Spatial exponential decay of the ground state of the renormalized Nelson model by Feynman-Kac formula

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1 Nelson model

This is the joint work with Oliver Matte. In [11] we discuss the pointwise spatial decay of the ground state of the *renormalized* Nelson model [14, 9, 13]. In this article we review only the standard Nelson model. Let Φ_{g} be the ground state of the Nelson Hamiltonian. An upper bound of the spatial decay of $\|\Phi_{\mathrm{g}}(x)\|_{\mathcal{F}}$ has been already shown in [10]. In this article the lower bound is shown in terms of an Agmon type metric.

We apply stochastic methods to measure the spatial exponential localization. This type of arguments have been done for Schrödinger type operators in a large number of papers, e.g., [1, 2, 3, 6, 5, 4, 15, 16].

1.1 Quantum mechanical matters

The particle Hamiltonian is defined by the 3-dimensional Schrödinger operator with external potential V:

$$H_{\mathrm{p}} = -\frac{1}{2}\Delta + V,$$

which acts in $L^2(\mathbb{R}^3)$. We introduce the Kato-decomposable class [2, Section 4] and [6].

Definition 1.1 *Let $V : \mathbb{R}^d \to \mathbb{R}$.*

(1) *V is a Kato-class potential if and only if $\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{\mathrm{B}_r(x)} |\kappa_d(x-y)V(y)|\,\mathrm{d}y = 0$ holds with function κ_d depending on the dimension d:*

$$\kappa_d(x) = \begin{cases} |x|, & d=1,\\ -\log|x|, & d=2,\\ |x|^{2-d}, & d\ge 3.\end{cases}$$

The set of Kato-class potentials is denoted by $\mathcal{K}(\mathbb{R}^d)$.

(2) $V \in \mathcal{K}_{\rm loc}(\mathbb{R}^d)$ *if and only if* $\mathbb{1}_K V \in \mathcal{K}(\mathbb{R}^d)$ *for any compact set* $K \subset \mathbb{R}^d$.

(3) V *is Kato-decomposable if and only if* $V = V_+ - V_-$ *with* $V_+(x) = \max\{V(x), 0\}$ *and* $V_-(x) = \max\{-V(x), 0\}$ *satisfy that* $V_+ \in \mathcal{K}_{\rm loc}(\mathbb{R}^d)$ *and* $V_- \in \mathcal{K}(\mathbb{R}^d)$. *The set of Kato-decomposable potentials is denoted by* $\mathcal{K}_d$.

The self-adjoint operator of the form $H_{\rm p} = -\frac{1}{2}\Delta + V$ with Kato-decomposable potential V is defined through a Feynman-Kac formula. Let $(B_t)_{t\geq 0}$ be 3-dimensional Brownian motion on a probability space $(\mathscr{X}, \mathcal{B}, \mathcal{W}^x)$, which starts from $x \in \mathbb{R}^3$ at $t = 0$. The expectation value with respect to the probability measure $\mathcal{W}^x$ is denoted by $\mathbb{E}^x[\ldots]$. In particular we set $\mathbb{E}$ for $\mathbb{E}^0$ for notational simplicity. Let V be bounded. Then $H_{\rm p}$ is self-adjoint on $D(\Delta)$ and we have

$$(f, e^{-tH_{\rm p}} g)_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \mathbb{E}^x[e^{-\int_0^t V(B_s){\rm d}s} \overline{f(B_0)} g(B_t)]{\rm d}x.$$

Replacing V on the right-hand side above with Kato-decomposable potentials, one can also see that the right-hand side is finite for any $f, g \in L^2(\mathbb{R}^3)$ and, by Riesz representation theorem, one defines a strongly continuous one-parameter semigroup S_t, $t \geq 0$ such that

$$(f, S_t g)_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \mathbb{E}^x[e^{-\int_0^t V(B_s){\rm d}s} \overline{f(B_0)} g(B_t)]{\rm d}x.$$

By the Stone theorem for semigroups, there exists the self-adjoint operator $H_{\rm p}$ such that $S_t = e^{-tH_{\rm p}}$ for $t \geq 0$. This is the definition of $H_{\rm p}$ with Kato-decomposable potentials V.

1.2 Nelson Hamiltonian

Let us define the quantum field part. Let $\mathcal{F}$ be the boson Fock space over $L^2(\mathbb{R}^3)$ defined by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_{(n)}$$

with n particle subspace $\mathcal{F}_{(n)} = L^2_{\rm sym}(\mathbb{R}^{3n})$ for $n \geq 1$ and $\mathcal{F}_{(0)} = \mathbb{C}$. Then $\Phi \in \mathcal{F}$ is denoted by $\Phi = \oplus_{n=0}^{\infty} \Phi^{(n)}$. The vector $\Omega = 1 \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{F}$ is called the Fock vacuum. Let $a^\dagger(g)$ and $a(g)$ be the creation operator and the annihilation operator smeared by $g \in L^2(\mathbb{R}^3)$, respectively, acting in $\mathcal{F}$. They satisfy that $a(g)^* = a^\dagger(\bar{g})$, $[a(g), a^\dagger(f)] = (\bar{g}, f)_{L^2(\mathbb{R}^3)}$ and $[a(g), a(f)] = 0 = [a^\dagger(g), a^\dagger(f)]$. Let $\omega(k) = |k|$ be the relativistic energy of a single massless boson with momentum $k \in \mathbb{R}^3$. The free field Hamiltonian $H_{\rm f}$ acting in $\mathcal{F}$ is given by

$$H_{\rm f} = {\rm d}\Gamma(\omega),$$

where

$$\left(\mathrm{d}\Gamma(\omega)\Phi\right)^{(n)}(k_1,\ldots,k_n) = \left(\sum_{j=1}^{n}\omega(k_j)\right)\Phi^{(n)}(k_1,\ldots,k_n), \quad n \geq 1,$$
$$\mathrm{d}\Gamma(\omega)\Omega = 0.$$

The total Hilbert space $\mathcal{H}$ for the Nelson model is defined by

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}.$$

Now let us define the Nelson Hamiltonian with a cutoff $\hat{\varphi}$. Let $\mathscr{S}'_{\mathbb{R}}(\mathbb{R}^3)$ be the set of real-valued Schwarz distributions on $\mathbb{R}^3$ and $\hat{\varphi}$ the Fourier transform of φ in the sense of distribution.

Assumption 1.2 *Let $\varphi \in \mathscr{S}'_{\mathbb{R}}(\mathbb{R}^3)$. We suppose that $\hat{\varphi} \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k)$ and $\hat{\varphi}/\sqrt{\omega}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^3)$.*

Throughout this paper we assume Assumption 1.2. Let $\tilde{\varphi} = (\hat{\varphi}/\sqrt{\omega})\check{}$, where $\check{f}$ denotes the inverse Fourier transform of f. The linear interaction H_{I} is defined by

$$H_{\mathrm{I}} = \int_{\mathbb{R}^3}^{\oplus} H_{\mathrm{I}}(x)\mathrm{d}x$$

with the action:

$$(H_{\mathrm{I}}\Phi)(x) = \phi(x)\Phi(x) \quad a.e. x \in \mathbb{R}^3.$$

Here for each $x \in \mathbb{R}^3$ $\phi(x)$ is given by

$$\phi(x) = \frac{1}{\sqrt{2}}\left\{a^\dagger\left(\frac{\hat{\varphi}}{\sqrt{\omega}}e^{-ik\cdot x}\right) + a\left(\frac{\tilde{\hat{\varphi}}}{\sqrt{\omega}}e^{ik\cdot x}\right)\right\}.$$

Here $\tilde{\hat{\varphi}}(k) = \hat{\varphi}(-k)$. Then the Nelson Hamiltonian with ultraviolet cutoff $\hat{\varphi}$ and Kato-decomposable potential V is defined by

$$H = H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + H_{\mathrm{I}}.$$

Under Assumption 1.2, H is symmetric and H_{I} is infinitesimally small with respect to $\mathbb{1} \otimes H_{\mathrm{f}}$. Then H is self-adjoint on $D(H_{\mathrm{p}} \otimes \mathbb{1}) \cap D(\mathbb{1} \otimes H_{\mathrm{f}})$.

1.3 FKF for e^{-tH}

Let us define the bounded operator $\mathrm{J}_{[0,t]}$ by

$$\mathrm{J}_{[0,t]} = \overline{e^{\frac{1}{2}W}e^{a^\dagger(U)}e^{-tH_{\mathrm{f}}}e^{a(\tilde{U})}},$$

where $\overline{\{\ldots\}}$ denotes the operator closure and

$$U(k) = -\int_0^t \frac{e^{-|s|\omega(k)}\hat{\varphi}(k)}{\sqrt{2\omega(k)}} e^{-ik\cdot B_s} \mathrm{d}s, \quad \tilde{U}(k) = -\int_0^t \frac{e^{-|s-t|\omega(k)}\hat{\varphi}(-k)}{\sqrt{2\omega(k)}} e^{ik\cdot B_s} \mathrm{d}s.$$

The exponent W is given by

$$W = \frac{1}{2}\int_0^t \mathrm{d}s \int_0^t \mathrm{d}r \int_{\mathbb{R}^3} \frac{e^{-|s-r|\omega(k)}|\hat{\varphi}(k)|^2}{\omega(k)} e^{-ik\cdot(B_s-B_r)} \mathrm{d}k.$$

Note that $e^{a^\sharp(f)} = \sum_{n=0}^\infty a^\sharp(f)^n/n!$ is an unbounded operator. On e can see that $\|\mathrm{J}_{[0,t]}\| \leq C_{\hat{\varphi}}(t)$, where

$$C_{\hat{\varphi}}(t) = \begin{cases} 2\exp\left\{\frac{t}{2}\|\hat{\varphi}/\omega\|^2 + 2t(t\vee 1)(\|\hat{\varphi}/\sqrt{\omega}\|^2 + \|\hat{\varphi}/\omega\|^2)\right\} & \hat{\varphi}/\omega^{3/2} \not\in L^2(\mathbb{R}^3), \\ 2\exp\left\{t\left(\frac{3}{2}\|\hat{\varphi}/\omega\|^2 + \|\hat{\varphi}/\omega^{3/2}\|^2 \vee \|\hat{\varphi}/\sqrt{\omega}\|^2\right)\right\} & \hat{\varphi}/\omega^{3/2} \in L^2(\mathbb{R}^3). \end{cases}$$

In particular we have

$$|(\Psi, \mathrm{J}_{[0,t]}\Phi)_{\mathcal{F}}| \leq C_{\hat{\varphi}}(t)\|\Psi\|_{\mathcal{F}}\|\Phi\|_{\mathcal{F}}. \tag{1.1}$$

It is important to see that $\mathrm{J}_{[0,t]}$ depends on $w \in \mathscr{X}$ but the right-hand sides of (1.1) are independent of $w \in \mathscr{X}$. Let $V \in \mathcal{K}_3$ and suppose Assumption 1.2. Then we have FKF:

$$(F, e^{-tH}G)_{\mathcal{H}} = \int_{\mathbb{R}^3} \mathbb{E}^x[e^{-\int_0^t V(B_s)\mathrm{d}s}(F(B_0), \mathrm{J}_{[0,t]}G(B_t))_{\mathcal{F}}]\mathrm{d}x.$$

We refer to e.g., [13, 10].

1.4 Ground state

The next proposition guarantees the existence and the uniqueness of the ground state of the Nelson Hamiltonian H.

Proposition 1.3 *Suppose that $\hat{\varphi}/\omega^{3/2} \in L^2(\mathbb{R}^3)$ and $V \in \mathcal{K}_3$. Assume that the binding condition holds true. Then the ground state of H exists and it is unique.*

Proof: See [8, (3) and Theorem 3.1] for the binding condition and [17, 7] for the existence of the ground state. □

Example 1.4 *Let V be such that $\lim_{|x|\to\infty} V(x) = \infty$. Then V satisfies the binding condition, and then the ground state of H exists and unique.*

Let $\Phi_{\rm b}$ be a bound state of H such that $H\Phi_{\rm b} = E_{\rm b}\Phi_{\rm b}$. Let $\Psi \in \mathcal{F}$ and $t \geq 0$. Since $e^{-tH}\Phi_{\rm b} = e^{-tE_{\rm b}}\Phi_{\rm b}$, we have

$$(\Psi, \Phi_{\rm b}(x))_{\mathcal{F}} = \mathbb{E}^x[e^{-\int_0^t (V(B_s)-E_{\rm b}){\rm d}s}(\Psi, {\rm J}_{[0,T]}\Phi_{\rm b}(B_t))_{\mathcal{F}}] \quad a.e. x \in \mathbb{R}^3. \tag{1.2}$$

Let $\Phi_{\rm g}$ be the ground state of H such that $H\Phi_{\rm g} = E_{\rm g}\Phi_{\rm g}$, where $E_{\rm g}$ denotes the infimum of the spectrum of H. We set

$$\ell_\Omega(x) = \mathbb{E}^x[e^{-\int_0^t (V(B_s)-E_{\rm g}){\rm d}s}(\Omega, {\rm J}_{[0,T]}\Phi_{\rm g}(B_t))_{\mathcal{F}}].$$

Lemma 1.5 *Let $V \in \mathcal{K}_3$. Then $\ell_\Omega(x)$ is continuous in x and $\ell_\Omega(x) > 0$ for all $x \in \mathbb{R}^3$.*

Proof: The continuity is shown in [12, 11] and the positivity in [13, 11]. $\Box$

By Lemma 1.5 and (1.2), ℓ_Ω can be regarded as the continuous version of $(\Omega, \Phi_{\rm g}(\cdot))_{\mathcal{F}}$.

2 Pointwise bounds

By using an Agmon metric type argument [1, 5], we can estimate the lower bound of $\|\Phi_{\rm g}(x)\|_{\mathcal{F}}$.

2.1 Geodesic distance for V

Assumption 2.1 *Suppose that V is continuous, $V(x) \geq \varepsilon$ for all $x \in \mathbb{R}^3$ with some $\varepsilon > 0$ and $\lim_{|x|\to\infty} V(x) = \infty$.*

Suppose Assumption 2.1. Let us set $W = V_{\rm sup}$. W is also continuous and satisfies that $W(x) \geq \varepsilon$ for all $x \in \mathbb{R}^3$ and $\lim_{|x|\to\infty} W(x) = \infty$. We fix $T > 0$. We estimate $\|\Phi_{\rm g}(x)\|_{\mathcal{F}}$ from below in terms of the exponent of an Agmon type metric. We define two C^1-path spaces:

$$\mathcal{C}^* = \{{\rm q} \in C^1([0,T];\mathbb{R}^3) \mid {\rm q}(0) = x, {\rm q}(T) = 0\},$$
$$\mathcal{C} = \{\gamma \in C^1([0,T];\mathbb{R}^3) \mid \gamma(0) = 0, \gamma(T) = x\}.$$

Let

$$\mathscr{A}({\rm q}, T) = \int_0^T \left(W({\rm q}(s)) + \frac{1}{2}|\dot{\rm q}(s)|^2\right){\rm d}s, \quad {\rm q} \in \mathcal{C}^*,$$
$$\mathscr{L}(\gamma, T) = \int_0^T \sqrt{2W(\gamma(s))}|\dot\gamma(s)|{\rm d}s, \quad \gamma \in \mathcal{C}.$$

We set $\gamma^{\rm q}(s) = {\rm q}(T-s)$ for ${\rm q} \in \mathcal{C}^*$ and ${\rm q}^\gamma(s) = \gamma(T-s)$ for $\gamma \in \mathcal{C}$. Then $\gamma^{\rm q} \in \mathcal{C}$, ${\rm q}^\gamma \in \mathcal{C}^*$ and $\mathscr{L}(\gamma^{\rm q}, T) \leq \mathscr{A}({\rm q}, T)$ and $\mathscr{L}(\gamma, T) \leq \mathscr{A}({\rm q}^\gamma, T)$ follow for any ${\rm q} \in \mathcal{C}^*$ and $\gamma \in \mathcal{C}$ by the arithmetic and geometric inequality: $2ab \leq a^2 + b^2$. We are interested in the existence of a minimizer γ^* of $\mathscr{L}(\gamma, T)$. We shall approximate W by a C^∞-function. For $b > 0$ there exists $Y \in C^\infty(\mathbb{R}^3)$ such that

$$(1-b)W(x) \leq Y(x) \leq (1+b)W(x).$$

For $x, y \in \mathbb{R}^3$ we define the geodesic distance for W by

$$\varrho(x,y) = \inf \left\{ \mathscr{L}(\gamma, T) | \, \gamma \in \mathcal{C}^1([0,T]; \mathbb{R}^3), \gamma(0) = x, \gamma(T) = y \right\}.$$

ϱ defines a metric on $\mathbb{R}^3$. Set $W_b = \frac{1}{1-b} Y$. Then $W(x) \leq W_b(x)$. Let

$$\mathscr{L}_b(\gamma, T) = \int_0^T \sqrt{2W_b(\gamma(s))} |\dot{\gamma}(s)| \mathrm{d}s.$$

Then $W_b \geq \varepsilon/(1-b) > 0$, $W_b \in C^\infty(\mathbb{R}^3)$ and $\lim_{|x|\to\infty} W_b(x) = \infty$. Let

$$\varrho_b(x,y) = \inf \left\{ \mathscr{L}_b(\gamma, T) \mid \gamma \in C^1([0,T]; \mathbb{R}^3), \gamma(0) = x, \gamma(T) = y \right\}.$$

$\varrho_b(0, X) = \inf_{x \in X} \varrho_b(0, x)$ is the distance from 0 to X. We can see that ϱ_b is geodesically complete. The geodesic completeness implies that there exists a length minimizing geodesic connecting any two points by Hopf-Rinow theorem. Then there exists a minimizer $\gamma^* \in C^\infty([0,T]; \mathbb{R}^3)$ of $\mathscr{L}_b(\gamma)$. We define

$$\mathscr{A}_b(\mathrm{q}, S) = \int_0^S \left(W_b(\mathrm{q}(s)) + \frac{1}{2} |\dot{\mathrm{q}}(s)|^2 \right) \mathrm{d}s.$$

We shall connect two minima:

$$\begin{aligned} &\inf \left\{ \mathscr{L}_b(\gamma, T) \mid \gamma(0) = 0, \gamma(T) = 0, \gamma \in C^\infty([0,T]; \mathbb{R}^3) \right\}, \\ &\inf \{ \mathscr{A}_b(\mathrm{q}, S) \mid S > 0, \mathrm{q}(0) = x, \mathrm{q}(s) = 0, \mathrm{q} \in C^\infty([0,S]; \mathbb{R}^3) \}. \end{aligned}$$

$\mathscr{L}_b(\gamma, T) = \int_0^T \sqrt{2W_b(\gamma(s))} |\dot{\gamma}(s)| \mathrm{d}s$ is invariant under re-parametrization: $\gamma \to \gamma \circ \phi$ by any smooth bijection $\phi : [0,T] \to [0,T]$. On the other hand in general $\mathscr{A}_b(\mathrm{q}, S)$ is not invariant. From this property one can construct a bijection ϕ such that

$$\sqrt{2W_b(\gamma \circ \phi(s))} |\gamma \dot{\circ} \phi(s)| = W_b(\gamma \circ \phi(s)) + \frac{1}{2} |\gamma \dot{\circ} \phi(s)|^2.$$

Then we have the lemma.

Lemma 2.2 *There exists minimizer* $(\mathrm{q}^*, S^*) \in C^\infty([0,T]; \mathbb{R}^3) \times (0, \infty)$ *of* $\mathscr{A}_b(\mathrm{q}, S)$ *and it holds that* $\mathscr{L}_b(\gamma^*, T) = \mathscr{A}_b(\mathrm{q}^*, S^*)$. *Moreover let* $\gamma_*(s) = \mathrm{q}^*(S^* - s)$. *Then*

$$\mathscr{L}_b(\gamma^*, T) = \int_0^{S^*} \sqrt{2W_b(\gamma_*(s))} |\dot{\gamma}_*(s)| \mathrm{d}s = \mathscr{A}_b(\mathrm{q}^*, S^*). \tag{2.1}$$

2.2 Exponential decay

Let $K \subset \mathbb{R}^3$ be a compact set. Since $\ell_\Omega(\cdot)$ is continuous and strictly positive on K, we can set $\chi_K = \inf_{y \in K} \ell_\Omega(y) > 0$. Let $\Phi_\infty = \sup_{y \in \mathbb{R}^3} \|\Phi_{\mathrm{g}}(y)\|_{\mathcal{F}}$.

Lemma 2.3 *Let $T > 0$. Then there exists $\tau > 0$ such that for any $\mathrm{q} \in \mathcal{C}^*$,*

$$\ell_\Omega(x) \geq \chi_K e^{-\int_0^T \left(V_{\sup}(\mathrm{q}(s)) + \frac{1}{2}\int_0^T |\dot{\mathrm{q}}(s)|^2\right)\mathrm{d}s} e^{-T\tau} e^{T(E_{\mathrm{g}} - \frac{\|\hat{\varphi}/\omega\|\Phi_\infty}{\sqrt{2}\chi_K})}. \tag{2.2}$$

Proof: By Jensen's inequality, we have

$$\begin{aligned}
\ell_\Omega(x) &= \mathbb{E}\left[e^{-\int_0^T (V(B_s+x)-E_{\mathrm{g}})\mathrm{d}s}(\Omega, e^{-\phi_{\mathcal{E}}(\int_0^T \mathrm{j}_s\tilde{\varphi}(\cdot - B_s)\mathrm{d}s)}\mathrm{J}_t\Phi_{\mathrm{g}}(B_T+x))_{\mathcal{F}}\right] \\
&\geq \mathbb{E}\left[e^{-\int_0^T (V(B_s+x)-E_{\mathrm{g}})\mathrm{d}s}\ell_\Omega(B_T+x)e^{-\frac{\|\hat{\varphi}/\omega\|\Phi_\infty}{\sqrt{2}\ell_\Omega(B_T+x)}}\right].
\end{aligned}$$

Let $\mathrm{q} \in \mathcal{C}^*$ and we define ξ by

$$\xi = e^{-\int_0^T \dot{\mathrm{q}}(s)\cdot \mathrm{d}B_s - \frac{1}{2}\int_0^T |\dot{\mathrm{q}}(s)|^2 \mathrm{d}s}.$$

Thus $\mathbb{E}[\xi] = 1$. By the Girsanov theorem, we see that

$$\mathbb{E}\left[e^{-\int_0^T (V(B_s+x)-E_{\mathrm{g}})\mathrm{d}s}\ell_\Omega(B_T+x)e^{-\frac{\|\hat{\varphi}/\omega\|\Phi_\infty}{\sqrt{2}\ell_\Omega(B_T+x)}}\right] = \mathbb{E}\left[\xi e^{-\int_0^T (V(B_s+\mathrm{q}(s))-E_{\mathrm{g}})\mathrm{d}s}\ell_\Omega(B_T)e^{-T\frac{\|\hat{\varphi}/\omega\|\Phi_\infty}{\sqrt{2}\ell_\Omega(B_T)}}\right].$$

Let $M = \{|B_s| \leq 1, 0 \leq s \leq T\}$ and K be the unit closed ball. Thus $\ell_\Omega(B_T) \geq \chi_K$ on M and

$$\ell_\Omega(x) \geq \chi_K e^{-\int_0^T V_{\sup}(\mathrm{q}(s))\mathrm{d}s} e^{T(E_{\mathrm{g}} - \frac{\|\hat{\varphi}/\omega\|\Phi_\infty}{\sqrt{2}\chi_K})}\mathbb{E}\left[\xi \mathbb{1}_M\right].$$

By Jensen's inequality again, we have

$$\mathbb{E}\left[\xi \mathbb{1}_M\right] \geq \chi_K e^{-\frac{1}{2}\int_0^T |\dot{\mathrm{q}}(s)|^2 \mathrm{d}s}\mathbb{E}[\mathbb{1}_M]e^{\frac{\mathbb{E}\left[\mathbb{1}_M\left(-\int_0^T \dot{\mathrm{q}}(s)\cdot \mathrm{d}B_s\right)\right]}{\mathbb{E}[\mathbb{1}_M]}} = \chi_K e^{-\frac{1}{2}\int_0^T |\dot{\mathrm{q}}(s)|^2 \mathrm{d}s}\mathbb{E}[\mathbb{1}_M].$$

Note that $\mathbb{E}[\mathbb{1}_M] \geq e^{-T\tau}$ with some $\tau > 0$ [5], where τ is the infimum of the spectrum of $-\Delta/2$ on the unit ball with Dirichlet boundary condition. Thus (2.2) follows. $\square$

Theorem 2.4 ([11]) *Let $\gamma \in \mathcal{C}$ and $\varepsilon > 0$. Then there exists $R > 0$ such that*

$$\chi_K e^{-(1+\varepsilon)\int_0^T \sqrt{2V_{\sup}(\gamma(s))}|\dot{\gamma}(s)|\mathrm{d}s} \leq \|\Phi_{\mathrm{g}}(x)\|_{\mathcal{F}}, \quad |x| \geq R.$$

Proof: By Lemma 2.2, there exists minimizer $(\mathrm{q}^*, S^*) \in C^\infty([0,T];\mathbb{R}^3) \times (0,\infty)$ of $\mathscr{A}_b(\mathrm{q}, S)$ and $\gamma^* \in C^\infty([0,T];\mathbb{R}^3)$ of $\mathscr{L}_b(\gamma, T)$. It also holds that $\mathscr{L}_b(\gamma^*, T) = \mathscr{A}_b(\mathrm{q}^*, S^*)$. It can be shown that there exists $R_{\delta,b}$ such that

$$S^* \leq \delta \int_0^{S^*}\left(W_b(\mathrm{q}^*(s)) + \frac{1}{2}\int_0^T |\dot{\mathrm{q}}^*(s)|^2\right)\mathrm{d}s$$

for $|x| \geq R_{\delta,b}$. Hence putting $T = S^*$ and $q = q^*$ in Lemma 2.3, we have for $|x| \geq R_{\delta,b}$,

$$\begin{aligned}\ell_\Omega(x) &\geq \chi_K \exp\{-(1+\delta)\mathscr{A}_b(\mathrm{q}^*, S^*)\} = \chi_K \exp\{-(1+\delta)\mathscr{L}_b(\gamma^*, T)\} \\ &\geq \chi_K \exp\left\{-(1+\delta)\sqrt{\frac{1+b}{1-b}}\mathscr{L}(\gamma, T)\right\}.\end{aligned}$$

Choose δ and b such that $1+\varepsilon = (1+\delta)\sqrt{\frac{1+b}{1-b}}$. Then the theorem follows. □

Corollary 2.5 ([11]) *(1) Suppose Assumption 2.1. Let $\varepsilon > 0$. Then there exists R such that*

$$\chi_K e^{-(1+\varepsilon)|x|\int_0^T \sqrt{2V_{\sup}(sx)}\mathrm{d}s} \leq \|\Phi_{\mathrm{b}}(x)\|_{\mathcal{F}}, \quad |x| \geq R.$$

(2) Assume that V obeys the lower bound

$$V(x) \geq \frac{a^2}{2}|x|^{2n} - b, \quad |x| \geq R$$

for some $a, b, n > 0$ and $R > 0$ so that $a^2 R^{2n}/2 - b > 0$. Then for all $\varepsilon > 0$, there exist $c_\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$C_\varepsilon e^{-(1+\varepsilon)\frac{a}{n+1}|x|^{n+1}} \leq \|\Phi_{\mathrm{b}}(x)\|_{\mathcal{F}} \leq c_\varepsilon e^{-(1-\varepsilon)\frac{a}{n+1}|x|^{n+1}}.$$

Proof: (1) Put $\gamma(s) = sx$ in Theorem 2.4. Then (1) follows. (2) The lower bound follows from (1) and the upper bound from [10]. □

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