Topological Complexity is a fibrewise L-S category

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Groups of Self-Homotopy Equivalences and Related Topics
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Contents

Topological complexity
- Robot Motion Planning
- Topological complexity and Schwartz genus
- Symmetric motion planning
- Monoidal motion planning
- Zero-divisors cup-length and TC weight

Fibrewise L-S category
- Fibrewise pointed space over B
- Fibrewise (pointed) L-S category
- Fibrewise unpointed L-S category

Results
- Topological complexity and fibrewise L-S category
- Topological complexity is a fibrewise L-S category
- Fibrewise cup-length and categorical weight
- Relationship among lower estimates

Upper estimates
- Fibrewise cone decomposition
- Fibrewise categorical sequence

Concluding Remarks

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Topological complexity and fibrewise L-S category

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Proof: The functor d is given by

$$d(B) = B{\times}B, \quad p_{d(B)} = \mathrm{pr}_2, \quad s_{d(B)} = \Delta.$$

[Skip Proof]

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Topological complexity and fibrewise L-S category

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1. $\mathrm{cup}^{\mathrm{B}}_{\mathrm{B}}(X; R) = \mathrm{Max}\left\{m{\geq}0 \,\middle|\, \exists_{\{u_1,\cdots,u_m \in I\}} \text{ s.t. } u_1 \cdots u_m \neq 0\right\}$
2. $\mathrm{wgt}^{\mathrm{B}}_{\mathrm{B}}(u; R) = \mathrm{Max}\left\{m{\geq}0 \,\middle|\, \forall_{f:Y\to X \in \underline{\underline{\mathcal{T}}}^B_B,\ \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(f)<m}\ f^*(u) = 0\right\}$
3. $\mathrm{Mwgt}^{\mathrm{B}}_{\mathrm{B}}(X; R) = \mathrm{Max}\left\{m{\geq}0 \,\middle|\, (e^X_m)^* \text{ is a split mono of modules over unstable cohomology operations}\right\}$

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