GANEA'S CONJECTURE ON LUSTERNIK-SCHNIRELMANN CATEGORY

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It is my honour to give a talk in this topology seminar on my recent work on Lusternik-Schnirelmann category, which I would like to abbreviate as LS category.

In Summer, a beautiful lecture by Ioan James on LS category forced me to consider Ganea's conjecture on LS category again, which was open for years.

1. LS category

I will begin my talk with defining the LS category. I have come to know very recently that there are lots of variant notion such as Cat, cat, g-cat, w-cat, cone-length, etc. But unfortunately, the only definition I know is what experts call *the nomalised* LS category:

$$\mathrm{cat}\,X = \text{the least integer } m \text{ so that } X \text{ is covered by } m+1 \text{ closed subset contractible in } X$$

So, I have to talk about mainly small cats without wiskers. For example, we know the following basic facts:

(1) $\mathrm{cat}\,\{*\} = 0$

(2) $\mathrm{cat}\,S^n = 1$ for $n \geq 0$.

(3) $\mathrm{cat}\,(X \times Y) \leq \mathrm{cat}\,X + \mathrm{cat}\,Y$

(4) $\mathrm{cat}\,X \leq$ (the number of critical points of a Morse function of X)-1

(5) $\mathrm{cat}\,X \leq$ (the dimension of X)

(6) $\mathrm{cat}\,X \geq$ (the number of elements in $\tilde{h}^*(X)$ which give a non-zero product in $\tilde{h}^*(X)$), for any multiplicative cohomology theory $\tilde{h}^*$.

2. The Ganea's conjecture on LS category

An american mathematician Tudor Ganea contributed much in this area and died in 1971. He left some problems some of which is known as Ganea's conjecture, e.g., The problem 10

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which is often called the Ganea conjecture is a fundamental conjecture for co-H-spaces, or spaces with normalised LS category 1.

The problem 2 is also a fundamental conjecture on LS category: $\operatorname{cat} X \times S^n = \operatorname{cat} X + 1$? This is the subject of this talk. There were some supporting evidences:

i) No counter examples found.

ii) (Jessup 1990 and Hess 1991) The rational version of the conjecture is true.

iii) (Singhof 1979 and Rudyak 1997) The conjecture is true for a large class of manifolds.

3. My approach to prove the conjecture

For over 10 years, my main interest area is Hopf spaces especially on its homotopy associativity and its associated projective spaces. An A_n-space is studied by Masahiro Sugawara and James Stasheff (1963) as a space with higher homotopy associativity: By Stasheff, an A_m-space M is defined to have projecitve spaces $P^1(M) \subset P^2(M) \subset P^3(M) \subset ... \subset P^m(M)$. These projective spaces are characterised by Stasheff in terms of homotopy theory. When $m = \infty$, $P^\infty(M)$ is often called the A_∞-structure of M. In this case, M has the homotopy type of a topological group by Stasheff and Milnor.

There looks nothing concerning about LS category. But this shows us another way to compute LS categories: for any space X, we know that $G = \Omega X$ is an A_∞-space, and hence there exists a filration

$$P^1(G) \subset P^2(G) \subset P^3(G) \subset ... \subset P^\infty(G) \simeq X$$

There are some fundamental results:

Theorem 3.1. *(Ganea) For any space X, $\operatorname{cat} X \leq m$ if and only if the canonical inclusion $e^X_m : P^m\Omega X \subset P^\infty\Omega X \simeq X$ has a homotopy section, i.e., there is a map $s : X \to P^m\Omega X$ which satisfies $e^X_m \circ s \simeq 1_X$.*

Theorem 3.2. *(Folk Theorem) For any spaces X and Y, $\operatorname{cat} X \times Y \leq m$ if and only if the canonical inclusion $\bigcup_{i+j=m} P^i\Omega X \times P^j\Omega Y \subset P^\infty\Omega X \times P^\infty\Omega Y \simeq X \times Y$ has a homotopy section.*

Corollary 3.2.1. $\operatorname{cat} X \times Y \leq \operatorname{cat} X + \operatorname{cat} Y$

Corollary 3.2.2. $\operatorname{cat} X\times S^n = \operatorname{cat} X$ *or* $\operatorname{cat} X + 1$

Corollary 3.2.3. $\operatorname{cat} X\times S^n = \operatorname{cat} X$ *if and only if the inclusion* $P^m\Omega X\times\{*\}\cup P^{m-1}\Omega X\times S^n \subset P^\infty\Omega X\times S^n \simeq X\times S^n$ *has a homotopy section.*

This is the way I found to prove the conjecture.

4. Obstructions

Under some connectedness condition on X, there is a homology decomposition $\{X^t\}$ of X satisfies that $0 = \operatorname{cat} X^1 \leq \operatorname{cat} X^2 \leq \ldots \leq \operatorname{cat} X^{k-1} = m-1 < \operatorname{cat} X^k = m \leq \ldots \leq \operatorname{cat} X$ with k'-invariant $\alpha_k : M(H_k(X), k-1) \to X^{k-1}$ for some k, where $M(H_k(X), k-1)$ is the Moore space of type $(H_k(X), k-1)$. In each stage, we can obtain the obstruction for X^k to satisfy $\operatorname{cat} X^k \leq m-1$, as a map from $M(H_k(X), k-1)$ to the total space of Stasheff's fibration, which is defined from α_k:

$$\begin{tikzcd} \Omega X \arrow[r, hook] & E^m\Omega X \arrow[r] & P^{m-1}\Omega X \\ & M(H_k(X), k-1). \arrow[u, "\beta_k"'] & \end{tikzcd}$$

To prove the conjecture, I tried to show the non-triviality of the obstruction β'_{k+n} for $X^k\times S^n$ to satisfy $\operatorname{cat} X^k\times S^n \leq m$ using the above k'-invariant α_k:

$$\begin{tikzcd} \Omega X\times\Omega S^n \arrow[r, hook] & E^m\Omega X*\Omega S^n \arrow[r] & P^m\Omega X\times\{*\}\cup P^{m-1}\Omega X\times S^n \\ & M(H_{k+n}(X\times S^n), k+n-1). \arrow[u, "\beta'_{k+n}"'] & \end{tikzcd}$$

At this moment, I thought I was wining to prove the conjecture. But it was just a dream. What was found out is that β'_{k+n} is essentially given by $\beta_k * 1_{S^{n-1}} \simeq \pm\Sigma^n\beta_k$ the n-fold suspension of the map β_k, while there are lots of maps whose higher suspension is trivial.

Theorem 4.1. *There is a series of complexes* Q_p *indexed by all primes* p *with* $\operatorname{cat} Q_p\times S^n = \operatorname{cat} Q_p = 2$ *for* $n \geq 2$.

5. Proof of the theorem 4.1

Let p be an odd prime and $Q_p = S^2 \cup e^{4p-2}$ with the map $\alpha = \eta\circ\alpha_1^2(3) : S^{4p-3} \to S^2$ as the only k'-invariant $\alpha : S^{4p-3} \to S^2$.

Lemma 5.1. *$\beta = \alpha_1^2(3)$ is not a suspension map but a co-H-map of order p, whose iterated suspensions $\Sigma^t\beta$ are trivial for $t \geq 2$ while $\Sigma\beta \neq 0$.*

Proof. The latter part is obtained by observing the table of homotopy groups by Toda [13]: Since $\pi_{4p-1}(S^5)$ has no p-torsion element, we have that $\Sigma^2\beta = 0$. Since S^3 is a Hopf space, the suspension homomorphism $\pi_{4p-3}(S^3) \to \pi_{4p-2}(S^4)$ is split injective. Thus it remains to show the first part of the lemma: since $\pi_{4p-4}(S^2) \cong \pi_{4p-4}(S^3)$ has no p-torsion element, β is not a suspension. Also one can easily compute that $H(\beta)$ is in $\pi(\Omega S^3 {*} \Omega S^3)$ which has no p-torsion element, where H is the generalised Hopf invariant homomorphism $H : [X, Y] \to [X, \Omega Y {*} \Omega Y]$ by Berstein-hilton [1] or Saito [8], for co-H-spaces X and Y. $\square$

Proposition 5.2. *$\alpha = \eta\beta = \eta\alpha_1^2(3)$ is not a co-H-map and the obstruction is described by the 2nd James-Hopf invariant $h_2(\alpha) = \beta$, which is a generator of the p-part of $\pi_{4p-3}(S^3)$:*

$$\mu_2\alpha \simeq (\alpha \vee \alpha)\mu_{4p-3} +_{4p-3} [i_1, i_2]\beta$$

where we denote by $\mu_k : S^k \to S^k \vee S^k$ the (unique) co-Hopf structure of the sphere S^k and by $+_k$ the multiplication induced by the co-Hopf structure of sphere S^k.

Proof. There is a well-known formula for the Hopf map η:

$$\mu_2\eta \simeq (\eta \vee \eta)\mu_3 +_3 [i_1, i_2]$$

in $\pi_3(S^2 \vee S^2)$ by $i_t : X \to X \vee X$ the inclusion to the t-th factor. Since $\alpha \simeq \eta\beta$, we have, in $\pi_{4p-3}(S^2 \vee S^2)$,

$$\mu_2\alpha \simeq \mu_2\eta\beta \simeq \{(\eta \vee \eta)\mu_3 +_3 [i_1, i_2]\}\beta.$$

Since β is a co-Hopf map by Lemma 5.1, this is homotopy equivalent to

$$(\eta \vee \eta)\mu_3\beta +_{4p-3} [i_1, i_2]\beta \simeq (\eta\beta \vee \eta\beta)\mu_{4p-3} +_{4p-3} [i_1, i_2]\beta \simeq (\alpha \vee \alpha)\mu_{4p-3} +_{4p-3} [i_1, i_2]\beta$$

This implies that $h_2(\alpha) \simeq \beta$ which gives the obstruction for α to be a co-Hopf map and $h_k(\alpha) = 0$ for $k \geq 3$. $\square$

Lemma 5.3. *The following diagram without the dotted arrow commutes up to homotopy.*

$$\begin{tikzcd}
S^{4p-3} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & S^2 \arrow[rr, hook, "i"] \arrow[dd, hook, "\Sigma\Omega i\Sigma j_1"] & & Q_p \arrow[dd, "1_{Q_p}"] \arrow[dddl, dashed, "\lambda"'] \\
S^1{*}S^1 \arrow[d, hook, "(\Omega i{*}\Omega i)(j_1{*}j_1)"'] & & & \\
\Omega Q_p{*}\Omega Q_p \arrow[r, "p_1^{Q_p}"] & \Sigma\Omega Q_p \arrow[rr, "ev_{Q_p}"] \arrow[dr, hook, "\iota_1^{Q_p}"'] & & Q_p \\
 & & P^2\Omega Q_p \arrow[ur, "e_2^{Q_p}"'] &
\end{tikzcd}$$

where $i : S^2 \to Q_p$ and $j_t : S^t \to \Omega S^{t+1}$ give the bottom cell inclusions and $p_1^{Q_p}$ denotes the Hopf construction of the loop addition of ΩQ_p, $\iota_1^{Q_p} : \Sigma\Omega Q_p \to P^2\Omega Q_p$ denotes the inclusion to the mapping cone of $p_1^{Q_p}$ and $e_t^{Q_p} : P^t\Omega Q_p \subset P^\infty\Omega Q_p \simeq Q_p$ denotes the canonical inclusion.

Remark 5.4. *The difference between the identity 1_{Q_p} and the map $e_2^{Q_p}\lambda$ is given by an element $ev_{Q_p}\gamma \in \pi_{4p-2}(Q_p)$, where $\gamma \in \pi_{4p-2}(\Sigma\Omega Q_p)$, since $\pi_{4p-2}(\Sigma\Omega Q_p) \to \pi_{4p-2}(Q_p)$ is a split surjection.*

This implies the following theorem.

Theorem 5.5. *(Berstein-Hilton)* $\operatorname{cat}_p Q_p = \operatorname{cat} Q_p = 2$ *but* $\operatorname{cat}_q Q_p = 1$ *for* $q \neq p$.

Proposition 5.6. *The following diagram except the dotted arrow commutes up to homotopy.*

$$\begin{tikzcd}
S^{4p-3}{*}S^{n-1} \arrow[r, "\hat{\alpha}"] \arrow[d, "\beta{*}1_{S^{n-1}}"] & Q_p{\times}\{*\} \cup S^2{\times}S^n \arrow[r, hook] \arrow[dd, hook, "\lambda{\times}\{*\}\cup(\Sigma\Omega i\Sigma j_1){\times}1_{S^n}"] & Q_p{\times}S^n \arrow[dd, "\lambda{\times}1_{S^n}"] \arrow[ddr, dashed, "1_{Q_p}{\times}1_{S^n}"] & \\
(S^1{*}S^1){*}S^{n-1} \arrow[d, hook, "((\Omega i{*}\Omega i)(j_1{*}j_1)){\times}1_{S^{n-1}}"] & & & \\
(\Omega Q_p{*}\Omega Q_p){*}S^{n-1} \arrow[r, "\hat{p}_1^{Q_p}"] & P^2\Omega Q_p \times \{*\} \cup \Sigma\Omega Q_p{\times}S^n \arrow[r, hook] & P^2\Omega Q_p{\times}S^n \arrow[r, "e_2^{Q_p}{\times}1_{S^n}"] & Q_p{\times}S^n
\end{tikzcd}$$

Since $\beta{*}1_{S^{n-1}} \simeq \pm\Sigma(\beta\wedge 1_{S^{n-1}}) \simeq \pm\Sigma^n\beta$, we have established the following result.

Proposition 5.7. $1_{Q_p}{\times}1_{S^n}$ *can be compressed into* $P^2\Omega Q_p \times \{*\} \cup \Sigma\Omega Q_p{\times}S^n$, *for* $n \geq 2$.

Proof. In the case when $n \geq 2$, $\beta * 1_{S^{n-1}}$ is trivial. Since the inclusion $P^2\Omega Q_p \times \{*\} \cup \Sigma\Omega Q_p \times S^n \to P^2\Omega Q_p \times S^n$ induces a split epimorphism in the homotopy groups, a similar argument to that used in the proof of Theorem 5.5 leads us the conclusion that there is a compression δ of $\lambda \times 1_{S^n}$ to $P^2\Omega Q_p \times \{*\} \cup \Sigma\Omega Q_p \times S^n$. Moreover, we may assume the compression homotopy leaves the subspace $Q_p \times \{*\} \cup S^2 \times S^n$ fixed. By Remark 5.4, the identity 1_{Q_p} is given from $e_2^{Q_p}\lambda$ by adding an element $ev_{Q_p}\gamma$, $\gamma \in \pi_{4p-2}(\Sigma\Omega Q_p)$. We define a map δ_2 by

$$\delta_2 : Q_p \times S^n \overset{\mu \times 1_{S^n}}{\to} (Q_p \vee S^{10}) \times S^n = Q_p \times S^n \cup S^{10} \times S^n \xrightarrow{\delta \cup (\gamma \times 1_{S^n})} P^2\Omega Q_p \times \{*\} \cup \Sigma\Omega Q_p \times S^n,$$

where μ denotes the co-action of S^{4p-2}. Since δ is homotopic to λ in $P^2\Omega Q_p \times S^n$ with the subspace $\{*\} \times S^n$ left fixed, δ_2 is homotopic to

$$(\lambda + \gamma) \times 1_{S^n} : Q_p \times S^n \xrightarrow{\mu \times 1_{S^n}} (Q_p \vee S^{10}) \times S^n = Q_p \times S^n \cup S^{10} \times S^n \xrightarrow{(\lambda \times 1_{S^n}) \cup (\gamma \times 1_{S^n})} P^2\Omega Q_p \times S^n,$$

in $P^2\Omega Q_p \times S^n$ which is a compression of $1_{Q_p} \times 1_{S^n}$. Thus $\delta_2 : Q_p \times S^n \to P^2\Omega Q_p \times \{*\} \cup \Sigma\Omega Q_p \times S^n$ gives the compression of $1_{Q_p} \times 1_{S^n}$. $\square$

Thus we have $2 = \operatorname{cat}_p Q_p \leq \operatorname{cat}_p Q_p \times S^n \leq \operatorname{cat} Q_p \times S^n \leq \operatorname{cat}(P^2\Omega Q_p \times \{*\} \cup \Sigma\Omega Q_p \times S^n) \leq 2$, for $n \geq 2$, and hence we have established our main theorem.

Theorem 5.8. $\operatorname{cat} Q_p \times S^n = \operatorname{cat}_p Q_p \times S^n = 2$, *for* $n \geq 2$, *while* $\operatorname{cat} Q_p \times S^1 = \operatorname{cat}_p Q_p \times S^1 = 3$.

Also one can see that there is a two cell complex $Q_2 = S^8 \cup e^{30}$ which satisfies $\operatorname{cat} Q_2 = \operatorname{cat} Q_2 \times S^n = 2$ for $n \geq 1$. So the conjecture was a kind of folk-lore.

That's all.

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