

# Topological Complexity is a fibrewise L-S category

Norio Iwase & Michihiro Sakai

Faculty of Mathematics, Kyushu University & Gifu National College of Technology

Groups of Self-Homotopy Equivalences and Related Topics  
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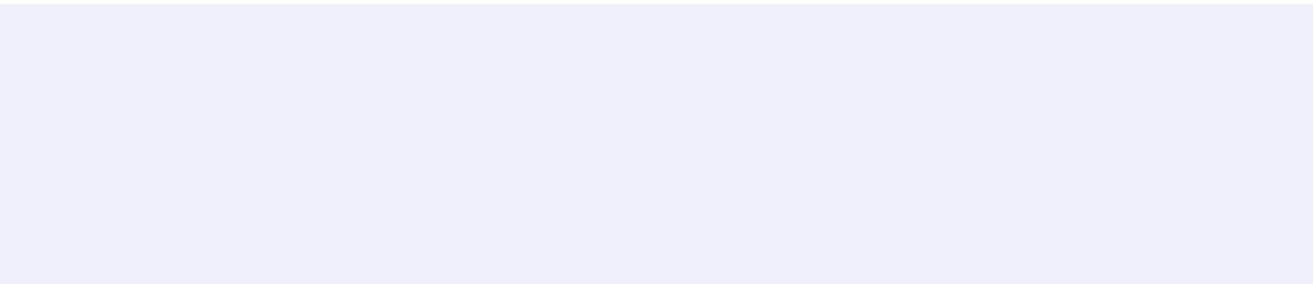
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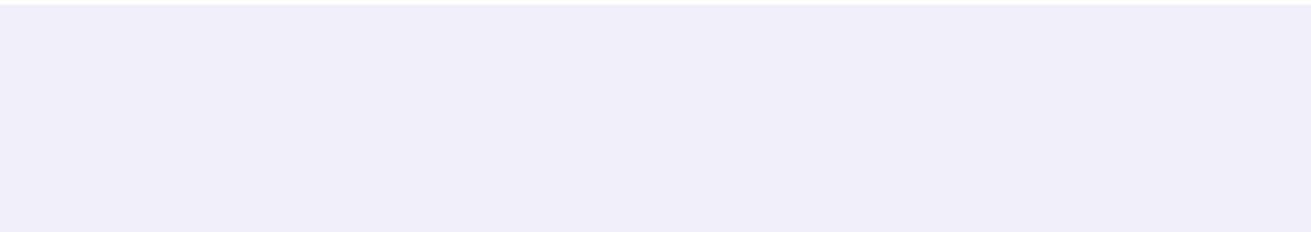
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# Zero-divisors cup-length and TC weight

Definition (Farber and Farber-Grant)

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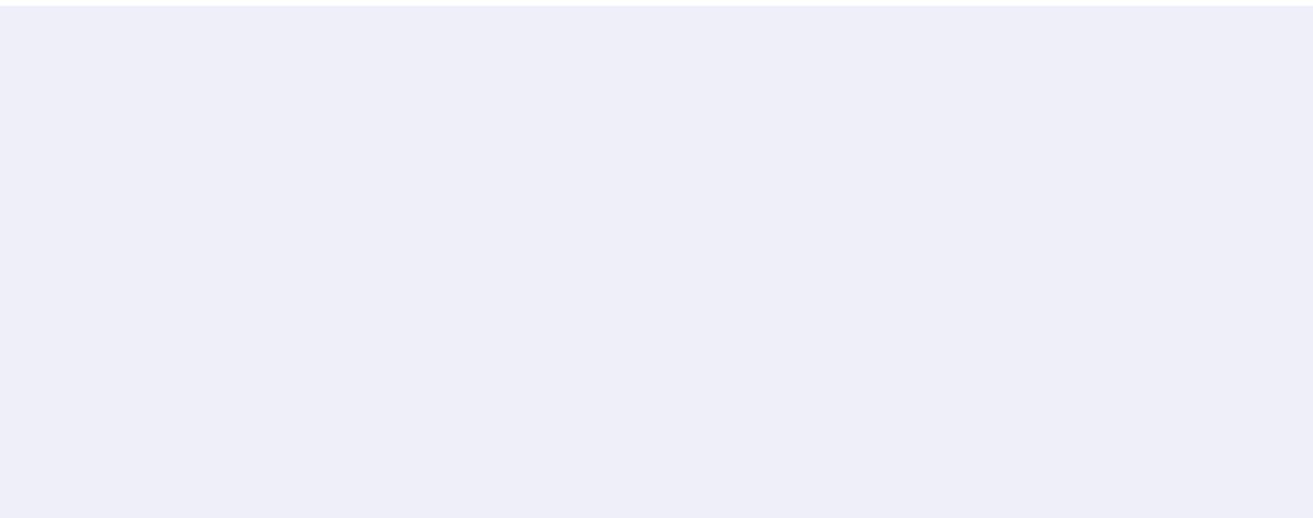
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