GANEA'S CONJECTURE ON A CO-H-SPACE

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Before starting my talk, je remercie (le organisateur spetialement Professeur) Jean Claude Thomas. Ca me fait grand plaisir que je donne un expose dans cette conference.

Maybe it's better to talk in English rather than in French or Japanese.

Tudor Ganea (1922 - 1971) worked mainly on L-S category, co-H-spaces and co-category. He posed 15 problems at a conference in 1971.

Problem 1: Compute L-S category of manifolds.

Problem 2: (conjecture) $\operatorname{cat} X \times S^n = \operatorname{cat} X + 1$

$\vdots$

Problem 10: (conjecture) A co-H-space X has the homotopy type of $B \vee C$, where B is a bouquet of circles and C is simply connected.

This Problem 10 is the subject of my talk. This conjecture is often called the Ganea conjecture in a number of literatures.

1. Definitions

A co-H-space and a co-H-map are conceptual dual of an H-space and an H-map in the sense of Eckmann and Hilton, where an H-space and an H-map are homotopy theoretical generalisations of a Lie group and a continuous homomorphism between them.

Definition 1.1. *Let X be a path-connected space. The space X is a co-H-space if there exists a co-multiplication $\mu_X : X \to X \vee X$ with two-sided homotopy co-unit $X \to \{*\}$, i.e., its compositions with projections to the t-th factors $X \xrightarrow{\mu} X \vee X \xrightarrow{pr_t} X$, $t = 1, 2$ are both homotopic to the identity.*

Definition 1.2. *Let $f : X \to Y$ be a map between co-H-spaces. The map f is a co-H-map if $(f \vee f) \circ \mu_X \sim \mu_Y \circ f$, where μ_X denotes the co-multiplication of X.*

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We can easily see the following facts.

Fact 1.3. 1. *The following two conditions are equivalent.*

(a) *X is a co-H-space.*

(b) *The homotopy set functor $[X,-]$ has a natural multiplications with natural unit.*

2. *The following two conditions are equivalent.*

(a) *$f : X \to Y$ is a co-H-map.*

(b) *f induces a map $f^* : [Y,-] \to [X,-]$ preserving the multiplications and the units.*

We state here the following classical result of a co-H-space.

Theorem 1.4. *(Fox) The fundamental group of a co-H-space is a free group.*

The following theorems are fundamental results on a co-H-space and a co-H-map.

Theorem 1.5. *(Ganea) For a non-contractible path-connected space X, X is a co-H-space if and only if the Lusternik-Schnirelmann category $\operatorname{cat} X$ is $\operatorname{cat}\{*\}+1$.*

Theorem 1.6. *(Berstein-Hilton and Saito) For a map f between co-H-spaces, f is a co-H-map if and only if its generalized Hopf invariant is trivial.*

These results are obtained by using the following result:

Theorem 1.7. *(Ganea) A co-H-space X is dominated by $\Sigma\Omega X$ which is a homotopy fibre of the folding map $X \vee X \to X$.*

2. A simply connected co-H-space

A simply connected co-H-space has nice properties:

Theorem 2.1. *(Arkowincz) If a simply connected co-H-space is co-associative, then the given co-multiplication is a co-group structure, i.e., the co-multiplication induces a natural group structure on the functor $[X,-]$.*

Theorem 2.2. *(Berstein) A simply connected co-H-space is rationally equivalent with a wedge of spheres of dimensions ≥ 2.*

By Eckmann and Hilton, a simply connected space X admits a homology decomposition, i.e., there is a filtration $\{X_i\}$ of $X' \simeq X$ such that

1. X' has the weak topology with respect to the filtration,
2. the homology of each X_i is of dimension i and the inclusion $X_i \to X$ induces an isomophism up to dimensions i, and
3. each X_i has the homotopy type of the mapping cone of a map (k'-invariant) $h_i : M_{i-1} \to X_{i-1}$ where M_{i-1} denotes the Moore space of type $(H_i(X, \mathbb{Z}), i-1)$.

Theorem 2.3. *(Curjel) If X is a simply connected co-H-space, then the inclusions $X_{i-1} \to X_i$ and $X_{i-1} \to X$ are co-H-maps and each k'-invariant is of finite order.*

3. A standard co-H-space

Let us call a co-H-space standard, if the Ganea conjecture holds for the co-H-space.

Theorem 3.1. *For a co-H-space X, the condition 1) below is equivalent with other six conditions below.*

1. *(Ganea) X is standard.*
2. *(Berstein-Dror) The co-action of B along $j : X \to B$ associated with the co-multiplication of X can be chosen as co-associative.*
3. *(Hilton-Mislin-Roitberg) Co-shear map $\Phi_R : X \vee X \to X \vee X$ given by $\Phi_R(x, *) = (x, *)$ and $\Phi_R(*, x) = \mu(x)$ is a homotopy equivalence.*
4. *(Hilton-Mislin-Roitberg) Co-shear map $\Phi_L : X \vee X \to X \vee X$ given by $\Phi_L(*, x) = (x, *)$ and $\Phi_L(x, *) = \mu(x)$ is a homotopy equivalence.*
5. *(Hilton-Mislin-Roitberg) The functor $[X, -]$ has natural algebraic loop structure.*
6. *(Hilton-Mislin-Roitberg) The co-H-structure of X can be chosen to make $e = i \circ j$ be loop-like from the left.*
7. *(Hilton-Mislin-Roitberg) The co-H-structure of X can be chosen to make $e = i \circ j$ be loop-like from the right.*

Corollary 3.1.1. *(Berstein-Dror) If a co-H-space is co-associative, then it has a co-group structure.*

The rational version of the Ganea conjecture was verified as follows.

Theorem 3.2. *(Henn) Let X be an almost rational co-H-space, i.e., a co-H-space X with rational higher homotopy groups $\pi_i(X)$, $i \geq 2$. Then X is standard and splits into a one-point-sum of circles and rational spheres of dimension ≥ 2.*

Also for complexes up to dimension 3, the Ganea conjecture was verified.

Theorem 3.3. *(Saito-Sumi-I.) If $H_*(X, \mathbb{Z})$ is concentrated in dimensions 1, $n+1$ and $n+2$ $(n \geq 1)$ with $H_{n+2}(X, \mathbb{Z})$ torsion free, then X is standard and splits into a one-point-sum of circles, spheres of dimension $n+1$ and $n+2$, and Moore spaces $S^{n+1} \cup_m e^{n+2}$, for some $m \geq 2$.*

The key lemma of this theorem is as follows:

Lemma 3.4. *Let $\widetilde{X}$ be the universal cover of X. Then $H_*(\widetilde{X}, \mathbb{Z})$ is isomorphic to the $\mathbb{Z}\pi$-module $\mathbb{Z}\pi \otimes H_*(X, \mathbb{Z})$, where π denotes the fundamental group of X.*

This can be shown by using a theorem due to Seshadri, Cohn and Bass.

This lemma also enables us to show the following result.

Theorem 3.5. *(I.) A co-H-space X admits an almost homology decomposition, i.e., there is a filtration $\{X_i\}$ of $X' \simeq X$ such that*

1. *X' has the weak topology w.r.t. the filtration,*
2. *the homology of each X_i is of dimension i and the inclusion $X_i \to X$ induces an isomophism up to dimension i,*
3. *each X_i has the homotopy type of the mapping cone of a map (k'-invariant) $h_i : M_{i-1} \to X_{i-1}$ where M_{i-1} denotes the Moore space of type $(H_i(X, \mathbb{Z}), i-1)$,*
4. *the inclusions $X_{i-1} \to X_i$ and $X_{i-1} \to X$ are co-H-maps and each k'-invariant is of finite order.*

We remark that this implies that there is a simple construction of the localisation of higher homnotopy groups (due to Bendersky).

So, one can proceed to show the almost p-local version of the Ganea conjecture using some more technique.

Conjecture 3.6. *Let X be an almost p-local space. If X is a co-H-space, then X is standard.*

But evenif one can succeed to show the almost p-local version, he needs to manipulate p-local equivalences to get a true homotopy equivalence for a proof of the original conjecture.

4. Construction of counter examples

Theorem 4.1. *There exists a series of complexes $\{R_n; n \geq 4\}$ such that the integral homology groups of R_n is concentrated in dimensions 1, $n+1$ and $n+5$.*

Let $R_n = (S^1 \vee S^{n+1}) \cup_\phi e^{n+5}$, where ϕ is given by

(4.1)

$$\phi : S^{n+4} \xrightarrow{\mu} S^{n+4} \vee S^{n+4} \xrightarrow{9 \vee 16} S^{n+4} \vee S^{n+4} \xrightarrow{\nu_{n+1} \vee \nu_{n+1}} e \cdot S^{n+1} \vee \tau \cdot S^{n+1} \subset \bigvee_{i \in \mathbb{Z}} \tau^i \cdot S^{n+1} \to S^1 \vee S^{n+1}.$$

5. Unsplittability of R_n

In this section, we show that R_n cannot be split. We state the following well-known result:

Proposition 5.1. *The set of invertible elements in the group ring $\mathbb{Z}\pi$ is $\pm\pi \subset \mathbb{Z}\pi$.*

Proof. Since π is the infinite cyclic group, $\mathbb{Z}\pi$ is isomorphic with $\mathbb{Z}[x, \frac{1}{x}]$ the ring of Laurent polynomials with coefficients in $\mathbb{Z}$. We can express each Laurent polynomial as the form $x^i(a_\ell x^\ell + a_{\ell-1}x^{\ell-1} + ... + a_1 x^1 + a_0)$ with $a_\ell a_0 \neq 0$, $\ell \geq 0$ and $i \in \mathbb{Z}$. If the product of any two such Laurent polynomials, say $x^i(a_\ell x^\ell + ... + a_0)$ and $x^j(b_m x^m + ... + b_0)$, is equal to the unity, then we have that $i + j = \ell = m = 0$ and $a_0 b_0 = 1$. Hence every invertible element can be expressed as $\pm x^i$ for some $i \in \mathbb{Z}$. □

The (non-trivial) right actions of the Steenrod algebra on the homology groups $\tilde{H}_*(\widetilde{R_n}; \mathbb{F}_p)$ and $\tilde{H}_*(\widetilde{B \vee C_n}; \mathbb{F}_p)$ for $p = 2$ and $p = 3$ are given by the following proposition.

Proposition 5.2. *(1) Let x'_q be the modulo 2 reduction of the element x_q. Then in $\tilde{H}_*(\widetilde{R_n}; \mathbb{F}_2)$, the only non-trivial relation is: $x'_{n+5}\mathcal{S}q^4 = x'_{n+1}$,*

(2) Let u'_q be the modulo 2 reduction of the element u_q. Then in $\tilde{H}_*(\widetilde{B \vee C_n}; \mathbb{F}_2)$,the only non-trivial relation is: $u'_{n+5}\mathcal{S}q^4 = u'_{n+1}$.

(3) Let x''_q be the modulo 3 reduction of the element x_q. Then in $\tilde{H}_*(\widetilde{R_n}; \mathbb{F}_3)$, the only non-trivial relation is: $x''_{n+5}\mathcal{P}^1 = \tau \cdot x''_{n+1}$.

(4) Let u''_q be the modulo 3 reduction of the element u_q. Then in $\tilde{H}_*(\widetilde{B\vee C_n};\mathbb{F}_3)$, the only non-trivial relation is: $u''_{n+5}\mathcal{P}^1 = u''_{n+1}$.

Thus in $\tilde{H}_{n+1}(\widetilde{B\vee C_n};\mathbb{F}_2)$ and $\tilde{H}_{n+1}(\widetilde{B\vee C_n};\mathbb{F}_3)$, we have the following equations:

$$u'_{n+1} = \widetilde{h}_*(x'_{n+1}) = \widetilde{h}_*(x'_{n+5}\mathcal{S}q^4) = \widetilde{h}_*(x'_{n+5})\mathcal{S}q^4 = \tau^j{\cdot}u'_{n+5}\mathcal{S}q^4 = \tau^j{\cdot}u'_{n+1},$$
$$u''_{n+1} = \pm\widetilde{h}_*(x''_{n+1}) = \pm\widetilde{h}_*(\tau^{-1}{\cdot}x''_{n+5}\mathcal{P}^1) = \pm\tau^{-1}{\cdot}\widetilde{h}_*(x''_{n+5})\mathcal{P}^1 = \pm\tau^{j-1}{\cdot}u''_{n+5}\mathcal{P}^1 = \pm\tau^{j-1}{\cdot}u''_{n+1}.$$

The upper line tells us that $j = 0$, while the lower line tells us that $j = 1$. It's a contradiction. Thus we obtain the following theorem.

Theorem 5.3. *R_n cannot be split into a one-point-sum of a simply connected space and a bunch of circles.*

6. Self maps of S_n

This section provides an easy but rather crucial property for R_n for $n \geq 4$. By putting $f : S_n \to S_n$ and $g : S_n \to S_n$ be maps of degrees -8 and 9, we obtain

$$f + g \sim 1_{S_n}. \tag{6.1}$$

Proposition 6.1. *The compositions of f and g with $\Sigma^{n-3}\alpha$ and $\Sigma^{n-3}\beta$ give the equations:*

(1) $f\circ\Sigma^{n-3}\alpha \sim *$, (2) $g\circ\Sigma^{n-3}\alpha \sim \Sigma^{n-3}\alpha$, (3) $g\circ\Sigma^{n-3}\beta \sim *$ and (4) $f\circ\Sigma^{n-3}\beta \sim \Sigma^{n-3}\beta$.

7. Homotopy section of $B\vee\widetilde{R_n} \to R_n$

In this section, we show that R_n is a co-H-space. To show this, it is sufficient to show the existence of a homotopy section of $p : B\vee\widetilde{R_n} \to R_n$. We define a map s_0 $B\vee S_n \to B\vee\widetilde{B\vee S_n} \simeq B\vee\bigvee_{i\in\mathbb{Z}}\tau^i{\cdot}S_n$ as follows:

$$s_0|_B = in_B : B \to B\vee\bigvee_{i\in\mathbb{Z}}\tau^i{\cdot}S_n,$$
$$s_0|_{S_n} : S_n \overset{\{f,g\}}{\longrightarrow} S_n\vee S_n \overset{\widetilde{\psi(\tau)}\vee 1_{S_n}}{\longrightarrow} \tau{\cdot}S_n\vee S_n \overset{\Psi(\tau^{-1})\vee 1_{S_n}}{\longrightarrow} B\vee\tau{\cdot}S_n\vee S_n \subset B\vee\bigvee_{i\in\mathbb{Z}}\tau^i{\cdot}S_n.$$

By (6.1), we have $p_0\circ s_0 \sim 1_B\vee(f+g) \sim 1_B\vee 1_{S_n} = 1_{B\vee S_n}$. Since $n \geq 4$, we know that $\pi_{n+4}(S_n\vee S_n) \cong \pi_{n+4}(S_n)\oplus\pi_{n+4}(S_n)$, for dimensional reasons. Then by Proposition 6.1, it follows that

$$s_0\circ\Sigma^{n-3}\alpha \sim in_{S_n}\circ\Sigma^{n-3}\alpha : S^{n+4} \to B\vee S_n \subset B\vee \bigvee_{i\in\mathbb{Z}} \tau^i\cdot S_n,$$

$$s_0\circ\Sigma^{n-3}\beta \sim \Psi(\tau^{-1})\circ in_{\tau\cdot S_n}\circ\widetilde{\psi(\tau)}\circ\Sigma^{n-3}\beta : S^{n+4} \to S_n \to \tau\cdot S_n \to B\vee\tau\cdot S_n \subset B\vee \bigvee_{i\in\mathbb{Z}} \tau^i\cdot S_n.$$

Hence we obtain that

$$s_0\circ(\Sigma^{n-3}\alpha + \psi(\tau)\Sigma^{n-3}\beta) = s_0\circ\Sigma^{n-3}\alpha + \Psi(\tau)s_0\circ\Sigma^{n-3}\beta$$
$$\sim in_{S_n}\circ\Sigma^{n-3}\alpha + in_{\tau\cdot S_n}\circ\widetilde{\psi(\tau)}\circ\Sigma^{n-3}\beta = in_{S_n\vee\tau\cdot S_n}\circ(\Sigma^{n-3}\alpha + \widetilde{\psi(\tau)}\circ\Sigma^{n-3}\beta).$$

Thus the map $s_0\circ(\Sigma^{n-3}\alpha + \psi(\tau)\Sigma^{n-3}\beta)$ is homotopic to the attaching map of the cell $1\cdot e^{n+5}$. Hence it induces a map $s : R_n \to B\vee\widetilde{R_n}$ so that $p\circ s$ is clearly the identity up to homotopy.

Since the universal cover of R_n is a suspension space for the dimensional reasons, we obtain the following theorem.

Theorem 7.1. *R_n is a co-H-space.*

Remark 7.2. *Although we know that R_n and $B\vee C_n$ have isomorphic homotopy groups in each dimension, because they have the same (almost) p-type for any prime p.*

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