Gibbs measures of the quantum Rabi model

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1 Quantum Rabi model

This is the joint work with Tomoyuki Shirai and a review of [8]. The quantum Rabi model describes a two-level atom coupled to a single mode photon by the dipole interaction term. The single photon is represented by the 1D harmonic oscillator. Suppose that the eigenvalues of the two-level atom is $\{-\triangle, \triangle\}$. Here $\triangle > 0$ is a constant. Let σ_x, σ_y and σ_z be the 2×2 Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the Hamiltonian of the two-level atom is represented by $\triangle\sigma_z$. On the other hand let a and $a^\dagger$ be the annihilation operator and the creation operator in $L^2(\mathbb{R})$, respectively. They are given by

$$a = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + x\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + x\right).$$

They satisfy the canonical commutation relation $[a, a^\dagger] = \mathbb{1}$, and $a^* = a^\dagger$, where a^* denotes the adjoint of a. The harmonic oscillator is given by $a^\dagger a$, i.e.,

$$a^\dagger a = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 - \frac{1}{2}.$$

The harmonic oscillator $a^\dagger a$ is self-adjoint on $D(\frac{\mathrm{d}^2}{\mathrm{d}x^2}) \cap D(x^2)$ and the spectrum of $a^\dagger a$ is $\mathrm{spec}(a^\dagger a) = \mathbb{N} \cup \{0\}$. The quantum Rabi Hamiltonian is defined as a self-adjoint operator on the tensor product Hilbert space $\mathbb{C}^2 \otimes L^2(\mathbb{R})$ by

$$K = \triangle\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes a^\dagger a + g\sigma_x \otimes (a + a^\dagger).$$

Here $g \in \mathbb{R}$ stands for a coupling constant. It can be seen that K has the parity symmetry:

$$[K, \sigma_z \otimes (-\mathbb{1})^{a^\dagger a}] = 0.$$

The parity symmetry is also referred to as $\mathbb{Z}_2$-symmetry. We discuss measures associated with the ground state of the quantum Rabi Hamiltonian. The quantum Rabi model can be regarded as the one mode version of the spin-boson model in quantum field theory. In [5] the path measure associated with the ground state of the spin-boson model is discussed. In this note we also show the existence of the measure Π_∞ associated with the ground state $\Phi_{\rm g}$ of the quantum Rabi Hamiltonian. Then under some condition we can see that

$$(\Phi_{\rm g}, \mathcal{O}\Phi_{\rm g}) = \mathbb{E}_{\Pi_\infty}[f_{\mathcal{O}}]$$

for some observable $\mathcal{O}$ with a function $f_{\mathcal{O}}$.

2 Probabilistic preparation

2.1 Unitary transformations

In this section we define a self-adjoint operators L. Let $\sigma = (\sigma_x, \sigma_y, \sigma_z)$. The rotation group in $\mathbb{R}^3$ has an adjoint representation on $su(2)$. Let $n \in \mathbb{R}^3$ be a unit vector and $\theta \in [0, 2\pi)$. Thus $e^{(i/2)\theta n\cdot\sigma}(x\cdot\sigma)e^{-(i/2)\theta n\cdot\sigma} = Rx\cdot\sigma$, where R denotes the 3×3 matrix representing the rotation around n by an angle θ. In particular for $n = (0,1,0)$ and $\theta = \pi/2$, we have $U\sigma_x U^{-1} = \sigma_z$ and $U\sigma_z U^{-1} = -\sigma_x$, where

$$U = e^{i\frac{\pi}{4}\sigma_y}. \tag{2.1}$$

Then

$$UKU^{-1} = \begin{pmatrix} -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 - \sqrt{2}gx - \frac{1}{2} & -\triangle \\ -\triangle & -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 + \sqrt{2}gx - \frac{1}{2} \end{pmatrix}.$$

Let us define the unitary operator $\mathcal{S}_g$. Let $p = -i\frac{\mathrm{d}}{\mathrm{d}x}$ and F denotes the Fourier transform on $L^2(\mathbb{R})$. Then $\mathcal{S}_g$ is defined by

$$\mathcal{S}_g = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} 0 & e^{i\sqrt{2}gp} \\ e^{-i\sqrt{2}gp} & 0 \end{pmatrix}. \tag{2.2}$$

Let $\varphi_{\rm g}$ be the normalized ground state of $a^\dagger a$, i.e., $a^\dagger a\varphi_{\rm g} = 0$ and it is explicitly given by

$$\varphi_{\rm g}(x) = \pi^{-1/4}e^{-|x|^2/2}.$$

Since $\varphi_{\rm g}$ is strictly positive, we can define the unitary operator $\mathcal{U}_{\varphi_{\rm g}} : L^2(\mathbb{R}) \to L^2(\mathbb{R}, \varphi_{\rm g}^2\mathrm{d}x)$ by

$$\mathcal{U}_{\varphi_{\rm g}} f = \varphi_{\rm g}^{-1} f. \tag{2.3}$$

We set the probability measure $\varphi_{\rm g}^2(x)\mathrm{d}x$ on $\mathbb{R}$ by $\mathrm{d}\mu$, i.e.,

$$\mathrm{d}\mu(x) = \frac{1}{\sqrt{\pi}}e^{-|x|^2}\mathrm{d}x.$$

Define

$$\mathcal{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R}, \mathrm{d}\mu).$$

Let $\mathcal{U} = \mathcal{U}_{\varphi_{\mathrm{g}}} U$. We define the self-adjoint operator L by

$$\begin{aligned} L &= \mathcal{U} K \mathcal{U}^{-1} = -\triangle \sigma_x \otimes \mathbb{1} + g\sigma_z \otimes (b^\dagger + b) + \mathbb{1} \otimes b^\dagger b \\ &= \begin{pmatrix} -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\frac{\mathrm{d}}{\mathrm{d}x} \end{pmatrix} - \begin{pmatrix} -\sqrt{2}gx & \triangle \\ \triangle & \sqrt{2}gx \end{pmatrix}. \end{aligned} \tag{2.4}$$

Here b and $b^\dagger$ are the annihilation operator and the creation operator in $L^2(\mathbb{R}, \mathrm{d}\mu)$, which are defined by $\varphi_{\mathrm{g}}^{-1} a^\sharp \varphi_{\mathrm{g}} = b^\sharp$. It is actually given by

$$b = a + \frac{x}{\sqrt{2}}, \quad b^\dagger = a^\dagger - \frac{x}{\sqrt{2}}.$$

2.2 Ornstein-Uhrenbeck process

Let $(X_t)_{t\geq 0}$ be the Ornstein-Uhrenbeck process on a probability space

$$(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mathrm{P}^x).$$

We see that $\mathrm{P}^x(X_0 = x) = 1$ and

$$\int_{\mathbb{R}} \mathbb{E}_{\mathrm{P}}^x \left[X_t\right] \mathrm{d}\mu(x) = 0, \quad \int_{\mathbb{R}} \mathbb{E}_{\mathrm{P}}^x \left[X_t X_s\right] \mathrm{d}\mu(x) = \frac{1}{2} e^{-|t-s|}.$$

Here $\mathbb{E}_{\mathrm{P}}^x[\cdots]$ denotes the expectation with respect to the probability measure P^x. Let $h = b^\dagger b$. The generator of X_t is given by $-h$ and

$$(\phi, e^{-th}\psi)_{L^2(\mathbb{R}, \mathrm{d}\mu)} = \int_{\mathbb{R}} \mathbb{E}_{\mathrm{P}}^x \left[\overline{\phi(X_0)} \psi(X_t)\right] \mathrm{d}\mu(x). \tag{2.5}$$

It is well known that the Ornstein-Uhrenbeck process can be represented by 1D-Brownian motion. Let $(B_t)_{t\geq 0}$ be 1D-Brownian motion starting from x at $t = 0$ on a probability space $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mathcal{W}^0)$. The distributions of X_s under P^x and $e^{-s}\left(x + \frac{1}{\sqrt{2}} B_{e^{2s}-1}\right)$ under $\mathcal{W}^0$ are identical. We denote this as

$$X_s \stackrel{d}{=} e^{-s}\left(x + \frac{1}{\sqrt{2}} B_{e^{2s}-1}\right) \quad s \geq 0. \tag{2.6}$$

We can compute the density function κ_t of X_t as

$$\mathbb{E}_{\mathrm{P}}^x[f(X_t)] = \int_{\mathbb{R}} f(y) \kappa_t(y, x) \mathrm{d}y,$$

where

$$\kappa_t(y, x) = \frac{1}{\sqrt{\pi(1 - e^{-2t})}} \exp\left(-\frac{|y - e^{-t}x|^2}{1 - e^{-2t}}\right). \tag{2.7}$$

The Mehler kernel M_t is defined by

$$M_t(x,y)=\frac{\varphi_{\rm g}(x)}{\varphi_{\rm g}(y)}\kappa_t(y,x)=\frac{1}{\sqrt{\pi(1-e^{-2t})}}\exp\left(-\frac{1}{2}\frac{(1+e^{-2t})(x^2+y^2)-4xye^{-t}}{1-e^{-2t}}\right).$$

For the later use we extend the Ornstein-Uhrenbeck process $(X_t)_{t\geq 0}$ to the Ornstein-Uhrenbeck process $(\hat{X}_t)_{t\in\mathbb{R}}$ on the whole real line on the probability space $(\bar{\mathcal{X}},\mathcal{B}_{\bar{\mathcal{X}}},\bar{\rm P}^x)$. Here $\bar{\mathcal{X}}=\mathcal{X}\times\mathcal{X}$, $\mathcal{B}_{\bar{\mathcal{X}}}=\mathcal{B}_{\mathcal{X}}\times\mathcal{B}_{\mathcal{X}}$ and $\bar{\rm P}^x={\rm P}^x\otimes{\rm P}^x$. Define for $w=(w_1,w_2)\in\mathcal{X}\times\mathcal{X}$

$$\hat{X}_t(w)=\begin{cases} X_t(w_1), & t\geq 0,\\ X_{-t}(w_2), & t<0.\end{cases}\tag{2.8}$$

Then $\hat{X}_t$ and $\hat{X}_{-s}$ for any $s,t>0$ are independent. We also see that

$$(\phi,e^{-th}\psi)_{L^2(\mathbb{R},{\rm d}\mu)}=\int_{\mathbb{R}}\mathbb{E}_{\bar{\rm P}^x}\left[\overline{\phi(\hat{X}_0)}\psi(\hat{X}_t)\right]{\rm d}\mu(x)=\int_{\mathbb{R}}\mathbb{E}_{\bar{\rm P}^x}\left[\overline{\phi(\hat{X}_{-s})}\psi(\hat{X}_{t-s})\right]{\rm d}\mu(x)\tag{2.9}$$

for any $0\leq s\leq t$.

2.3 Spin process

In order to show the spin part by a path measure we introduce a Poisson process. Let $(N_t)_{t\geq 0}$ be a Poisson process on a probability space

$$(\mathcal{Y},\mathcal{B}_{\mathcal{Y}},\Pi)$$

with the unit intensity, i.e.,

$$\mathbb{E}_\Pi\left[\mathbb{1}_{\{N_t=n\}}\right]=\frac{t^n}{n!}e^{-t},\quad n\geq 0.$$

Note that N_t is a nonnegative integer-valued random process, $N_0=0$ and $t\mapsto N_t$ is not decreasing. Furthermore $t\mapsto N_t$ is right continuous and its left limit exists (cádlág). Let

$$\mathbb{Z}_2=\{-1,+1\}.$$

Then for $u\in L^2(\mathbb{Z}_2)$,

$$\|u\|^2_{L^2(\mathbb{Z}_2)}=\sum_{\alpha\in\mathbb{Z}_2}|u(\alpha)|^2.$$

Introducing the norm on $\mathbb{C}^2$ by $(u,v)_{\mathbb{C}^2}=\sum_{i=1}^2\bar{u}_iv_i$, we identify $\mathbb{C}^2\cong L^2(\mathbb{Z}_2)$ by $\mathbb{C}^2\ni u=\begin{pmatrix}u_1\\u_2\end{pmatrix}\cong u(\alpha)$ with $u(+1)=u_1$ and $u(-1)=u_2$. Note that

$$(u,v)_{\mathbb{C}^2}=(u,v)_{L^2(\mathbb{Z}_2)}.$$

Under this identification σ_x,σ_y and σ_z are represented as the operators U_x,U_y and U_z, respectively on $L^2(\mathbb{Z}_2)$ by

$$U_xu(\alpha)=u(-\alpha),\quad U_yu(\alpha)=-i\alpha u(-\alpha),\quad U_zu(\alpha)=\alpha u(\alpha),\quad u\in L^2(\mathbb{Z}_2).\tag{2.10}$$

We define

$$S_t = (-1)^{N_t}\alpha, \quad \alpha \in \mathbb{Z}_2.$$

Here $(S_t)_{t\geq 0}$ is a dichotomous process which is referred to as a spin process in this note. Let $\sigma_F = \frac{1}{2}(\sigma_z + i\sigma_y)(\sigma_z - i\sigma_y) = -\sigma_x + \mathbb{1}$ be the fermionic harmonic oscillator. Then it is known that for $u, v \in \mathbb{C}^2$, $(u, e^{-t\sigma_F} v)_{\mathbb{C}^2} = \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}_\Pi[\overline{u(S_0)} v(S_t)]$. Hence

$$(u, e^{t\sigma_x} v)_{\mathbb{C}^2} = e^t \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}_\Pi[\overline{u(S_0)} v(S_t)]. \tag{2.11}$$

We also extend the Poisson process $(N_t)_{t\geq 0}$ to the Poisson process $(\hat{N}_t)_{t\in\mathbb{R}}$ on the whole real line on a probability space $(\bar{\mathcal{Y}}, \mathcal{B}_{\bar{\mathcal{Y}}}, \bar{\Pi})$, where $\bar{\mathcal{Y}} = \mathcal{Y} \times \mathcal{Y}$, $\mathcal{B}_{\bar{\mathcal{Y}}} = \mathcal{B}_\mathcal{Y} \times \mathcal{B}_\mathcal{Y}$ and $\bar{\Pi} = \Pi \otimes \Pi$. Let $(\bar{N}_t)_{t\geq 0}$ be a Poisson process on $(\mathcal{Y}, \mathcal{B}_\mathcal{Y}, \Pi)$ such that $t \mapsto \bar{N}_t$ is left continuous and its right limit exists (càglàd). Define for $w = (w_1, w_2) \in \mathcal{Y} \times \mathcal{Y}$,

$$\hat{N}_t(w) = \begin{cases} N_t(w_1), & t \geq 0, \\ \bar{N}_{-t}(w_2), & t < 0. \end{cases}$$

Then $\mathbb{R} \ni t \mapsto \hat{N}_t$ is a càdlàg path. Note that $\hat{N}_t$ is independent of $\hat{N}_{-s}$ for any $s, t > 0$. We define

$$\hat{S}_t = (-1)^{\hat{N}_t}\alpha, \quad \alpha \in \mathbb{Z}_2.$$

By the shift invariance of $\hat{S}_s$ [9, Proposition 3.44] we can see that for $u, v \in \mathbb{C}^2$,

$$(u, e^{t\sigma_x} v)_{\mathbb{C}^2} = e^t \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}_{\bar{\Pi}}[\overline{u(\hat{S}_0)} v(\hat{S}_t)] = e^t \sum_{\alpha\in\mathbb{Z}_2} \mathbb{E}_{\bar{\Pi}}[\overline{u(\hat{S}_{-s})} v(\hat{S}_{t-s})]$$

for any $0 \leq s \leq t$.

3 Path measure associated with the ground state

In this section we construct the path measure associated with the ground state of the quantum Rabi model. We recall that $L = -\triangle\sigma_x \otimes \mathbb{1} + \mathbb{1} \otimes b^\dagger b + g\sigma_z \otimes (b + b^\dagger)$. Let Φ_g be the ground state of L such that

$$L\Phi_\mathrm{g} = E\Phi_\mathrm{g}$$

with $E = \inf \mathrm{spec}(L)$. It is shown that $\Phi_\mathrm{g} > 0$ in [4] under the identification (3.1). Hence $(\mathbb{1}, \Phi_\mathrm{g})_\mathcal{H} \neq 0$. Then

$$\Phi_\mathrm{g} = \lim_{t\to\infty} \frac{e^{-tL}\mathbb{1}}{\|e^{-tL}\mathbb{1}\|_\mathcal{H}}.$$

Let us set

$$\langle \mathcal{O} \rangle = (\Phi_\mathrm{g}, \mathcal{O}\Phi_\mathrm{g})_\mathcal{H}$$

for a bounded operator $\mathcal{O}$. Then we have

$$\langle \mathcal{O} \rangle = \lim_{t\to\infty} \frac{(e^{-tL}\mathbb{1}, \mathcal{O}e^{-tL}\mathbb{1})_\mathcal{H}}{\|e^{-tL}\mathbb{1}\|^2_\mathcal{H}}.$$

The right-hand side can be represented in terms of Feynman-Kac formula, and under some condition we can also see that

$$\langle \mathcal{O} \rangle = \mathbb{E}_{\Pi_\infty}[f_{\mathcal{O}}]$$

with some probability measure Π_∞ and a function $f_{\mathcal{O}}$. The probability measure Π_∞ is referred to as the path measure associated with the ground state $\Phi_{\rm g}$. The similar results are investigated in models in quantum field theory [10, 1, 5, 6, 7], but as far as we know there is no example in quantum mechanics.

3.1 Feynman-Kac formula

Combining (2.5) and (2.11) we can represent $(\phi, e^{-tH}\psi)$ by a path measure. Let

$$q_s = (S_s, X_s) \quad s \geq 0$$

be the $(\mathbb{Z}_2 \times \mathbb{R})$-valued random process on the probability space $(\mathcal{X} \otimes \mathcal{Y}, \mathcal{B}_{\mathcal{X}} \otimes \mathcal{B}_{\mathcal{Y}}, \mathrm{P}^x \otimes \Pi)$. We introduce the identification:

$$\mathcal{H} \cong L^2(\mathbb{Z}_2 \times \mathbb{R}) \tag{3.1}$$

by

$$\begin{pmatrix} \phi_+(x) \\ \phi_-(x) \end{pmatrix} \cong \phi(\alpha, x) = \delta_{+1\alpha}\phi_+(x) + \delta_{-1\alpha}\phi_-(x), \quad (\alpha, x) \in \mathbb{Z}_2 \times \mathbb{R}. \tag{3.2}$$

Here $\delta_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$. We use identification (3.1) without notices unless no confusion arises. Let $W : \mathbb{Z}_2 \times \mathbb{R} \to \mathbb{R}$ be defined by

$$W(\alpha, x) = \sqrt{2}\alpha x.$$

Thus $W(q_{s-}) = \sqrt{2}S_{s-}X_s$. The Poisson integral $\int_0^{t+} W(q_{s-})\mathrm{d}N_s$ is a random process on the probability space $(\mathcal{X} \otimes \mathcal{Y}, \mathcal{B}_{\mathcal{X}} \otimes \mathcal{B}_{\mathcal{Y}}, \mathrm{P}^x \otimes \Pi)$, which is defined by

$$\left(\int_0^{t+} W(q_{s-})\mathrm{d}N_s\right)(w_1, w_2) = \sum_{j=1}^n W(q_{s_j}(w_1, w_2)) = \sqrt{2}\sum_{j=1}^n S_{s_j-}(w_1)X_{s_j}(w_2).$$

Here $\{s_j\}$ is the set of jump points such that $N_{s_j-}(w_1) \neq N_{s_j+}(w_1)$ for $0 \leq s_j \leq t$. Let

$$\mathbf{E}\left[\ldots\right] = \frac{1}{2}\sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\mathrm{P}}^x \mathbb{E}_\Pi\left[\ldots\right] \mathrm{d}\mu(x).$$

Lemma 3.1 *Let $\phi, \psi \in \mathcal{H}$. Then under the identification* (3.2)*, it follows that*

$$(\phi, e^{-tL}\psi) = 2e^t\mathbf{E}\left[\overline{\phi(q_0)}\psi(q_t)\triangle^{N_t}e^{-g\int_0^t W(q_s)\mathrm{d}s}\right]. \tag{3.3}$$

Proof: We refer the reader to [8]. ■

Lemma 3.1 can be extended to the path integral representations of Euclidean Green functions. Let $h=-\Delta/2$ and $(B_t)_{t\geq 0}$ be 1D Brownian motion on $(\mathcal{X},\mathcal{B}_{\mathcal{X}},\mathcal{W}^x)$. Suppose that $0<t_0<t_1<\ldots<t_n$. Let $C^{\{t_0,t_1,\ldots,t_n\}}(A_0\times\cdots\times A_n)=\{\omega\in\mathcal{X}\mid \omega(t_j)\in A_j, j=0,1,\ldots,n\}$ be a cylinder set. Then it is known that

$$\mathcal{W}^x(C^{\{t_0,t_1,\ldots,t_n\}}(A_0\times\cdots\times A_n))=\mathbb{E}^x\left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(B_{t_j})\right)\right].$$

We know furthermore that for $f,g\in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}}\mathbb{E}^x\left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(B_{t_j})\right)\bar{f}(B_0)g(B_t)\right]\mathrm{d}x=(f,e^{-t_0h}\mathbb{1}_{A_0}e^{-(t_1-t_0)h}\cdots e^{-(t_n-t_{n-1})h}\mathbb{1}_{A_n}e^{-(t-t_n)h}g).$$

Lemma 3.2 *Let $f_j=f_j(\alpha,x)$ be bounded function on $\mathbb{Z}_2\times\mathbb{R}$ for $j=0,1,\ldots,n$. Suppose that $0<t_0<t_1<\ldots<t_n$. Then*

$$\begin{aligned}&(\phi,e^{-t_0L}f_0e^{-(t_1-t_0)L}f_1e^{-(t_2-t_1)L}\cdots e^{-(t_n-t_{n-1})L}f_ne^{-(t-t_n)L}\psi)\\&=2e^t\mathbf{E}\left[\bar{\phi}(q_0)\psi(q_t)\left(\prod_{j=0}^n f_j(q_{t_j})\right)e^{-g\int_0^t W(q_s)\mathrm{d}s}\right].\end{aligned}$$

Proof: Denote the natural filtrations of $(N_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ by $\mathscr{N}_s=\boldsymbol{\sigma}(N_r,0\leq r\leq s)$ and $\mathscr{M}_s=\boldsymbol{\sigma}(X_r,0\leq r\leq s)$, respectively. The Markov properties of $(N_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ yield that

$$\begin{aligned}&\left(e^{-sL}fe^{-tL}\phi\right)(\alpha,x)\\&=e^{s+t}\mathbb{E}_\Pi\mathbb{E}_{\mathrm{P}}^x\left[e^{-g\int_0^s W(q_r)\mathrm{d}r}f(q_s)\mathbb{E}_\Pi^{S_s}\mathbb{E}_{\mathrm{P}}^{X_s}\left[e^{-g\int_0^t W(q_r)\mathrm{d}r}\phi(q_t)\right]\right]\\&=e^{s+t}\mathbb{E}_\Pi\mathbb{E}_{\mathrm{P}}^x\left[e^{-g\int_0^s W(q_r)\mathrm{d}r}f(q_s)\mathbb{E}_\Pi\mathbb{E}_{\mathrm{P}}^x\left[e^{-g\int_0^t W(q_{r+s})\mathrm{d}r}\phi(q_{t+s})\Big|\mathscr{N}_s\times\mathscr{M}_s\right]\right]\\&=e^{s+t}\mathbb{E}_\Pi\mathbb{E}_{\mathrm{P}}^x\left[e^{-g\int_0^s W(q_r)\mathrm{d}r}f(q_s)e^{-g\int_0^t W(q_{r+s})\mathrm{d}r}\phi(q_{t+s})\right]\\&=e^{s+t}\mathbb{E}_\Pi\mathbb{E}_{\mathrm{P}}^x\left[e^{-g\int_0^{s+t} W(q_r)\mathrm{d}r}f(q_s)\phi(q_{t+s})\right].\end{aligned}$$

Repeating these procedures we have the lemma. ■

3.2 Probability measure Π_∞ associated with the ground state

We set $T_s=S_{\triangle s}$ and $q_s^\triangle=(T_s,X_s)$. We assume that $\triangle>0$ in what follows.

Lemma 3.3 *Let $\phi,\psi\in\mathcal{H}$. Then*

$$(\phi,e^{-tL}\psi)=e^{\triangle t}\sum_{\alpha\in\mathbb{Z}_2}\int_{\mathbb{R}}\mathbb{E}_\Pi\mathbb{E}_{\mathrm{P}}^x\left[\overline{\phi(q_0^\triangle)}\psi(q_t^\triangle)e^{-g\int_0^t W(q_s^\triangle)\mathrm{d}s}\right]\mathrm{d}\mu(x). \tag{3.4}$$

Proof: Since

$$\frac{1}{\triangle}L = -\sigma_x \otimes 1\!\!1 + 1\!\!1 \otimes \frac{1}{\triangle} b^\dagger b + \frac{g}{\triangle}\sigma_z \otimes (b^\dagger + b),$$

the Feynman-Kac formula (3.3) yields that

$$(\phi, e^{-tL}\phi) = (\phi, e^{-\triangle t \frac{1}{\triangle}L}\phi) = e^{\triangle t}\sum_{\alpha\in\mathbb{Z}_2}\int_{\mathbb{R}} \mathbb{E}_\Pi \mathbb{E}_{\mathrm{P}}^x \left[\overline{\phi(S_0, X_0)}\psi(S_{\triangle t}, X_t) e^{-\frac{g}{\triangle}\int_0^{\triangle t}\sqrt{2}S_s X_{s/\triangle}\mathrm{d}s}\right]\mathrm{d}\mu(x).$$

By the change of variable s to $\triangle s$ in $\frac{g}{\triangle}\int_0^{\triangle t}\sqrt{2}S_s X_{s/\triangle}\mathrm{d}s$, we see (3.4). ■

For the later use we have a technical lemma below.

Lemma 3.4 *We have*

$$\mathbb{E}_{\mathrm{P}}^x\left[e^{-g\int_0^t W(\hat{q}_s^\triangle)\mathrm{d}s}\right] = e^{-g\left(\int_0^t e^{-s}(-1)^{N_{\triangle s}}\mathrm{d}s\right)x} e^{\frac{g^2}{4}\int_0^{(1-e^{-2t})/2}\left|\int_y^t (-1)^{N_{\triangle s}}\mathrm{d}s\right|^2\mathrm{d}y}.$$

In particular

$$\mathbb{E}_{\mathrm{P}}^x\left[e^{-g\int_0^t W(\hat{q}_s^\triangle)\mathrm{d}s}\right] \le e^{|g|(1-e^{-t})x} e^{\frac{g^2}{4}\int_0^{(1-e^{-2t})/2}|t-y|^2\mathrm{d}y}.$$

Proof: We have

$$\begin{aligned}
\mathbb{E}_{\mathrm{P}}^x\left[e^{-g\int_0^t W(\hat{q}_s^\triangle)\mathrm{d}s}\right] &= \mathbb{E}_{\mathcal{W}}^0\left[e^{-g\int_0^t e^{-s}(x+\frac{1}{\sqrt{2}}B_{e^{2s}-1})(-1)^{N_{\triangle s}}\mathrm{d}s}\right]\\
&= e^{-g\left(\int_0^t e^{-s}(-1)^{N_{\triangle s}}\mathrm{d}s\right)x}\mathbb{E}_{\mathcal{W}}^0\left[e^{-g\int_0^t B_{(1-e^{-2s})/2}(-1)^{N_{\triangle s}}\mathrm{d}s}\right]\\
&= e^{-g\left(\int_0^t e^{-s}(-1)^{N_{\triangle s}}\mathrm{d}s\right)x} e^{\frac{g^2}{4}\left\|\int_0^t 1\!\!1_{(1-e^{-2s})/2}(\cdot)(-1)^{N_{\triangle s}}\mathrm{d}s\right\|_{L^2(\mathbb{R})}^2}.
\end{aligned}$$

Then the lemma is proven. ■

Now we extend $(T_t)_{t\ge 0}$ to the process on the whole real line. Let

$$\hat{T}_t = (-1)^{\hat{N}_{\triangle t}}\alpha \quad t\in\mathbb{R}.$$

We can realize $(\hat{T}_t)_{t\in\mathbb{R}}$ as a coordinate process as usual. Let $\mathcal{D} = D(\mathbb{R})$ be the space of càdlàg paths on $\mathbb{R}$. There exists a topology d° on $\mathcal{D}$ such that $(\mathcal{D}, d^\circ)$ is a separable and complete metric space (e.g. [3, Section 3.5] and [2, Section 16]). Let $\mathcal{B}_\mathcal{D}$ be the Borel sigma-field of $\mathcal{D}$. Thus

$$\hat{T}_\bullet : (\bar{\mathcal{Y}}, \mathcal{B}_{\bar{\mathcal{Y}}}, \bar{\Pi}) \to (\mathcal{D}, \mathcal{B}_\mathcal{D})$$

is an $\mathcal{D}$-valued random variable. We denote its image measure on $(\mathcal{D}, \mathcal{B}_\mathcal{D})$ by Q^α, i.e., $\mathrm{Q}^\alpha(A) = \bar{\Pi}(\hat{T}_\bullet^{-1}(A))$ for $A\in\mathcal{B}_\mathcal{D}$, and the coordinate process on $(\mathcal{D}, \mathcal{B}_\mathcal{D})$ by the same symbol $(\hat{T}_t)_{t\ge 0}$, i.e., $\hat{T}_t(\omega) = \omega(t)$ for $\omega\in\mathcal{D}$. Let $\pi_\Lambda : \mathcal{D}\to\mathbb{R}^\Lambda$ be the projection defined by $\pi_\Lambda(\omega) = (\omega(t_0), \ldots, \omega(t_n))$ for $\omega\in\mathcal{D}$ and $\Lambda = \{t_0, \ldots, t_n\}$. Then

$$\mathcal{A} = \{\pi_\Lambda^{-1}(E) \,|\, \Lambda\subset\mathbb{R}, \#\Lambda < \infty, E\in\mathcal{B}(\mathbb{R}^\Lambda)\}$$

is the family of cylinder sets. It is known that the sigma-field generated by cylinder sets coincides with $\mathcal{B}_\mathcal{D}$. Moreover let $\mathcal{D}_T = D([-T, T])$ be the space of càdlàg paths on $[-T, T]$

and $\pi_T : \mathcal{D} \to \mathcal{D}_T$ be the projection defined by $\pi_T\omega = \omega\lceil_{[-T,T]}$. Let $\mathcal{B}_T$ be the Borel sigma-field of $\mathcal{D}_T$. Let $\pi_\Lambda : \mathcal{D}_T \to \mathbb{R}^\Lambda$ be the projection defined by $\pi_\Lambda(\omega) = (\omega(t_0), \ldots, \omega(t_n))$ for $\omega \in \mathcal{D}_T$ and $\Lambda = \{t_0, \ldots, t_n\}$. Note that we use the same notation π as the projection from $\mathcal{D}$ to $\mathbb{R}^\Lambda$. Then

$$\mathcal{A}_T = \{\pi_\Lambda^{-1}(E) \,|\, \Lambda \subset [-T, T], \#\Lambda < \infty, E \in \mathcal{B}(\mathbb{R}^\Lambda)\}$$

is the family of cylinder sets. We set

$$\overset{\circ}{\mathcal{B}} = \bigcup_{s \geq 0} \pi_s^{-1}(\mathcal{B}_s), \quad \overset{\circ}{\mathcal{B}}_T = \bigcup_{0 \leq s \leq T} \pi_s^{-1}(\mathcal{B}_s).$$

It is also seen that the sigma-field generated by $\overset{\circ}{\mathcal{B}}$ (resp. $\overset{\circ}{\mathcal{B}}_T$) coincides with $\mathcal{B}_\mathcal{D}$ (resp. $\mathcal{B}_T$). Together with them we have

$$\mathcal{B}_\mathcal{D} = \boldsymbol{\sigma}(\mathcal{A}) = \boldsymbol{\sigma}(\overset{\circ}{\mathcal{B}}), \quad \mathcal{B}_T = \boldsymbol{\sigma}(\mathcal{A}_T) = \boldsymbol{\sigma}(\overset{\circ}{\mathcal{B}}_T). \tag{3.5}$$

Hence (3.3) can be reformulated in terms of the coordinate process $(\hat{T}_t)_{t \geq 0}$ on $(\mathcal{D}, \mathcal{B}_\mathcal{D}, \mathrm{Q}^\alpha)$ instead of $(\bar{\mathcal{Y}}, \mathcal{B}_{\bar{\mathcal{Y}}}, \bar{\Pi})$ as

$$(\phi, e^{-tL}\psi) = e^{\triangle t} \sum_{\alpha \in \mathbb{Z}_2} \int_\mathbb{R} \mathbb{E}_\mathrm{Q}^\alpha \mathbb{E}_{\bar{\mathrm{P}}}^x \left[\overline{\phi(\hat{q}_0^\triangle)} e^{-g\int_0^t W(\hat{q}_s^\triangle) \mathrm{d}s} \psi(\hat{q}_t^\triangle) \right] \mathrm{d}\mu(x). \tag{3.6}$$

Here

$$\hat{q}_s^\triangle = (\hat{T}_s, \hat{X}_s) \quad s \in \mathbb{R},$$

where $\hat{X}_t$ is the Ornstein-Uhlenbeck process on the whole real line. The advantage of (3.4) is that $\triangle^{N_t}$ disappears. $\triangle^{N_t}$ is not shift invariant but $\hat{T}_s$ in (3.4) is shift invariant. Then

$$\begin{aligned} &\sum_{\alpha \in \mathbb{Z}_2} \int_\mathbb{R} \mathbb{E}_\mathrm{Q}^\alpha \mathbb{E}_{\bar{\mathrm{P}}}^x \left[\overline{\phi(\hat{q}_0^\triangle)} e^{-g\int_0^t W(\hat{q}_s^\triangle) \mathrm{d}s} \psi(\hat{q}_t^\triangle) \right] \mathrm{d}\mu(x) \\ &= \sum_{\alpha \in \mathbb{Z}_2} \int_\mathbb{R} \mathbb{E}_\mathrm{Q}^\alpha \mathbb{E}_{\bar{\mathrm{P}}}^x \left[\overline{\phi(\hat{q}_{-r}^\triangle)} e^{-g\int_0^t W(\hat{q}_{s-r}^\triangle) \mathrm{d}s} \psi(\hat{q}_{t-r}^\triangle) \right] \mathrm{d}\mu(x) \end{aligned}$$

for any $0 \leq r \leq t$. Let

$$W_\triangle(t, s) = \hat{T}_t \hat{T}_s e^{-|t-s|}. \tag{3.7}$$

Lemma 3.5 *We have*

$$(\mathbb{1}, e^{-tL}\mathbb{1}) = 2e^{\triangle t} \mathbb{E}_\mathrm{Q}^\alpha \left[\exp\left(\frac{g^2}{2} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}r W_\triangle(s, r) \right) \right].$$

Proof: By the Feynman-Kac formula given by (3.4) and inserting (2.6), we can see that

$$(\mathbb{1}, e^{-tL}\mathbb{1}) = e^{\triangle t} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_\mathrm{Q}^\alpha \left[\mathbb{E}_{\bar{\mathrm{P}}}^x \left[e^{-g\int_0^t \hat{T}_s e^{-s} B_{e^{2s}-1} \mathrm{d}s} \right] \int_\mathbb{R} e^{-\left(\sqrt{2}g\int_0^t \hat{T}_s e^{-s} \mathrm{d}s\right)x} \mathrm{d}\mu(x) \right]$$

Since

$$\mathbb{E}_{\mathrm{P}}^{x}\left[e^{-g\int_0^t \hat{T}_s e^{-s}B_{e^{2s}-1}\mathrm{d}s}\right]=\exp\left(\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r\hat{T}_s\hat{T}_r e^{-(s+r)}(e^{2(s\wedge r)}-1)\right)$$
$$\int_{\mathbb{R}} e^{-\left(\sqrt{2}g\int_0^t \hat{T}_s e^{-s}\mathrm{d}s\right)x}\mathrm{d}\mu(x)=\exp\left\{\frac{g^2}{2}\left(\int_0^t \hat{T}_s e^{-s}\mathrm{d}s\right)^2\right\},$$

we obtain that

$$(\mathbb{1},e^{-tL}\mathbb{1})=e^{\triangle t}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[\exp\left(\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r\hat{T}_s\hat{T}_r e^{-|s-r|}\right)\right].$$

Hence the lemma follows. ■

Remark 3.6 *(1) Since $W_\triangle(s,r)$ is independent of α, $\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[\exp\left(\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}rW_\triangle(s,r)\right)\right]$ is also independent of σ.*
(2) By the shift invariance of $\hat{T}_s$ we can also see that

$$\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[\exp\left(\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}rW_\triangle(s,r)\right)\right]=\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[\exp\left(\frac{g^2}{2}\int_{-u}^{t-u} \mathrm{d}s\int_{-u}^{t-u} \mathrm{d}rW_\triangle(s,r)\right)\right]$$

for any $0\le u\le t$. Thus we see that

$$\begin{aligned}(e^{-tL}\mathbb{1},e^{-tL}\mathbb{1})&=2e^{2\triangle t}\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[\exp\left(\frac{g^2}{2}\int_0^{2t} \mathrm{d}s\int_0^{2t} \mathrm{d}rW_\triangle(s,r)\right)\right]\\&=2e^{2\triangle t}\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[\exp\left(\frac{g^2}{2}\int_{-t}^{t} \mathrm{d}s\int_{-t}^{t} \mathrm{d}rW_\triangle(s,r)\right)\right].\end{aligned}\tag{3.8}$$

We can also compute $(e^{-tL}\mathbb{1},e^{-\beta b^\dagger b}e^{-tL}\mathbb{1})$ for $\beta>0$.

Lemma 3.7 *Let $\beta>0$. Then*

$$\begin{aligned}&(e^{-tL}\mathbb{1},e^{-\beta b^\dagger b}e^{-tL}\mathbb{1})\\&=2e^{2\triangle t}\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[\exp\left(\frac{g^2}{2}\int_{-t}^{t}\int_{-t}^{t}W_\triangle(s,r)\mathrm{d}s\mathrm{d}r-g^2(1-e^{-\beta})\int_{-t}^{0}\int_{0}^{t}W_\triangle(s,r)\mathrm{d}s\mathrm{d}r\right)\right].\end{aligned}$$

Proof: Since

$$(\phi,e^{-\beta b^\dagger b}\psi)=\sum_{\alpha\in\mathbb{Z}_2}\int_{\mathbb{R}}\bar{\phi}(\alpha,X_0)\mathbb{E}_{\mathrm{P}}^{x}[\psi(\alpha,X_\beta)]\mathrm{d}\mu(x),$$

we see that

$$(e^{-tL}\mathbb{1},e^{-\beta b^\dagger b}e^{-tL}\mathbb{1})=\sum_{\alpha\in\mathbb{Z}_2}\int_{\mathbb{R}}(e^{-tL}\mathbb{1})(\alpha,X_0)\mathbb{E}_{\mathrm{P}}^{x}[(e^{-tL}\mathbb{1})(\alpha,X_\beta)]\mathrm{d}\mu(x).$$

It is straightforward to compute $(e^{-tL}\mathbb{1})(\alpha, X_0)$ and $(e^{-tL}\mathbb{1})(\alpha, X_\beta)$. We have

$$
\begin{aligned}
(e^{-tL}\mathbb{1})(\alpha, X_0) &= e^{\triangle t}\mathbb{E}_{\mathrm{Q}}^{\alpha}\mathbb{E}_{\mathrm{P}}^{x}\left[e^{-\sqrt{2}g\int_0^t \hat{T}_s X_s^x \mathrm{d}s}\right] \\
&= e^{\triangle t}\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[e^{-\sqrt{2}g\int_0^t \hat{T}_s e^{-s}\mathrm{d}s x}\mathbb{E}_{\mathcal{W}}^{0}\left[e^{-g\int_0^t \hat{T}_s e^{-s} B_{e^{2s}-1}\mathrm{d}s}\right]\right] \\
&= e^{\triangle t}\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[e^{-\sqrt{2}g\int_0^t \hat{T}_s e^{-s}\mathrm{d}s x} e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \hat{T}_s\hat{T}_r e^{-(s+r)}(e^{2(s\wedge r)}-1)}\right].
\end{aligned}
$$

The computation of $\mathbb{E}_{\mathrm{P}}^{x}\left[(e^{-tL}\mathbb{1})(\alpha, X_\beta)\right]$ is more complicated than that of $(e^{-tL}\mathbb{1})(\alpha, X_0)$. We have

$$
\mathbb{E}_{\mathrm{P}}^{x}\left[(e^{-tL}\mathbb{1})(\alpha, X_\beta)\right] = e^{\triangle t}\mathbb{E}_{\mathrm{P}}^{x}\left[\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[e^{-\sqrt{2}g\int_0^t \hat{T}_s e^{-s}\mathrm{d}s X_\beta} e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \hat{T}_s\hat{T}_r e^{-(s+r)}(e^{2(s\wedge r)}-1)}\right]\right].
$$

Inserting (2.6) to X_β above again, we obtain that

$$
\begin{aligned}
&= e^{\triangle t}\mathbb{E}_{\mathcal{W}}^{0}\left[\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[e^{-\sqrt{2}g\int_0^t \hat{T}_s e^{-s}\mathrm{d}s e^{-\beta}\left(x+\frac{1}{\sqrt{2}}B_{e^{2\beta}-1}\right)} e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \hat{T}_s\hat{T}_r e^{-(s+r)}(e^{2(s\wedge r)}-1)}\right]\right] \\
&= e^{\triangle t}\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[e^{-\left(\sqrt{2}g\int_0^t \hat{T}_s e^{-s}\mathrm{d}s\right)e^{-\beta}x} e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \hat{T}_s\hat{T}_r e^{-(s+r)}(1-e^{-2\beta})} e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \hat{T}_s\hat{T}_r e^{-(s+r)}(e^{2(s\wedge r)}-1)}\right] \\
&= e^{\triangle t}\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[e^{-\left(\sqrt{2}g\int_0^t \hat{T}_{s-t} e^{-s}\mathrm{d}s\right)e^{-\beta}x} e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \hat{T}_{s-t}\hat{T}_{r-t} e^{-(s+r)}(1-e^{-2\beta})} e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \hat{T}_{s-t}\hat{T}_{r-t} e^{-(s+r)}(e^{2(s\wedge r)}-1)}\right].
\end{aligned}
$$

In the last line above we shift $\hat{T}_s$ by t. Since $\hat{T}_u$ for $0 \le u \le t$ and $\hat{T}_{s-t}$ for $0 \le s \le t$ are independent, combining above computations, we have

$$
\begin{aligned}
&(e^{-tL}\mathbb{1}, e^{-\beta b^\dagger b}e^{-tL}\mathbb{1}) \\
&= \sum_{\alpha\in\mathbb{Z}_2} e^{2\triangle t}\int_{\mathbb{R}} \frac{e^{-x^2}}{\sqrt{\pi}}\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[e^{-\left(\sqrt{2}g\int_0^t \hat{T}_s e^{-s}\mathrm{d}s\right)x} e^{-\left(\sqrt{2}g\int_0^t \hat{T}_{s-t} e^{-s}\mathrm{d}s\right)e^{-\beta}x} e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \hat{T}_s\hat{T}_r e^{-(s+r)}(e^{2(s\wedge r)}-1)}\right. \\
&\qquad \left.\times e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \hat{T}_{s-t}\hat{T}_{r-t} e^{-(s+r)}(1-e^{-2\beta})} e^{\frac{g^2}{2}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}r \hat{T}_{s-t}\hat{T}_{r-t} e^{-(s+r)}(e^{2(s\wedge r)}-1)}\right]\mathrm{d}x. \qquad (3.9)
\end{aligned}
$$

Terms dependent on x on the exponent above can be computed as

$$
\begin{aligned}
&-x^2 - \sqrt{2}g\left(e^{-\beta}\int_0^t \hat{T}_{s-t}e^{-s}\mathrm{d}s + \int_0^t \hat{T}_s e^{-s}\mathrm{d}s\right)x \\
&= -\left(x + \frac{g}{\sqrt{2}}\int_0^t \hat{T}_s e^{-s}\mathrm{d}s + \frac{g}{\sqrt{2}}e^{-\beta}\int_0^t \hat{T}_{s-t}e^{-s}\mathrm{d}s\right)^2 + \frac{g^2}{2}\left(\int_0^t \hat{T}_s e^{-s}\mathrm{d}s + e^{-\beta}\int_0^t \hat{T}_{s-t}e^{-s}\mathrm{d}s\right)^2.
\end{aligned}
$$

The first term on the right-hand side can be integrated with respect to $\mathrm{d}x$ as

$$
\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\left(x+\frac{g}{\sqrt{2}}\int_0^t \hat{T}_s e^{-s}\mathrm{d}s+\frac{g}{\sqrt{2}}e^{-\beta}\int_0^t \hat{T}_{s-t}e^{-s}\mathrm{d}s\right)^2}\mathrm{d}x = 1.
$$

The second term on the right-hand side can be computed as

$$\left(\int_0^t \hat{T}_s e^{-s}\mathrm{d}s + e^{-\beta}\int_0^t \hat{T}_{s-t}e^{-s}\mathrm{d}s\right)^2$$
$$= \int_0^t\int_0^t \hat{T}_s\hat{T}_r e^{-(s+r)}\mathrm{d}s\mathrm{d}r + 2e^{-\beta}\int_0^t\int_0^t \hat{T}_{s-t}\hat{T}_r e^{-(s+r)}\mathrm{d}s\mathrm{d}r + e^{-2\beta}\int_0^t\int_0^t \hat{T}_{s-t}\hat{T}_{r-t}e^{-(s+r)}\mathrm{d}s\mathrm{d}r. \tag{3.10}$$

Terms independent of x on (3.9) are

$$\int_0^t \mathrm{d}s\int_0^t \hat{T}_s\hat{T}_r e^{-(s+r)}(e^{2(s\wedge r)}-1)\mathrm{d}r + \int_0^t \mathrm{d}s\int_0^t \hat{T}_{s-t}\hat{T}_{r-t}e^{-(s+r)}(1-e^{-2\beta})\mathrm{d}r$$
$$+\int_0^t \mathrm{d}s\int_0^t \hat{T}_{s-t}\hat{T}_{r-t}e^{-(s+r)}(e^{2(s\wedge r)}-1)\mathrm{d}r. \tag{3.11}$$

Then the sum of (3.10) and (3.11) is

$$(3.10)+(3.11)$$
$$= \int_0^t \mathrm{d}s\int_0^t \hat{T}_{s-t}\hat{T}_{r-t}e^{-|s-r|}\mathrm{d}r + \int_0^t \mathrm{d}s\int_0^t \hat{T}_s\hat{T}_r e^{-|s-r|}\mathrm{d}r + 2e^{-\beta}\int_0^t\int_0^t \hat{T}_{s-t}\hat{T}_r e^{-(s+r)}\mathrm{d}s\mathrm{d}r$$
$$= \int_{-t}^0 \mathrm{d}s\int_{-t}^0 \hat{T}_s\hat{T}_r e^{-|s-r|}\mathrm{d}r + \int_0^t \mathrm{d}s\int_0^t \hat{T}_s\hat{T}_r e^{-|s-r|}\mathrm{d}r + 2e^{-\beta}\int_{-t}^0\int_0^t \hat{T}_s\hat{T}_r e^{-|s-r|}\mathrm{d}s\mathrm{d}r.$$

By the trick $\int_{-t}^t\int_{-t}^t = \int_{-t}^0\int_{-t}^0 + \int_0^t\int_0^t + 2\int_{-t}^0\int_0^t$, we see that

$$(3.10)+(3.11) = \int_{-t}^t \mathrm{d}s\int_{-t}^t \hat{T}_s\hat{T}_r e^{-|s-r|}\mathrm{d}r - 2(1-e^{-\beta})\int_{-t}^0\int_0^t \hat{T}_s\hat{T}_r e^{-|s-r|}\mathrm{d}s\mathrm{d}r.$$

Then the lemma follows. ■

Define the probability measure Π_T on $(\mathcal{D},\mathcal{B}_{\mathcal{D}})$ by

$$\Pi_T(A) = \frac{1}{Z_T}\frac{1}{2}e^{2T\triangle}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[\mathbb{1}_A e^{\frac{g^2}{2}\int_{-T}^T\mathrm{d}t\int_{-T}^T\mathrm{d}sW_\triangle(t,s)}\right],\quad A\in\mathcal{B}_{\mathcal{D}}, \tag{3.12}$$

where $Z_T = \frac{1}{2}e^{2T\triangle}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}_{\mathrm{Q}}^{\alpha}\left[e^{\frac{g^2}{2}\int_{-T}^T\mathrm{d}t\int_{-T}^T\mathrm{d}sW_\triangle(t,s)}\right]$ is the normalizing constant. Note that pair interaction $W_\triangle(t,s)$ is independent of σ and hence one can replace $\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}_{\mathrm{Q}}^{\alpha}$ with $2\mathbb{E}_{\mathrm{Q}}^{\alpha}$ in (3.12). We also notice that $1 = \|\Phi_{\mathrm{g}}\|_{\mathcal{H}}^2 = \sum_{\alpha\in\mathbb{Z}_2}\int_{\mathbb{R}}|\Phi_{\mathrm{g}}(\alpha,x)|^2\mathrm{d}\mu(x)$, $2 = \|\mathbb{1}\|_{\mathcal{H}}^2 = \sum_{\alpha\in\mathbb{Z}_2}\int_{\mathbb{R}}\mathrm{d}\mu(x)$ and $2Z_T = \|e^{-TL}\mathbb{1}\|^2$.

Let $A_j\in\mathcal{B}(\mathbb{R})$ for $j=0,1,\ldots,n$ and $\Lambda=\{t_0,t_1,\ldots,t_n\}\subset[-T,T]$. The cylinder set is defined by

$$C_T^\Lambda(A_0\times\cdots\times A_n) = \{\omega\in\mathcal{D}_T \mid \omega(t_j)\in A_j, j=0,1,\ldots,n\}.$$

Recall that the family of cylinder sets is denoted by $\mathcal{A}_T$. We also note that $\boldsymbol{\sigma}(\mathcal{A}_T)=\mathcal{B}_T$. Let

$$m_t(A) = e^{2Et}e^{2\triangle t}\sum_{\alpha\in\mathbb{Z}_2}\int_{\mathbb{R}}\mathbb{E}_{\mathrm{Q}}^{\alpha}\mathbb{E}_{\mathrm{P}}^{x}\left[\mathbb{1}_A\Phi_{\mathrm{g}}(\hat{q}_{-t}^{\triangle})\Phi_{\mathrm{g}}(\hat{q}_t^{\triangle})e^{-g\int_{-t}^t W(\hat{q}_s^{\triangle})\mathrm{d}s}\right]\mathrm{d}\mu(x). \tag{3.13}$$

Since $\overset{\circ}{\mathcal{B}}$ is a finitely additive family of sets, we define the finitely additive set function ν on $(\mathcal{D}, \overset{\circ}{\mathcal{B}})$ by $\nu(A) = m_t(A)$ for $A \in \pi_t^{-1}(\mathcal{B}_t)$.

Lemma 3.8 *ν is well defined, i.e., $m_t(A) = m_s(A)$ for $A \in \pi_t^{-1}(\mathcal{B}_t) \subset \pi_s^{-1}(\mathcal{B}_s)$.*

Proof: Notice that $m_t \circ \pi_t^{-1}$ and $m_s \circ \pi_t^{-1}$ are probability measures on $(\mathcal{D}_t, \mathcal{B}_t)$. We compute finite dimensional distributions of $m_t \circ \pi_t^{-1}$ and $m_s \circ \pi_t^{-1}$. Let $\Lambda = \{t_0, t_1, \ldots, t_n\} \subset [-t, t] \subset [-s, s]$. Since $e^{-r\bar{L}}\Phi_{\rm g} = \Phi_{\rm g}$ for any $r \geq 0$, we have by (4.4),

$$
\begin{aligned}
& m_t \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \cdots \times A_n)) \\
&= e^{2Et} e^{2\triangle t} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\rm Q}^\alpha \mathbb{E}_{\rm P}^x \left[\left(\prod_{j=0}^n 1\!\!1_{A_j}(\hat{T}_{t_j}) \right) \Phi_{\rm g}(\hat{q}_{-t}^\triangle) \Phi_{\rm g}(\hat{q}_t^\triangle) e^{-g\int_{-t}^t W(\hat{q}_s^\triangle) {\rm d}s} \right] {\rm d}\mu(x) \\
&= (e^{-(t_0+t)\bar{L}}\Phi_{\rm g}, 1\!\!1_{A_0} e^{-(t_1-t_0)\bar{L}} 1\!\!1_{A_1} \cdots e^{-(t_n-t_{n-1})\bar{L}} 1\!\!1_{A_n} e^{-(t-t_n)\bar{L}} \Phi_{\rm g}) \\
&= (\Phi_{\rm g}, 1\!\!1_{A_0} e^{-(t_1-t_0)\bar{L}} 1\!\!1_{A_1} \cdots e^{-(t_n-t_{n-1})\bar{L}} 1\!\!1_{A_n} \Phi_{\rm g}) \\
&= (e^{-(t_0+s)\bar{L}}\Phi_{\rm g}, 1\!\!1_{A_0} e^{-(t_1-t_0)\bar{L}} 1\!\!1_{A_1} \cdots e^{-(t_n-t_{n-1})\bar{L}} 1\!\!1_{A_n} e^{-(s-t_n)\bar{L}} \Phi_{\rm g}) \\
&= m_s \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \cdots \times A_n)).
\end{aligned}
$$

It is straightforward to see that the Kolmogorov consistency condition also holds true:

$$
m_t \circ \pi_t^{-1} \left(C_t^{\{\Lambda, s_1, \ldots, s_m\}} \left(A_0 \times \cdots \times A_n \times \prod^m \mathbb{R} \right) \right) = m_t \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \cdots \times A_n)).
$$

Let $\pi_\Lambda : [-t, t]^{\mathbb{R}} \to \mathbb{R}^\Lambda$ be the projection such that for $\omega \in [-t, t]^{\mathbb{R}}$, $\pi_\Lambda \omega = (\omega(t_0), \ldots, \omega(t_n))$. Thus by the Kolmogorov extension theorem there exists a unique probability measure $\bar{m}_t$ on $([-t, t]^{\mathbb{R}}, \boldsymbol{\sigma}(\mathcal{A}_t))$ such that

$$
\bar{m}_t(\pi_\Lambda^{-1}(A_0 \times \cdots \times A_n)) = m_t \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \cdots \times A_n)) \tag{3.14}
$$

for all $\Lambda \subset [-t, t]$ with $\#\Lambda < \infty$ and $A_j \in \mathcal{B}(\mathbb{R})$. Since the extension is unique, $m_t \circ \pi_t^{-1} = \bar{m}_t$. Similarly there exists a unique probability measure $\bar{m}_s$ on $([-t, t]^{\mathbb{R}}, \boldsymbol{\sigma}(\mathcal{A}_t))$ such that $m_s \circ \pi_t^{-1} = \bar{m}_s$. Then $m_s \circ \pi_t^{-1} = m_t \circ \pi_t^{-1}$ on $\mathcal{B}_t$, which implies the lemma. ■

The first task is to extend ν to a probability measure by the Hopf extension theorem.

Lemma 3.9 *ν can be uniquely extended to a probability measure Π_∞ on $(\mathcal{D}, \mathcal{B}_{\mathcal{D}})$.*

Proof: Suppose that $E_n \in \overset{\circ}{\mathcal{B}}$ such that $E_n \supset E_{n+1} \supset \ldots$ and $\lim_{n\to\infty} \nu(E_n) = \alpha > 0$. It is enough to show that $\bigcap_n E_n \neq \emptyset$ by the Hopf extension theorem. Let $E_n = \pi_{T_n}^{-1}(E_n')$ with $E_n' \in \mathcal{B}_{T_n}$. We can assume that $T_n < T_{n+1} <\to \infty$. Let $\mu_T = \nu \circ \pi_T^{-1}$ be a probability measure on $\mathcal{D}_T$. Since $\mathcal{D}_T$ is a Polish space, μ_T is regular, i.e., for $A \in \mathcal{B}_T$ and $\epsilon > 0$ there exist a compact set K and an open set O in $\mathcal{D}_T$ such that $K \subset A \subset O$ and $\mu_T(O \setminus K) < \epsilon$. There exists a compact set $K_n' \subset \mathcal{D}_{T_n}$ such that $\mu_{T_n}(E_n' \setminus K_n') < \alpha/2^n$. Let $K_n = \pi_{T_n}^{-1}(K_n')$, $D_n = \bigcap_{j=1}^n K_j$ and $D = \bigcap_{n=1}^\infty D_n$. Since $D \subset \bigcap_n E_n$, it is enough to show that $D \neq \emptyset$. We see that

$$\begin{aligned}\alpha-\nu(D_n) &\le \nu(E_n)-\nu(D_n)\le \nu(E_n\setminus D_n)\\ &=\nu(\cup_{j=1}^n E_n\setminus K_j)=\nu(\pi_{T_n}^{-1}\cup_{j=1}^n E_n'\setminus K_j')=\mu_{T_n}(\cup_{j=1}^n E_n'\setminus K_j')\\ &=\sum_{j=1}^n \mu_{T_n}(E_n'\setminus K_j')\le \sum_{j=1}^n \mu_{T_n}(E_j'\setminus K_j')\le \sum_{j=1}^n \alpha/2^j.\end{aligned}$$

Then $0<\nu(D_n)$ and we see that $D_n\neq\emptyset$. Let $f_n\in D_n$, i.e., $f_n\in\bigcap_{j=1}^n K_j$. Thus

$$f_n\in K_\ell \quad \text{for any} \quad n\ge\ell.$$

Let $\ell=1$. Then $\pi_{T_1}(f_n)\in K_1'$ for any $n\ge 1$. Taking a subsequence n', we see that $\lim_{n'\to\infty}\pi_{T_1}(f_{n'})=h_1\in K_1'$ exists. Let $\ell=2$. Then $\pi_{T_2}(f_{n'})\in K_2'$ for any $n'\ge 2$. Take a subsequence n'' of n' again, then $\lim_{n''\to\infty}\pi_{T_2}(f_{n''})=h_2\in K_2'$ exists. Proceeding this procedure, we can obtain a subsequence $\{m\}$ that $\lim_{m\to\infty}\pi_{T_\ell}(f_m)=h_\ell\in K_\ell'$ exists for any ℓ. Let $g_\ell=\pi_{T_\ell}^{-1}(h_\ell)\in L_\ell$. Define $g\in\mathcal{D}$ by $g(x)=g_\ell(x)$ for $x\in[-T_\ell,T_\ell]$. By the construction this is well defined, i.e., $g_\ell(x)=g_{\ell+1}(x)$ for $x\in[-T_\ell,T_\ell]$. We see that $g\in D$ and $D\neq\emptyset$. ■

For probability measures Π_T and Π_∞ on $(\mathcal{D},\mathcal{B}_\mathcal{D})$ in order to show that $\Pi_T(A)\to\Pi_\infty(A)$ for every $A\in\mathring{\mathcal{B}}$, we define the finitely additive set function ρ_T on $(\mathcal{D}_T,\mathring{\mathcal{B}}_T)$. Let $\mathbb{1}_T=e^{-T\bar{L}}\mathbb{1}$ for $t\ge 0$. Then $s\text{-}\lim_{T\to\infty}\mathbb{1}_T=\Phi_\mathrm{g}$ and $\|\mathbb{1}_T\|^2=2e^{2TE}Z_T$. The finitely additive set function ρ_T on $(\mathcal{D}_T,\mathring{\mathcal{B}}_T)$ is defined by

$$\rho_T(A)=e^{2Et}e^{2t\triangle}\frac{1}{\|\mathbb{1}_T\|^2}\sum_{\alpha\in\mathbb{Z}_2}\int_\mathbb{R}\mathbb{E}_\mathrm{Q}^\alpha\mathbb{E}_\mathrm{P}^x\left[\mathbb{1}_A\mathbb{1}_{T-t}(\hat{q}_{-t}^\triangle)\mathbb{1}_{T-t}(\hat{q}_t^\triangle)e^{-g\int_{-t}^t W(\hat{q}_s^\triangle)\mathrm{d}s}\right]\mathrm{d}\mu(x) \tag{3.15}$$

for $A\in\pi_t^{-1}(\mathcal{B}_t)$ but $t\le T$. The right-hand side of (3.15) is denoted by $M_{T,t}(A)$.

Lemma 3.10 *ρ_T is well defined, i.e, $M_{T,t}(A)=M_{T,s}(A)$ for $A\in\pi_t^{-1}(\mathcal{B}_t)\subset\pi_s^{-1}(\mathcal{B}_r)$.*

Proof: This is shown in a similar manner to Lemma 3.8. Let

$$M_{T,t}(A)=e^{2Et}e^{2t\triangle}\sum_{\alpha\in\mathbb{Z}_2}\int_\mathbb{R}\mathbb{E}_\mathrm{Q}^\alpha\mathbb{E}_\mathrm{P}^x\left[\mathbb{1}_A\mathbb{1}_{T-t}(\hat{q}_{-t}^\triangle)\mathbb{1}_{T-t}(\hat{q}_t^\triangle)e^{-g\int_{-t}^t W(\hat{q}_s^\triangle)\mathrm{d}s}\right]\mathrm{d}\mu(x).$$

Then $M_{T,t}\circ\pi_t^{-1}$ and $M_{T,s}\circ\pi_t^{-1}$ are probability measures on $(\mathcal{D}_t,\mathcal{B}_t)$. Let $\Lambda=\{t_0,t_1,\ldots,t_n\}\subset[-t,t]\subset[-s,s]$. We have by (4.4),

$$\begin{aligned}&M_{T,t}\circ\pi_t^{-1}(C_t^\Lambda(A_0\times\cdots\times A_n))\\ &=e^{2Et}e^{2\triangle t}\sum_{\alpha\in\mathbb{Z}_2}\int_\mathbb{R}\mathbb{E}_\mathrm{Q}^\alpha\mathbb{E}_\mathrm{P}^x\left[\left(\prod_{j=0}^n\mathbb{1}_{A_j}(\hat{T}_{t_j})\right)\mathbb{1}_{T-t}(\hat{q}_{-t}^\triangle)\mathbb{1}_{T-t}(\hat{q}_t^\triangle)e^{-g\int_{-t}^t W(\hat{q}_r^\triangle)\mathrm{d}r}\right]\mathrm{d}\mu(x)\\ &=(e^{-(t_0+t)\bar{L}}\mathbb{1}_{T-t},\mathbb{1}_{A_0}e^{-(t_1-t_0)\bar{L}}\mathbb{1}_{A_1}\cdots e^{-(t_n-t_{n-1})\bar{L}}\mathbb{1}_{A_n}e^{-(t-t_n)\bar{L}}\mathbb{1}_{T-t})\\ &=(e^{-(t_0+s)\bar{L}}\mathbb{1}_{T-s},\mathbb{1}_{A_0}e^{-(t_1-t_0)\bar{L}}\mathbb{1}_{A_1}\cdots e^{-(t_n-t_{n-1})\bar{L}}\mathbb{1}_{A_n}e^{-(s-t_n)\bar{L}}\mathbb{1}_{T-s})\\ &=M_{T,s}\circ\pi_t^{-1}(C_t^\Lambda(A_0\times\cdots\times A_n)).\end{aligned}$$

It is straightforward to see that the Kolmogorov consistency condition also holds true:

$$M_{T,t}\circ\pi_t^{-1}\left(C_t^{\{\Lambda,s_1,\ldots,s_m\}}\left(A_0\times\cdots\times A_n\times\prod^m\mathbb{R}\right)\right)=M_{T,t}\circ\pi_t^{-1}(C_t^\Lambda(A_0\times\cdots\times A_n)).$$

Thus by the Kolmogorov extension theorem there exists a unique probability measure $\bar{M}_{T,t}$ on $([-t,t]^{\mathbb{R}},\boldsymbol{\sigma}(\mathcal{A}_t))$ such that

$$\bar{M}_{T,t}(\pi_\Lambda^{-1}(A_0\times\cdots\times A_n))=M_{T,t}\circ\pi_t^{-1}(C_t^\Lambda(A_0\times\cdots\times A_n)) \tag{3.16}$$

for all $\Lambda\subset[-T,T]$ with $\#\Lambda<\infty$ and $A_j\in\mathcal{B}(\mathbb{R})$. Since the extension is unique, $M_{T,t}\circ\pi_t^{-1}=\bar{M}_{T,t}$. Similarly there exists a unique probability measure $\bar{M}_{T,s}$ on $([-t,t]^{\mathbb{R}},\boldsymbol{\sigma}(\mathcal{A}_t))$ such that $M_{T,s}\circ\pi_t^{-1}=\bar{M}_{T,s}$. Then $M_{T,s}\circ\pi_t^{-1}=M_{T,t}\circ\pi_t^{-1}$ on $\mathcal{B}_t$, which implies the lemma. ■

We shall show that $\rho_T=\Pi_T$ on $\mathring{\mathcal{B}}_T$ for any $T>0$.

Lemma 3.11 *We have $\rho_T=\Pi_T$ on $\mathring{\mathcal{B}}_T$.*

Proof: Let $t\le T$. It is enough to show that $\Pi_T(A)=\rho_T(A)$ for $A\in\pi_t^{-1}(\mathcal{B}_t)$. Let $\Lambda=\{t_0,t_1,...,t_n\}\subset[-t,t]\subset[-T,T]$ and $A_0\times\cdots\times A_n\in\mathcal{B}(\mathbb{R}^\Lambda)$. We have

$$\Pi_T\circ\pi_t^{-1}(C_t^\Lambda(A_0\times\cdots\times A_n))=\frac{1}{Z_T}e^{2T\triangle}\frac{1}{2}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}_{\mathrm{Q}}^\alpha\left[\left(\prod_{j=0}^n\mathbb{1}_{A_j}(\hat{T}_{t_j})\right)e^{\frac{g^2}{2}\int_{-T}^T\mathrm{d}t\int_{-T}^T\mathrm{d}sW_\triangle(t,s)}\right], \tag{3.17}$$

$$\begin{aligned}&\rho_T\circ\pi_t^{-1}(C_t^\Lambda(A_0\times\cdots\times A_n))\\&=e^{2Et}e^{2\triangle t}\frac{1}{\|\mathbb{1}_T\|^2}\sum_{\alpha\in\mathbb{Z}_2}\int_{\mathbb{R}}\mathbb{E}_{\mathrm{Q}}^\alpha\mathbb{E}_{\bar{\mathrm{P}}}^x\left[\left(\prod_{j=0}^n\mathbb{1}_{A_j}(\hat{T}_{t_j})\right)\mathbb{1}_{T-t}(\hat{q}_{-t}^\triangle)\mathbb{1}_{T-t}(\hat{q}_t^\triangle)e^{-g\int_{-t}^tW(\hat{q}_s^\triangle)\mathrm{d}s}\right]\mathrm{d}\mu(x).\end{aligned} \tag{3.18}$$

By (4.4) we see that

$$\begin{aligned}(3.17)&=\frac{1}{\|\mathbb{1}_T\|^2}(\mathbb{1},e^{-(t_0+T)L}\mathbb{1}_{A_0}e^{-(t_1-t_0)L}\mathbb{1}_{A_1}\cdots\mathbb{1}_{A_n}e^{-(T-t_n)L}\mathbb{1})\\&=\frac{e^{2Et}}{\|\mathbb{1}_T\|^2}(\mathbb{1}_{T-t},e^{-(t_0+t)L}\mathbb{1}_{A_0}e^{-(t_1-t_0)L}\mathbb{1}_{A_1}\cdots\mathbb{1}_{A_n}e^{-(t-t_n)L}\mathbb{1}_{T-t})=(3.18).\end{aligned}$$

Then we have

$$\Pi_T\circ\pi_t^{-1}(C_t^\Lambda(A_0\times\cdots\times A_n))=\rho_T\circ\pi_t^{-1}(C_t^\Lambda(A_0\times\cdots\times A_n)). \tag{3.19}$$

Since both sides of (3.19) satisfy the Kolmogorov consistency condition, there exists a unique probability measure μ on $(\mathcal{D}_T,\mathcal{B}_t)$ such that

$$\mu(\pi_\Lambda^{-1}(A_0\times\cdots\times A_n))=\Pi_T\circ\pi_t^{-1}(C_t^\Lambda(A_0\times\cdots\times A_n))=\rho_T\circ\pi_t^{-1}(C_t^\Lambda(A_0\times\cdots\times A_n)).$$

$\Pi_T\circ\pi_t^{-1}$ and $\rho_T\circ\pi_t^{-1}$ are probability measures on $(\mathcal{D}_t,\mathcal{B}_t)$, and $\Pi_T\circ\pi_t^{-1}(C_t^\Lambda(A_0\times\cdots\times A_n)) = \Pi_T\circ\pi_t^{-1}(\pi_\Lambda^{-1}(A_0\times\cdots\times A_n)) = \rho_T\circ\pi_t^{-1}(C_t^\Lambda(A_0\times\cdots\times A_n)) = \rho_T\circ\pi_t^{-1}(\pi_\Lambda^{-1}(A_0\times\cdots\times A_n))$. Since the extension is unique, $\Pi_T\circ\pi_t^{-1}=\mu=\rho_T\circ\pi_t^{-1}$ on $(\mathcal{D}_t,\mathcal{B}_t)$ follows. ■

The following proposition is shown for spin boson model in [5, Theorem 3.8] and for relativistic Pauli-Fierz model in [6, Lemma 7.6], and the proof for the quantum Rabi Hamiltonian is a minor modification of [5, 6].

Proposition 3.12 *There exists a probability measure* Π_∞ *on* $(\mathcal{D},\mathcal{B}_\mathcal{D})$ *such that*

$$\lim_{T\to\infty}\Pi_T(A)=\Pi_\infty(A)\quad A\in\overset{\circ}{\mathcal{B}}.$$

Proof: By $s\text{-}\lim_{T\to\infty}\mathbb{1}_T=\Phi_\mathrm{g}$ we obtain that $s\text{-}\lim_{T\to\infty}\mathbb{1}_{T-t}=\Phi_\mathrm{g}$ and $\lim_{T\to\infty}\|\mathbb{1}_T\|=1$. Then for each $\alpha\in\mathbb{Z}_2$, $(\mathbb{1}_{T-t}/\|\mathbb{1}_T\|)(\cdot,\sigma)\to\varphi_\mathrm{g}(\cdot,\sigma)$ as $T\to\infty$ in $L^2(\mathbb{R},\mathrm{d}\mu)$. Let $\Phi_\mathrm{g}^T=\frac{\mathbb{1}_{T-t}}{\|\mathbb{1}_T\|}$. Note that $\Phi_\mathrm{g},\Phi_\mathrm{g}^T\in L^\infty(\mathbb{Z}_2\times\mathbb{R})$. Let $A\in\pi_t^{-1}(\mathcal{B}_t)$. Then $\Pi_T(A)=\rho_T(A)$ by Lemma 3.11 and $\nu(A)=\Pi_\infty(A)$ by Lemma 3.8. We have

$$\begin{aligned}&\Pi_T(A)-\Pi_\infty(A)=\rho_T(A)-\nu(A)\\&=e^{2Et}e^{2\Delta t}\sum_{\alpha\in\mathbb{Z}_2}\mathbb{E}_\mathrm{Q}^\alpha\left[\mathbb{1}_A\int_\mathbb{R}\mathbb{E}_{\bar{\mathrm{P}}}^x\left[\left(\Phi_\mathrm{g}(\hat q_{-t}^\triangle)\Phi_\mathrm{g}(\hat q_t^\triangle)-\Phi_\mathrm{g}^T(\hat q_{-t}^\triangle)\Phi_\mathrm{g}^T(\hat q_t^\triangle)\right)e^{-g\int_{-t}^tW(\hat q_s^\triangle)\mathrm{d}s}\right]\mathrm{d}\mu(x)\right].\end{aligned}$$

Then

$$\begin{aligned}&\int_\mathbb{R}\mathbb{E}_{\bar{\mathrm{P}}}^x\left[\left|\Phi_\mathrm{g}(\hat q_{-t}^\triangle)\Phi_\mathrm{g}(\hat q_t^\triangle)-\Phi_\mathrm{g}^T(\hat q_{-t}^\triangle)\Phi_\mathrm{g}^T(\hat q_t^\triangle)\right|e^{-g\int_{-t}^tW(\hat q_s^\triangle)\mathrm{d}s}\right]\mathrm{d}\mu(x)\\&\le\int_\mathbb{R}\mathbb{E}_{\bar{\mathrm{P}}}^x\left[\left|\Phi_\mathrm{g}(\hat q_{-t}^\triangle)-\Phi_\mathrm{g}^T(\hat q_{-t}^\triangle)|\Phi_\mathrm{g}(\hat q_t^\triangle)\right|e^{-g\int_{-t}^tW(\hat q_s^\triangle)\mathrm{d}s}\right]\mathrm{d}\mu(x)\\&+\int_\mathbb{R}\mathbb{E}_{\bar{\mathrm{P}}}^x\left[|\Phi_\mathrm{g}^T(\hat q_{-t}^\triangle)|\left|\Phi_\mathrm{g}(\hat q_t^\triangle)-\Phi_\mathrm{g}^T(\hat q_t^\triangle)\right|e^{-g\int_{-t}^tW(\hat q_s^\triangle)\mathrm{d}s}\right]\mathrm{d}\mu(x).\end{aligned}$$

We estimate $\int_\mathbb{R}\mathbb{E}_{\bar{\mathrm{P}}}^x\left[\left|\Phi_\mathrm{g}(\hat q_{-t}^\triangle)-\Phi_\mathrm{g}^T(\hat q_{-t}^\triangle)|\Phi_\mathrm{g}(\hat q_t^\triangle)\right|e^{-g\int_{-t}^tW(\hat q_s^\triangle)\mathrm{d}s}\right]\mathrm{d}\mu(x)$. By the shift invariance we have

$$\begin{aligned}&\int_\mathbb{R}\mathbb{E}_{\bar{\mathrm{P}}}^x\left[\left|(\Phi_\mathrm{g}(\hat q_{-t}^\triangle)-\Phi_\mathrm{g}^T(\hat q_{-t}^\triangle))\Phi_\mathrm{g}(\hat q_t^\triangle)\right|e^{-g\int_{-t}^tW(\hat q_s^\triangle)\mathrm{d}s}\right]\mathrm{d}\mu(x)\\&=\int_\mathbb{R}|\Phi_\mathrm{g}(\hat q_0^\triangle)-\Phi_\mathrm{g}^T(\hat q_0^\triangle)|\mathbb{E}_{\bar{\mathrm{P}}}^x\left[\left|\Phi_\mathrm{g}(\hat q_{2t}^\triangle)\right|e^{-g\int_0^{2t}W(\hat q_s^\triangle)\mathrm{d}s}\right]\mathrm{d}\mu(x).\end{aligned}$$

By the Schwarz inequality we also have

$$\le\left(\int_\mathbb{R}|\Phi_\mathrm{g}(\hat q_0^\triangle)-\Phi_\mathrm{g}^T(\hat q_0^\triangle)|^2\mathrm{d}\mu(x)\right)^{1/2}\left(\int_\mathbb{R}\mathbb{E}_{\bar{\mathrm{P}}}^x\left[\left|\Phi_\mathrm{g}(\hat q_{2t}^\triangle)\right|^2\right]\mathrm{d}\mu(x)\right)^{1/2}\left(\mathbb{E}_{\bar{\mathrm{P}}}^x\left[e^{-2g\int_0^{2t}W(\hat q_s^\triangle)\mathrm{d}s}\right]\right)^{1/2}.$$

Since by Lemma 3.4,

$$\mathbb{E}_{\bar{\mathrm{P}}}^x\left[e^{-2g\int_0^{2t}W(\hat q_s^\triangle)\mathrm{d}s}\right]\le e^{|g|(1-e^{-2t})|x|}e^{g^2\int_0^{(1-e^{-4t})/2}|2t-y|^2\mathrm{d}y},$$

we have

$$\int_{\mathbb{R}} \mathbb{E}_{\mathrm{P}}^x \left[\left| (\Phi_{\mathrm{g}}(\hat{q}_{-t}^{\triangle}) - \Phi_{\mathrm{g}}^T(\hat{q}_{-t}^{\triangle}))\Phi_{\mathrm{g}}(\hat{q}_t^{\triangle}) \right| e^{-g\int_{-t}^t W(\hat{q}_s^{\triangle})\mathrm{d}s} \right] \mathrm{d}\mu(x)$$
$$\leq C \left(\int_{\mathbb{R}} |\Phi_{\mathrm{g}}(\hat{q}_0^{\triangle}) - \Phi_{\mathrm{g}}^T(\hat{q}_0^{\triangle})|^2 \mathrm{d}\mu(x) \right)^{1/2} \left(\int_{\mathbb{R}} e^{|g|(1-e^{-2t})|x|} \mathrm{d}\mu(x) \right)^{1/2}.$$

Here we employed that $\Phi_{\mathrm{g}} \in L^\infty(\mathbb{Z}_2 \times \mathbb{R})$. Since $\int_{\mathbb{R}} |\Phi_{\mathrm{g}}(\hat{q}_0^{\triangle}) - \Phi_{\mathrm{g}}^T(\hat{q}_0^{\triangle})|^2 \mathrm{d}\mu(x) \to 0$ as $T \to \infty$,

$$\int_{\mathbb{R}} \mathbb{E}_{\mathrm{P}}^x \left[\left| \Phi_{\mathrm{g}}(\hat{q}_{-t}^{\triangle}) - \Phi_{\mathrm{g}}^T(\hat{q}_{-t}^{\triangle}) |\Phi_{\mathrm{g}}(\hat{q}_t^{\triangle}) \right| e^{-g\int_{-t}^t W(\hat{q}_s^{\triangle})\mathrm{d}s} \right] \mathrm{d}\mu(x) \to 0$$

as $T \to \infty$. Similarly we can also show that

$$\int_{\mathbb{R}} \mathbb{E}_{\mathrm{P}}^x \left[|\Phi_{\mathrm{g}}^T(\hat{q}_{-t}^{\triangle})| \left| \Phi_{\mathrm{g}}(\hat{q}_t^{\triangle}) - \Phi_{\mathrm{g}}^T(\hat{q}_t^{\triangle}) \right| e^{-g\int_{-t}^t W(\hat{q}_s^{\triangle})\mathrm{d}s} \right] \mathrm{d}\mu(x) \to 0$$

as $T \to \infty$. Then the proof is complete. ■

The sequence of probability measures $(\Pi_T)_{T>0}$ is said to locally converge to the probability measure Π_∞ whenever $\lim_{T\to\infty} \Pi_T(A) = \Pi_\infty(A)$ for all $A \in \pi_t^{-1}(\mathcal{B}_t)$ and for all $t \geq 0$.

Corollary 3.13 *Let f be a $\mathcal{B}_t$-measurable and bounded function. Then*

$$\lim_{T\to\infty} \mathbb{E}_{\Pi_T}[f] = \mathbb{E}_{\Pi_\infty}[f].$$

Proof: It is enough to show the corollary for a nonnegative function f. Since f is bounded and $\mathcal{B}_t$-measurable, there exists a sequence $\{f_n\}$ such that $\lim_{n\to\infty} \sup_{x\in\mathcal{D}} |f_n(x) - f(x)| = 0$. Here f_n is of the form $f_n = \sum_{j=1}^{m_n} a_j \mathbb{1}_{A_j}$ with $A_j \in \mathcal{B}_t$ and $a_j > 0$. Let $\epsilon > 0$ be arbitrary. We assume that $\sup_{x\in\mathcal{D}} |f_n(x) - f(x)| \leq \epsilon$. Then we see that

$$\begin{aligned} |\mathbb{E}_{\Pi_T}[f] - \mathbb{E}_{\Pi_\infty}[f]| &\leq \mathbb{E}_{\Pi_T}[|f - f_n|] + |\mathbb{E}_{\Pi_T}[f_n] - \mathbb{E}_{\Pi_\infty}[f_n]| + \mathbb{E}_{\Pi_\infty}[|f_n - f|] \\ &\leq 2\epsilon + |\mathbb{E}_{\Pi_T}[f_n] - \mathbb{E}_{\Pi_\infty}[f_n]| \end{aligned}$$

and from Proposition 3.12 it follows that $\lim_{T\to\infty} |\mathbb{E}_{\Pi_T}[f] - \mathbb{E}_{\Pi_\infty}[f]| \leq 2\epsilon$. Then the corollary follows. ■

4 Expectations by Π_∞

In this section we give some examples of application of Π_∞. These examples are one mode versions of the spin boson model [10, 5]. Then we show only outlines of proofs.

4.1 Number operator $b^\dagger b$

Theorem 4.1 *Let $\beta \in \mathbb{C}$. Then*

$$\langle e^{\beta b^\dagger b} \rangle = \mathbb{E}_{\Pi_\infty} \left[e^{-g^2(1-e^\beta)\int_{-\infty}^0 \int_0^\infty W_\triangle(s,r)\mathrm{d}s\mathrm{d}r} \right], \tag{4.1}$$

$$\langle (b^\dagger b)^m \rangle = \sum_{l=1}^m a_l(m) g^{2l} \mathbb{E}_{\Pi_\infty} \left[\left(\int_{-\infty}^0 \int_0^\infty W_\triangle(s,r)\mathrm{d}s\mathrm{d}r \right)^l \right]. \tag{4.2}$$

Here $a_l(m) = \frac{(-1)^l}{l!}\sum_{s=1}^l (-1)^s \binom{l}{s}$ *are the Stirling numbers. In particular* $\langle (b^\dagger b)^m\rangle \le e^{2g^2}-1$ *for any* $m \ge 0$.

Simple but non trivial application is as follows. We know that $\langle \sigma_x \otimes (-\mathbb{1})^{b^\dagger b}\rangle < 0$ since the parity of $\Phi_{\rm g}$ is -1. As a corollary of Theorem 4.1 we can show that $\langle (-\mathbb{1})^{b^\dagger b}\rangle > 0$.

Corollary 4.2 *We have*

$$\langle (-\mathbb{1})^{b^\dagger b}\rangle = \mathbb{E}_{\Pi_\infty}\left[e^{-2g^2\int_{-\infty}^0\int_0^\infty W_\triangle(s,r){\rm d}s{\rm d}r}\right] > 0.$$

Proof: Put $\beta = i\pi$ in Theorem 4.1. Then the corollary follows. ■

4.2 Gaussian functions

We construct a path integral representation of $\langle e^{i\beta x}\rangle$.

Theorem 4.3 *We have*

$$\langle e^{i\beta x}\rangle = e^{-\beta^2/4}\mathbb{E}_{\Pi_\infty}\left[e^{i\beta K}\right],$$

where

$$K = -\frac{g}{\sqrt{2}}\int_{-\infty}^{\infty}\hat{T}_s e^{-|s|}{\rm d}s.$$

Corollary 4.4 *Let* $\beta\in\mathbb{C}$ *such that* $|\beta|<1$. *Then*

$$\langle e^{\beta x^2}\rangle = \frac{1}{\sqrt{1-\beta}}\mathbb{E}_{\Pi_\infty}\left[e^{\frac{\beta K^2}{1-\beta}}\right]. \tag{4.3}$$

In particular $\lim_{\beta\uparrow 1}\|e^{\beta x^2/2}\Phi_{\rm g}\|^2 = \infty$.

Proof: By Theorem 4.2 we see that

$$\begin{aligned}\langle e^{-\beta^2x^2/2}\rangle &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\Phi_{\rm g}, e^{ik\beta x}\Phi_{\rm g})e^{-k^2/2}{\rm d}k = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-k^2\beta^2/4}\mathbb{E}_{\Pi_\infty}\left[e^{ik\beta K}\right]e^{-k^2/2}{\rm d}k\\ &= \mathbb{E}_{\Pi_\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-k^2\beta^2/4}e^{ik\beta K}e^{-k^2/2}{\rm d}k\right] = \frac{1}{\sqrt{1+\beta^2/2}}\mathbb{E}_{\Pi_\infty}\left[e^{-\frac{\beta^2K^2}{\beta^2+2}}\right].\end{aligned}$$

By an analytic continuation we obtain (4.3) for $\beta\in\mathbb{C}$ such that $|\beta|<1$. Then the corollary follows. ■

4.3 Spin σ_z

Let $\bar{L} = L - E$. Path integral representations of Euclidean Green functions by Lemma 3.2 can be rewritten as follows.

Corollary 4.5 *(1) Suppose that $\phi, \psi \in \mathcal{H}$ and $f_j = f_j(\alpha, x) \in L^\infty(\mathbb{Z}_2 \times \mathbb{R})$ for $j = 0, 1, \ldots, n$, and $0 < t_0 < t_1 < \ldots < t_n < t$. Then*

$$\begin{aligned}&(\phi, e^{-t_0\bar{L}} f_0 e^{-(t_1-t_0)\bar{L}} f_1 e^{-(t_2-t_1)\bar{L}} \cdots e^{-(t_n-t_{n-1})\bar{L}} f_n e^{-(t-t_n)\bar{L}} \psi)\\ &= e^{\triangle t} e^{Et} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\mathrm{Q}}^\alpha \mathbb{E}_{\mathrm{P}}^x \left[\bar{\phi}(\hat{q}_0^\triangle) \psi(\hat{q}_t^\triangle) \left(\prod_{j=0}^n f_j(\hat{q}_{t_j}^\triangle) \right) e^{-g \int_0^t W(\hat{q}_s^\triangle) \mathrm{d}s} \right]. \end{aligned} \tag{4.4}$$

(2) Suppose that $g_j = g_j(\alpha) \in L^\infty(\mathbb{Z}_2)$ for $j = 0, 1, \ldots, n$ and $0 < t_0 < t_1 < \ldots < t_n < t$. Then

$$\begin{aligned}&(1\!\!1, e^{-t_0\bar{L}} g_0(\sigma_z) e^{-(t_1-t_0)\bar{L}} g_1(\sigma_z) e^{-(t_2-t_1)\bar{L}} \cdots e^{-(t_n-t_{n-1})\bar{L}} g_n(\sigma_z) e^{-(t-t_n)\bar{L}} 1\!\!1)\\ &= e^{\triangle t} e^{Et} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\mathrm{Q}}^\alpha \left[\left(\prod_{j=0}^n g_j(\hat{T}_{t_j}) \right) \int_{\mathbb{R}} \mathbb{E}_{\mathrm{P}}^x \left[e^{-g \int_0^t W(\hat{q}_s^\triangle) \mathrm{d}s} \right] \mathrm{d}\mu(x) \right]. \end{aligned} \tag{4.5}$$

Proof: (1) is a simple reworking of Lemma 3.2 and (2) is a special case of (1). ■

One can see that the integrand in (4.5) is

$$\mathbb{E}_{\mathrm{P}}^x \left[e^{-g \int_0^t W(\hat{q}_s^\triangle) \mathrm{d}s} \right] = e^{-g\left(\int_0^t e^{-s} (-1)^{N_{\triangle s}} \mathrm{d}s\right) x} e^{\frac{g^2}{4} \int_0^{(1-e^{-2t})/2} \left| \int_y^t (-1)^{N_{\triangle s}} \mathrm{d}s \right|^2 \mathrm{d}y}.$$

by Lemma 3.4.

Theorem 4.6 *We have $\langle \sigma_z e^{-|t-s|\bar{L}} \sigma_z \rangle = \mathbb{E}_{\Pi_\infty}[\hat{T}_t \hat{T}_s]$ for any $t, s \in \mathbb{R}$.*

Proof: By Lemma 4.5 and a limiting argument, we see that

$$\begin{aligned}(\sigma_z \Phi_{\mathrm{g}}, e^{-t\bar{L}} \sigma_z \Phi_{\mathrm{g}}) &= \lim_{T \to \infty} \frac{1}{\|1\!\!1_{T-t/2}\|^2} (\sigma_z 1\!\!1_{T-t/2}, e^{-t\bar{L}} \sigma_z 1\!\!1_{T-t/2})\\ &= \lim_{T \to \infty} \frac{e^{2ET} e^{2T\triangle}}{\|1\!\!1_{T-t/2}\|^2} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\mathrm{Q}}^\alpha \left[\hat{T}_{-t/2} \hat{T}_{t/2} e^{\frac{g^2}{2} \int_{-T}^T \mathrm{d}t \int_{-T}^T \mathrm{d}s W_\triangle(t,s)} \right].\end{aligned}$$

Then we have

$$\begin{aligned}(\sigma_z \Phi_{\mathrm{g}}, e^{-t\bar{L}} \sigma_z \Phi_{\mathrm{g}}) &= \lim_{T \to \infty} \frac{\|1\!\!1_T\|^2}{\|1\!\!1_{T-t/2}\|^2} \frac{e^{2ET} e^{2T\triangle}}{\|1\!\!1_T\|^2} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\mathrm{Q}}^\alpha \left[\hat{T}_{-t/2} \hat{T}_{t/2} e^{\frac{g^2}{2} \int_{-T}^T \mathrm{d}t \int_{-T}^T \mathrm{d}s W_\triangle(t,s)} \right]\\ &= \lim_{T \to \infty} \frac{\|1\!\!1_T\|^2}{\|1\!\!1_{T-t/2}\|^2} \frac{\mathbb{E}_{\mathrm{Q}}^\alpha \left[\hat{T}_{-t/2} \hat{T}_{t/2} e^{\frac{g^2}{2} \int_{-T}^T \mathrm{d}t \int_{-T}^T \mathrm{d}s W_\triangle(t,s)} \right]}{\mathbb{E}_{\mathrm{Q}}^\alpha \left[e^{\frac{g^2}{2} \int_{-T}^T \mathrm{d}t \int_{-T}^T \mathrm{d}s W_\triangle(t,s)} \right]} = \mathbb{E}_{\Pi_\infty}[\hat{T}_{-t/2} \hat{T}_{t/2}].\end{aligned}$$

Hence for $t > s$,

$$(\sigma_z \Phi_{\mathrm{g}}, e^{-(t-s)\bar{L}} \sigma_z \Phi_{\mathrm{g}}) = \mathbb{E}_{\Pi_\infty}[\hat{T}_{-(t-s)/2} \hat{T}_{(t-s)/2}] = \mathbb{E}_{\Pi_\infty}[\hat{T}_t \hat{T}_s]$$

by the shift invariance. ■

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