

L-S category and a categorical sequence

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Henan University (Kaifeng, 15–18th October 2006)

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代数的 位相幾何学 国際会議
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‘Cone Decomposition’ + ‘Higher Hopf invariant’

||

‘Categorical Sequence’

Definition (Lusternik-Schnirelmann)

$$\text{cat}(M) = \text{Min} \left\{ m \geq 0 \left| \begin{array}{l} \exists \{A_0, \dots, A_m; \text{ closed in } M\} \\ M = \bigcup_{i=0}^m A_i, \text{ where each } A_i \text{ is} \\ \text{contractible in } M. \end{array} \right. \right\}$$

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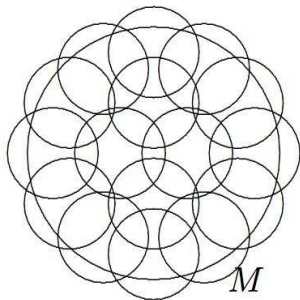


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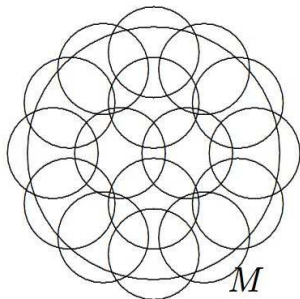


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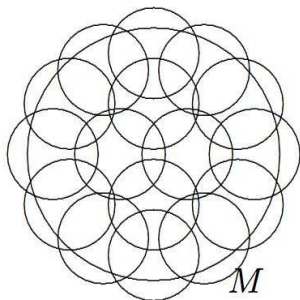


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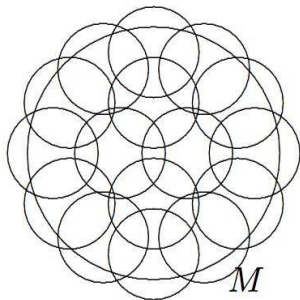


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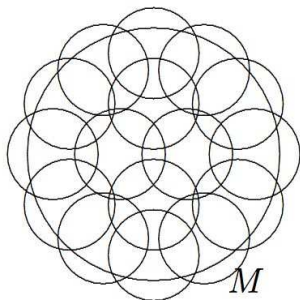


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this definition gives only an **upper bound** for $\text{cat}(M)$.

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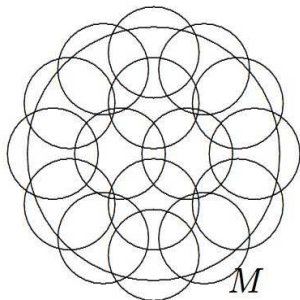


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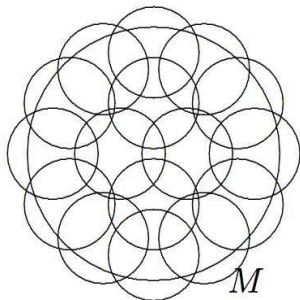


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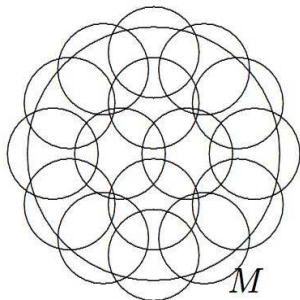


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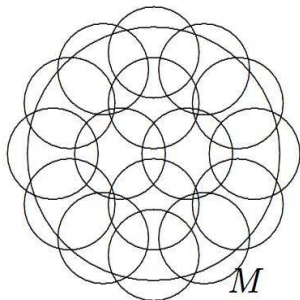


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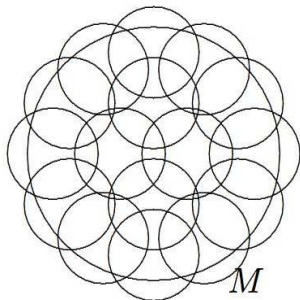


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For small n , sometimes the n -sphere S^n becomes a Lie group

Fact

① $S^0 = O(1)$,

② $S^1 = O(2) = SO(2)$.

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Hopf invariant and Hopf structure

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There is an element of Hopf invariant 1 in $\pi_{2n+1}(S^{n+1})$, if S^n is an H-space.

Theorem (Toda)

There is no element of Hopf invariant one in $\pi_{31}(S^{16})$.

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We may write them as $X = S^r \cup_f e^{q+1}$.

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Definition (Bernstein-Hilton)

For a map f from S^q to a space X with $\text{cat}(X) = m$,

$$H_m^s(f) \in \pi_q\left(\prod^{m+1} X, \mathbb{T}^{m+1} X\right),$$

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For any space X , the space of all loops at the base point of X admits a natural A_∞ -structure,

associated with projective spaces $P^m\Omega(X)$ and natural maps

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Using this criterion, we can now determine more L-S categories...

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Theorem (Ganea, I, Sakai)

$$\text{cat}(X) \leq m \iff \exists \sigma : X \rightarrow P^m\Omega(X) \text{ such that } e_m^X \sigma \sim 1_X.$$

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For any space X , the space of all loops at the base point of X admits a natural A_∞ -structure,

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Thus the difference $d_m^{\sigma(X)}(f)$ has a unique lift

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One advantage of this definition is that we can use the properties of projective spaces to determine a higher Hopf invariant.

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Let us go back to Fox's paper on L-S category of a space X . Fox introduced a notion of categorical sequence 'catseq(X)' to give an upper bound to the original L-S category $\text{cat}(X)$.

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By combining a cone decomposition with a higher Hopf invariant, Kikuchi and I obtain a categorical sequence, and eventually get a better upper bound of L-S category.

Proposition (Kikuchi, I)

$$\text{catseq}(\text{SO}(10)) \leq 21.$$

On the other hand, we know

$$H^*(\text{SO}(10); \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_3, x_5, x_7, x_9]/(x_1^{16}, x_3^4, x_5^2, x_7^2, x_9^2),$$

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$$\{*\} = F_0 \subset F_1 \subset \cdots \subset F_{19} \subset F_{20} = \text{Spin}(7).$$

Outline of the proof of $\text{catseq}(\text{SO}(10)) = 21$

We construct a sequence $\{E_i; 0 \leq i \leq 21\}$ of subspaces of $\text{SO}(10)$ of length 21, using the cone decomposition $\{F_i; 0 \leq i \leq 20\}$ of $\text{Spin}(7)$ together with the multiplication of $\text{SO}(9)$ again by using Yokota's CW decomposition of classical groups.

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Using $H_1(\alpha) = 0$, we obtain that the sequence $\{E_i; 0 \leq i \leq 21\}$ of subspaces of $\text{SO}(10)$ is categorical and hence we have

$$\text{catseq}(\text{SO}(10)) \leq 21.$$

While we do not know whether the above sequence $\{E_i; 0 \leq i \leq 21\}$ itself is a cone decomposition of $\text{SO}(10)$ or not,

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Problem

Do the following three invariants for $\text{SO}(n)$ coincide with each other? i.e.,

$$\text{cup}(\text{SO}(n)) =? \text{cat}(\text{SO}(n)) =? \text{Cat}(\text{SO}(n))$$

Thank you.