Hiroshima's talk in RIMS 2018

- $\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^d)) = \bigoplus_{n=0}^{\infty} \left[ \bigotimes_s^n L^2 \right] \ni \{\Phi^{(n)}\}$

$\|\Phi\|_{\mathcal{F}}^2 = \sum_n \|\Psi^{(n)}\|^2, \; \sum (\Psi^{(n)}, \Psi^{(n)}) = (\Phi, \Psi)$

$$\begin{cases} (a^+(f)\Psi)^{(n)} = \sqrt{n}\, S_n(f \otimes \Psi^{(n-1)}) & n \geq 1 \\ (a^+(f)\Psi)^{(0)} = 0 \end{cases}$$

$a(f) = (a^+(\bar{f}))^* \qquad f \in L^2$

CCR $[a(f), a^+(g)] = (\bar{f}, g)$

- $T : L^2 \to L^2$ contraction $\qquad \overset{0}{\otimes} T = 1$

$\Gamma(T) : \mathcal{F} \to \mathcal{F} \qquad \Gamma(T) = \bigoplus_{n=0}^{\infty} \left[ \overset{n}{\otimes} T \right]$

$\Gamma(S)\Gamma(T) = \Gamma(ST)$ and $\Gamma(T)^* = \Gamma(T^*)$

$h$ s.a in $L^2$

$\Gamma(e^{-ith}) = e^{-it\, d\Gamma(h)}$

$\omega = \sqrt{|k|^2 + \nu^2} \qquad \nu \geq 0 \qquad \begin{matrix} \nu = 0 & \text{massless} \\ \nu > 0 & \text{massive} \end{matrix}$

$d\Gamma(\omega) = H_f$

$\| a(f)\Psi \| \leq \| f/\sqrt{\omega} \| \, \| H_f^{1/2} \Psi \|$

$\| a^+(f)\Psi \| \leq \| f/\sqrt{\omega} \| \, \| H_f^{1/2}\Psi \| + \| f \| \, \| \Psi \|$

$$\left\| \prod_{j=1}^{n} a(f_j)\,\Phi \right\| \le \prod_{j}^{n} \| f_j/\sqrt{\omega} \| \, \| H_f^{n/2}\Phi \|$$

$$\left\| \prod_{j=1}^{n} a^{\dagger}(f_j)\,\Phi \right\| \le \sqrt{n!}\; 2^{n/2} \prod_{j=1}^{n} \| f_j \|_{\omega} \left( \sum^{n} \frac{1}{m!} \| H_f^{m/2}\Phi\|^2 \right)^{1/2}$$

$$e^{a^{\#}(f)} = \sum_{n=0}^{\infty} \frac{1}{n!} a^{\#}(f)^n$$

$e^{a^{\dagger}(f)}\Omega$ coherent vectors

$\underset{\sim\sim\sim}{\sigma(a(f)) = \sigma_p(a(f)) = \mathbb{C}}$

$$\left\| e^{a^{\dagger}(f)} e^{-\frac{t}{2}H_f} \Phi \right\| \le a \left\| e^{-\frac{1}{2}(t-1)H_f} \Phi \right\| \qquad t \ge 1$$

$$\| \qquad '' \qquad \| \le a_s \left\| e^{-\frac{1}{2}(t-s)H_f} \Phi \right\| \qquad \underline{0 < s < t < 1}.$$

where $a = a_1$, $a_s = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} (2/s)^{n/2} \|f\|_{\omega}^{n}$

$\underset{\sim\sim\sim}{a_s \le \sqrt{2}\, e^{(2/s)\|f\|_{\omega}^2}}$

Hence we have $\left\| e^{a^{\dagger}(f)} e^{-tH_f} e^{a(f)} \right\| \le 2 e^{(\frac{4}{s})\|f\|_{\omega}^2}$ $\quad 0 < s < t < 1$

$$\le 2 e^{4\|f\|_{\omega}^2} \qquad 1 \le t$$

$$\| \qquad '' \qquad \| \le \frac{q}{q-1} e^{(2q/s)\|f\|_{\omega}^2} \qquad q > 1$$

The $\sim$ or $s+v$, the $\sim$er $s+v$. as much as you like

Nelson model

$\phi(f) \quad f \in L^2 \qquad$ GRV $\qquad (Q, \Sigma, \mu)$

$$\begin{cases} \phi(f) \simeq \frac{1}{\sqrt{2}}\left[a^+(\hat{f}) + a(\hat{f})\right] \\ L^2(Q) \cong \mathcal{F} \\ \hat{H_f} \cong H_f \end{cases} \qquad \hat{H}_f = d\Gamma\left(\omega(-i\nabla)\right)$$

$\phi_E(f) \quad f \in L^2(\mathbb{R}^4) \qquad (Q_E, \Sigma_E, \mu_E)$

$J_t : L^2(Q) \to L^2(Q_E) \quad$ s.t. $\quad J_0^* J_t = e^{-tH_f}.$

$(F, e^{-tH_f} G) = (F, J_0^* J_t G) \qquad$ ok

Ex $\quad H_f + \phi(f)^{2n} = H$

$$(F, e^{-tH} G) = (F, J_0^* e^{-\int_0^t \phi^{2n}(\tau_s f)ds} J_t G)$$

FKN-formula

$T_s : L^2 \to L^2$

$T_s^* T_t = e^{-(t-s)\omega(-i\nabla)}.$

van Hove Hamiltonian $\quad H_f + \phi(f)$

$$(F, e^{-tH} G) = (F, \underbrace{J_0^* e^{-\int_0^t \phi_E(\tau_s f)\, ds} J_t}\, G)$$

Hypercontractivity $\quad e^{-\int_0^T \phi_E(\tau_s f)} = e^{-\phi_E\left(\int_0^T \tau_s f\, ds\right)}$

$$e^{-\phi_E(x)} \in \bigcap_{p>0} L^p(Q_E) \qquad = e^{-\phi_E(X)}$$

$\rightarrow J_0^* e^{-\phi_E(X)} J_t$ is bdd and

$$\| \cdots \| \le \| e^{-\phi_E(x)} \|_{L^q}$$

$$q = \frac{2}{1 - e^{-\nu t}} \qquad (\nu > 0) \text{ massive}$$

BCH formula

Intertwining property

factorization

$$e^{-\phi_E} \cong e^{-\frac{1}{\sqrt{2}} a_E^+(x)} e^{-\frac{1}{\sqrt{2}} a_E(x)} e^{+\frac{1}{4}\|X\|^2}$$

$$J_0^* e^{-\frac{1}{\sqrt{2}} a_E^+(x)} = e^{-\frac{1}{\sqrt{2}} a^+(\tau_0^* X)} J_0^*$$

$$e^{-\frac{1}{\sqrt{2}} a_E(x)} J_T = J_T e^{-\frac{1}{\sqrt{2}} a_E(x)}$$

$$\therefore J_0^* e^{-\phi_E(x)} J_t = \underline{e^{-\frac{1}{\sqrt{2}} a^+(\tau_0^* X)}}\, e^{-tH}\, \underline{e^{-\frac{1}{\sqrt{2}} a(\tau_t^* x)}}\, e^{\frac{1}{4}\|x\|^2}$$

b' b'dc

Nelson model

$$H = H_p + H_f + \phi(x) \qquad \phi(x) = \phi(\underset{\text{cutoff fct.}}{\varphi}(\cdot - x))$$

1964 E. Nelson $\quad \mathrm{Spec}(H)$ ? $\quad \nu = 0$ ? $\quad \nu > 0$ ?

$$(f, e^{-tH_p} g) = \int dx\, \mathbb{E}^x\left[ f(B_0) g(B_t) e^{-\int_0^t V(B_s) ds} \right]$$

$$(F, e^{-tH} G) = \int dx\, \mathbb{E}^x\left[ e^{-\int_0^t V} \left( F(B_0), J_0^* e^{-\phi_E(X)} J_t\, G(B_t) \right) \right]$$

$$\int_0^t T_s \varphi(\cdot - B_s) ds = X$$

$$= \int dx\, \mathbb{E}^x\left[ e^{-\int_0^t V} \left( F(B_0), \underbrace{e^{U} e^{-tH_f} e^{\widetilde{U}} e^{\frac{t}{4}\|X\|^2}}_{A} G(B_t) \right) \right]$$

$$\|A\| \leq 2 \exp\left( \frac{t}{2} \|\varphi/\omega\|^2 + 2t(t \vee 1)\left( \|\varphi/\sqrt{\omega}\|^2 + \|\varphi/\omega\|^2 \right) \right)$$

or

$$\leq 2 \exp t E(\hat{\varphi}) \qquad \text{IR regular のとき}$$

$$E(\hat{\varphi}) = \max \left\{ \frac{1}{2} \|\hat{\varphi}/\omega\|^2 + 2 \left( \qquad \right) + 2 \left( \qquad \right)^2 \right.$$

① Diamagnetic ineq.

$$|(F, e^{-tH} G)| \le (\|F\|, e^{-t(H_p - E(p))} \|G\|)$$

$$\|V(-\tfrac{1}{2}\Delta + z)^{-1}\| < a \Leftrightarrow \|V(H+z)^{-1}\| < a.$$

② UV renormalization (Møller + Matte 17) (Gubinelli + H + Lörinczi 14)

$$\hat{g} = e^{-\varepsilon |k|^2/2}$$

$$(F, e^{-t(H_\varepsilon - E_\varepsilon)} G) \overset{\varepsilon \downarrow 0}{=} \int dx\, \mathbb{E}^x \left[ e^{SV} (F, U_\infty e^{-tH_f} \tilde{U}_\infty G) \right]$$

$$E_\varepsilon = -\int \frac{e^{-\varepsilon |k|^2}}{|k|^2/2 + \omega} \frac{1}{\omega} dk$$

③ Existence of ground state. $\int |\hat{\varphi}|^2 / \omega^3 < \infty$

$\varphi_g$

④

$$\|e^{+\beta N} \varphi_g\| < \infty \quad \forall \beta > 0 \ ?$$

$$N = N_s + N_H = \underbrace{\int_{|k|<1} a^\dagger a}_{\text{difficult part}} + \underbrace{\int_{|k|\ge 1} a^\dagger a}_{\text{easy part}}$$

$$e^{-tN_H} \le e^{-tH_f}$$

$$\varphi_g = e^{-tH_\infty} e^{tE_\infty} \varphi = e^{tE_\infty} U_\infty e^{-tH_f} \tilde{U}_\infty \varphi_g$$

$$\varphi_g^T = \frac{e^{-TH_\varepsilon} f \otimes 1}{e^{-TH_\varepsilon} f \otimes 1} \xrightarrow[\varepsilon \downarrow 0]{} \frac{e^{-TH} f \otimes 1}{e^{-TH} f \otimes 1} \xrightarrow[T \uparrow \infty]{} \varphi_g$$

$$(\varphi_g, e^{-\beta N} \varphi_g) = \lim_T \lim_\varepsilon (\varphi_g^{T\varepsilon}, e^{-\beta N} \varphi_g^{T\varepsilon})$$

$$= \lim_T \frac{\int dx\, \mathbb{E}\left[ f(B_{-T}) f(B_T)\, e^{\frac{1}{2} S}\, e^{-(1-e^{-\beta}) \int_{-T}^0 ds \int_0^T dt\, W_\Lambda} \right]}{Z_T}$$

$$= \lim_T \mathbb{E}_{\mu_T}\left[ e^{-(1-e^{-\beta}) \int_{-T}^0 ds \int_0^T dt\, W_\Lambda} \right]$$

$$W_\Lambda = \int_{\lambda < |k| < \underline{\underline{\Lambda}}} \frac{1}{\omega} e^{-|t-s|\omega} e^{-ik(B_T - B_s)}$$

$$\to \mathbb{E}_\mu\left[ e^{(1-e^{-\beta}) \int_{-\infty}^0 ds \int_0^\infty dt\, \underline{W_\Lambda}} \right] < \infty$$

$\sim$ analytic cont.

もう一回 下書きに $\bigoplus_{n=0}^{n} \Phi_s^{2}$

- $\mathcal{F}(L^2(\mathbb{R}^d)) = $ $a^+(f)$, $a(f)$ creation annihilation

CCR $[a(f), a^+(g)] = (\bar{f}, g)$

- $T$ contraction $\bigoplus_{n=0}^{\infty} T = \Gamma(T)$ $\Gamma(T)\Gamma(S) = \Gamma(TS)$

$h$ : s.a. in $L^2$ $\Gamma(S)^* = \Gamma(S^*)$

$\Gamma(e^{-ith}) = e^{-it\, d\Gamma(h)}$ $h = \omega = \sqrt{|k|^2 + \nu^2}$ multiplication

massive $\nu > 0$

$\nu = 0$ massless

$d\Gamma(\omega) = H_f$ free field H.

$d\Gamma(1) = N$ number operator

- $\| a(f)\Phi \| \le \| f/\sqrt{\omega} \| \, \| H_f^{1/2}\Phi \|$

$\| a^+(f)\Phi \| \le \| f/\sqrt{\omega} \| \, \| H_f^{1/2}\Phi \| + \| f \| \, \| \Phi \|$

$\Rightarrow \| \prod^n a(f_j)\Phi \| \le \prod \| f_j/\sqrt{\omega} \| \, \| H_f^{n/2}\Phi \|$

$\| \prod^n a^+(f_j)\Phi \| \le \sqrt{n!}\,(2/s)^{n/2} \left( \sum_{m=0}^{n} \frac{1}{m!} \| (sH)^{m/2}\Phi \|^2 \right)^{1/2}$

Fock space $\quad \mathcal{F} = \mathcal{F}(L^2) = \bigoplus_n \left( \bigotimes_s^n L^2 \right)$

$a(f),\ a^+(g) \qquad CCR \quad [a(f), a(g)] = (\bar{f}, g)$

$T : L^2 \to L^2 \qquad \| T \| \leq 1 \qquad \Longrightarrow \quad \Gamma(T) = \bigoplus_n \left[ \bigotimes^n T \right]$ is also contraction

$\Gamma(T)\Gamma(S) = \Gamma(TS),\quad \Gamma(T)^* = \Gamma(T^*) \qquad$ $*$-algebra

$h : L^2 \to L^2$ s.a. $\quad \Gamma(e^{-ith}) = e^{-it\, d\Gamma(h)}$.

$h = \omega \quad \longrightarrow \quad d\Gamma(\omega) = H_f, \qquad h = 1 \quad \longrightarrow \quad d\Gamma(1) = N$

$\| a(f)\Phi \| \leq \| H_f^{1/2} \Phi \| \, \| f/\sqrt{\omega} \|$

$\| a^+(f)\Phi \| \leq \qquad " \qquad + \| f \| \, \| \Phi \|$

Extension

$\| \prod a(f_j) \Phi \| \leq \prod \| f_j/\sqrt{\omega} \| \, \| H_f^{n/2} \Phi \|$

$\| \prod a^+(f_j) \Phi \| \leq \sqrt{n!}\ 2^{n/2} \prod \| f_j \|_\omega \left( \sum_{m=0}^{n} \frac{1}{m!} \| H_f^{m/2} \Phi \|^2 \right)^{1/2}$

$$e^{a^\#(f)} = \sum \frac{1}{n!} a^\#(f)^n$$

$\| e^{a^+(f)} e^{-\frac{t}{2} H_f} \| \leq A_1 \| e^{-\frac{1}{2}(t-1)H_f} \| \qquad t \geq 1$

$\qquad \leq A_s \| e^{-\frac{1}{2}(t-s)H_f} \| \qquad 0 < s < t < 1$

where $A_s = \sum_n (2/s)^n \| f \|_\omega^n \leq \sqrt{2}\ e^{\frac{2}{s} \| f \|_\omega^2}$

So $\| e^{a^+(f)} e^{-tH_f} e^{a(f)} \| \leq 2\, e^{\frac{4}{s} \| f \|_\omega^2} \quad \begin{pmatrix} 0 < s \leq t < 1 \\ s = 1,\quad t \geq 1 \end{pmatrix}$

$\Phi(f)$ $f \in L^2$ GRV on $(Q, \Sigma, \mu)$

$$\begin{cases} \phi(f) \sim \frac{1}{\sqrt{2}}\left(a^+(\hat{f}) + a(\tilde{\hat{f}})\right) \\ L^2(Q) \cong \mathcal{F} \\ \hat{H}_f \cong H_f \end{cases}$$

$\phi_E(f)$ $f \in L^2(\mathbb{R}^4)$

$T_t : L^2 \to L^2(\mathbb{R}^4)$

s.t. $T_t^* T_s = e^{-|t-s|\omega}$

$\Gamma(T_t) = J_t$ $\therefore J_t^* J_s = e^{-|t-s|H_f}$

Thm

$(F, e^{-tH_f} G) = (F, J_0^* J_t G)$

Ex $H = H_f + \phi(f)^{2n}$ $\longrightarrow$ ok

$H = H_f + \phi(f)^n$

$(F, e^{-tH} G) = (F, J_0^* e^{\phi_E(X)} J_t G)$

- $J_0^* e^{\phi_E(x)} J_t$ Hypercontractivity ($\nu > 0$)

- $\nu = 0$ ? BCH, Factorization Intertw...

$$J_0^* e^{\phi_E(x)} J_t = \underbrace{e^{\frac{1}{\sqrt{2}} a^\dagger(T_0^* x)} e^{-tH_f} e^{\frac{1}{\sqrt{2}} a(T_t x)}}_{\text{b'dd.}} e^{\frac{1}{4}\|x\|^2}$$

# Nelson model

$$H = H_p + H_f + \phi(\tilde{\varphi}(\cdot - x))$$ no spectral gap!

$V=0$

$$(f, e^{-tH_p} g) = \int \mathbb{E}^x\left[f(B_0) g(B_t) e^{\int_0^t V(B_s) ds}\right]$$

$$(F, e^{-tH} G) = \int dx\, \mathbb{E}^x\left(C^{\int_0^t V}\left(F(B_0), K G(B_t)\right)\right)$$

$$K = J_0^* e^{\phi_E(X)} J_t \qquad X = -\int_0^t \tau_s \hat{\varphi}(\cdot - B_s) ds$$

$$K = e^{a^*(W_t)} e^{-tH_f} e^{a(W_t)} e^{\frac{t}{4}\|X\|^2}.$$

$$\|K\| \le e^{tE(\varphi)}.$$

## Application

① $|(F, e^{-tH} G)| \le (\|F\|, e^{-t(H_p - E(\varphi))} \|G\|)$

$\therefore \|Vf\| \le a\|(-\frac{1}{2}\Delta f\| + b\|f\|$

$\to \|VF\| \le a\| \, Kf\| + b\|f\|$ exp. decay

② UV renormalization

$$\hat{\varphi} = e^{-\varepsilon|k|^2/2} \qquad \therefore \qquad E_\varepsilon = -\int \frac{1}{\omega + |k|^2/2} \frac{|\hat{\varphi}|^2}{\omega}$$

$$(F, e^{-t(H_\varepsilon - E_\varepsilon)} G) \to (F, e^{-tH_\infty} G)$$

uniformly.

(3) $\exists$ ground state

$$\frac{\text{binding condition} > 0}{\text{uniform conv}}$$

(4) super exponential decay of boson numbers

$$N = \int a^\dagger a \, dk = \int_{|k|<1} + \int_{|k|\geq 1} = N_0 + N_\infty$$

$e^{+\beta N_\infty} e^{-tH_f}$ bdd.

$$\underline{e^{\beta N_0} \, e^{-tH_f} \quad ?} \qquad \varphi_g^T = \frac{e^{-TH_\infty} \otimes 1}{\| \quad " \quad \|} \to \varphi_g$$

$$(\varphi_g^T, e^{-\beta N_0} \varphi_g^T) = \mathbb{E}_{\mu_T}\left[ e^{-(1-e^{-\beta}) \int_{-T}^{0} \int_0^T W} \right]$$

$$W = \int_{\lambda \leq |k| \leq 1} \frac{1}{\omega} e^{-|t-s|\omega} e^{-ik(B_t - B_s)}$$

(5) Gaussian domination