H \)-space if and only if it is a Lie group or \( S^7 \), by [8, Theorems 1.1, 1.2] and [9, Corollary 0.6].

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