Spectrum of a Scalar Quantum Field Model on a Lorentzian Manifold

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This is the joint work with

Christian Gérard, Annalisa Panati and Akito Suzuki

- *Infrared divergence of a scalar quantum field model on a pseudo Riemannian manifold*, **IIS 15(2009) 399-421**
- *Infrared problem for the Nelson mode on a static space-times*, to appear in **CMP**
- *Absence of ground state for the Nelson model on a static-space-times*, **ArXiv 1012.2655**, to appear in **JFA**
- *Removal of UV cutoff for the Nelson model with variable coefficients*, **preprint 2011**
- *Existence and absence of ground states for a particle interacting through the quantized scalar field on a static spacetime*, **RIMS Kôkyûroku Bessatsu B21 (2010) 15-24**

Hilbert Space $\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathscr{F}$ $\mathscr{F} = \bigoplus_{n=0}^{\infty} L^2_{\mathrm{sym}}(\mathbb{R}^{3n})$

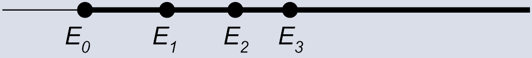
Standard Nelson model

$$H = \left(-\frac{1}{2}\Delta_X + V(X)\right) \otimes 1\!\!1 + 1\!\!1 \otimes d\Gamma(\omega) + \phi_\rho(X)$$

- (dispersion relation) $\omega = \omega(-i\nabla_x) = \sqrt{-\Delta_x + m^2}$, $m \geq 0$
- $d\Gamma(\omega)\Phi^{(n)}(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} \omega(-i\nabla_{x_j})\right) \Phi^{(n)}(x_1, \ldots, x_n)$
- $\phi(f) = \frac{1}{\sqrt{2}}(a^\dagger(\bar{f}) + a(f))$
- $\phi_\rho(X) = \phi(\omega^{-1/2}\rho(\cdot - X))$
- UV cutoff $0 \leq \rho \in \mathscr{S}$

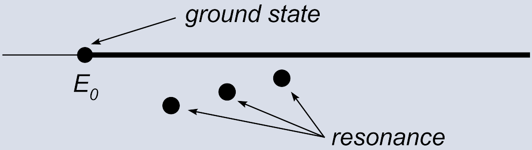
Ex. $\sigma(-\frac{1}{2}\Delta + V) = \{E_j\}$, $\sigma(\mathrm{d}\Gamma(\omega)) = [0, \infty)$ $(m = 0)$

- Embedded eigenvalues (no interaction)



$$\Downarrow +\phi_\rho(X)$$

- Resonances and ground state



Infrared problem

Existence and absence of ground states of standard Nelson model

$$\mathbf{I}_{\mathrm{IR}} = \int_{\mathbb{R}^3} \frac{|\hat{\rho}(k)|^2}{\omega(k)^3} dk$$

E.g., $\omega(k) = \sqrt{|k|^2 + m^2}$ and $\hat{\rho}(0) > 0$

- $m > 0 \Leftrightarrow \mathbf{I}_{\mathrm{IR}} < \infty$
- $m = 0 \Leftrightarrow \mathbf{I}_{\mathrm{IR}} = \infty$

THEOREM Bach-Fröhlich-Sigal(AdvMath98), Arai-Hirokawa(JFA98), Spohn(LMP99), Gérard(AHP00), Griesemer-Lieb-Loss(InvMath01), Lőrinczi-Minlos-Spohn(AHP03), Hirokawa-H.-Spohn(AdvMath04))....

- $\mathbf{I}_{\mathrm{IR}} < \infty \Longrightarrow \exists$**ground state**
- $\mathbf{I}_{\mathrm{IR}} = \infty \Longrightarrow$ **no ground state**

cf. $(\Phi_{\mathrm{g}}, N\Phi_{\mathrm{g}}) \leq \frac{1}{2}\mathbf{I}_{\mathrm{IR}}$

Nelson model on static Lorentzian manifold

Standard Nelson model

- $e^{-itH}\phi(f)e^{itH} = \int \phi(t,x)f(x)dx$
- $e^{-itH}Xe^{itH} = X_t$

Standard Nelson model satisfies that

$$(\partial_t^2 - \Delta_X + m^2)\phi(t,x) = \rho(x - X_t)$$

$$\partial_t^2 X_t = -\nabla V(X_t) - \int \phi(t,x)\nabla_X \rho(x - X_t)dx$$

Nelson model on Lorentzian mfd

Static (time independent) Lorentzian mfd on $\mathbb{R} \times \mathbb{R}^3$

$$g = (g_{\mu\nu}) = \begin{pmatrix} \lambda & \\ & -\gamma \end{pmatrix}$$

- $\lambda(x) > 0$,
- $\gamma(x)$ is a Riemannian metric on $\mathbb{R}^3$.
- $\Box_g = \sum |g|^{-1/2} \partial_\mu |g|^{1/2} g^{\mu\nu} \partial_\nu$

Wave equation on Lorentzian mfd

$$(\Box_g + m^2)\phi(t,x) = 0 \quad on \;\; L^2(\mathbb{R}^3, |g|^{1/2}dx)$$

Transformation: $U : L^2(\mathbb{R}^3, |g|^{1/2}dx) \ni u \mapsto |g|^{1/2}u \in L^2(\mathbb{R}^3)$:

- $U\phi(t,x) = \tilde{\phi}(t,x)$
- $U(\Box_g + m^2)U^{-1} = \partial_t^2 + h$
- $h = -\sum_{ij} \frac{1}{c}\partial_i a^{ij}(x)\partial_j \frac{1}{c} + m(x)^2 \to$ **dispersion relation** $h^{1/2}$
- Variable mass $m(x)$ appears even when bare mass $m = 0$.

Nelson model on static Lorentzian manifold

$$H = K \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \phi_\rho(X)$$

where

- $K = -\sum \partial_i A^{ij}(X)\partial_j + V(X)$
- $\omega = h^{1/2}$
- $\phi(X) = \phi(\omega^{-1/2}\rho(\cdot - X))$

How about variable mass $m(x)$?

Conjecture

- $m(x) \downarrow 0$ fast $\Longrightarrow$ no ground state
- $m(x) \downarrow 0$ slowly $\Longrightarrow \exists$ ground state

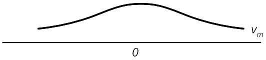


Figure: Long range variable mass $\sim$ massive

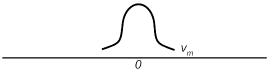


Figure: Short range variable mass $\sim$ massless

Existence of ground state

Assumptions

(F)
- (Elliptic cond) $C_0 \mathbb{1} \leq \left[a^{ij}(x)\right] \leq C_1 \mathbb{1}$
- $\partial^\alpha a^{ij}(x) \in O(\langle x\rangle^{-1})$, $|\alpha| \leq 1$
- $C_0 \leq c(x) \leq C_1$, $\partial^\alpha c(x) \in O(1)$, $|\alpha| \leq 2$
- (Massless)$\partial^\alpha m(x) \in O(1)$, $|\alpha| \leq 1$

(P)
- (Elliptic cond) $C_0 \mathbb{1} \leq \left[A^{ij}(X)\right] \leq C_1 \mathbb{1}$
- (Binding potential) $V(X) \geq C_0 \langle X\rangle^{2\delta} - C_1$

THEOREM (GHPS) Existence of ground state

Suppose $m(x) \geq a\langle x\rangle^{-1}$ for some $a > 0$, and $\delta > 3/2$. Then H has a ground state.

Proposition (Bruneau-Dereziński) general ω and K

Suppose that

(1) $\omega \geq 0$, Ker $\omega = 0$ (2) $\sup_X \|\omega^{-1/2}\rho(\cdot - X)\| < \infty$

(3) $(K+1\!\!1)^{-1/2}$ is compact (4) $\omega^{-1}\rho(\cdot - X)(K+1\!\!1)^{-1/2}$ is compact

(5) $\omega^{-3/2}\rho(\cdot - X)(K+1\!\!1)^{-1/2}$ **is compact** ($\sim \mathbf{I}_{\mathrm{IR}} < \infty$).

Then $K \otimes 1\!\!1 + 1\!\!1 \otimes d\Gamma(\omega) + \phi_\rho(X)$ has a ground state.

Proof of THEOREM: Check (5).

- Key estimate $m(x) \geq a\langle x\rangle^{-1} \Longrightarrow \omega^{-3/2}\langle x\rangle^{-3/2-\varepsilon}$ is bounded
- $V(X) > \langle X\rangle^{3+\varepsilon'} \Longrightarrow \langle X\rangle^{3/2+\varepsilon}(K+1\!\!1)^{-1/2}$ is compact.

Then

$\omega^{-3/2}\underline{\langle x\rangle^{-3/2-\varepsilon}\langle x\rangle^{3/2+\varepsilon}}\rho(x-X)\underline{\langle X\rangle^{-3/2-\varepsilon}\langle X\rangle^{3/2+\varepsilon}}(K+1\!\!1)^{-1/2}$ is compact.

Absence of ground state

Probabilistic approach

Particle Part

- Let $\Phi_{\rm p} > 0$ be the normalized ground state of K, $\Phi_{\rm p}(x) \le C_0 e^{-C_1|x|^{\delta+1}}$
- (ground state transform) $U : L^2(\Phi_{\rm p}^2 dx) \to L^2(dx), f \mapsto \Phi_{\rm p} f$
- $L = U(K - \inf\sigma(K))U^{-1}$

Field Part

- $\mathscr{F} \cong L^2(\mathscr{S}'_{\mathbb{R}}, d\nu)$, ν Gaussian measure such that $\int_{\mathscr{S}'_{\mathbb{R}}} e^{\alpha\phi(f)} d\nu = e^{(\alpha^2/4)\|f\|^2}$

Total Hilbert space and Hamiltonian

- $L^2(\mathbb{R}^3) \otimes \mathscr{F} \cong L^2(\mathbb{R}^3 \times \mathscr{S}'_{\mathbb{R}}, \Phi_{\rm p}^2 dx \otimes d\nu)$
- $H \cong L \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(\omega) + \phi_\rho(X)$

THEOREM (GHPS) Absence of ground state

Suppose $m(x) \le a\langle x\rangle^{-1-\varepsilon}$, $\forall\varepsilon > 0$, and $\delta > 0$. Then H has no ground state.

Outline of proof

- e^{-TH} is positivity improving.
- If H has a ground state $\Phi_g \Longrightarrow \Phi_g > 0$.
- $\mathbb{1} = \mathbb{1}_{L^2} \otimes \Omega$. $\Longrightarrow \Phi_g^T = e^{-TH}\mathbb{1}/\|e^{-TH}\mathbb{1}\| \to \Phi_g \ (T \to \infty)$.

$$\gamma = \lim_{T\to\infty} (\mathbb{1}, \Phi_g^T)^2 = \lim_{T\to\infty} \frac{(\mathbb{1}, e^{-TH}\mathbb{1})^2}{(\mathbb{1}, e^{-2TH}\mathbb{1})}$$

Lemma (Lőrinczi-Minlos-Spohn 03)

- $\gamma > 0 \Longrightarrow H$ has a ground state
- $\gamma = 0 \Longrightarrow H$ has no ground state

Feynman-Kac formula $\mathscr{X} = C(\mathbb{R}, \mathbb{R}^3)$

$\exists$diffusion $(X_t)_{t\in\mathbb{R}}$ on a prob.space $(\mathscr{X}, B(\mathscr{X}), \exists P^x)$ st

$$(f, e^{-tL} g)_{L^2(\Phi_{\rm g}^2 dx)} = \mathbb{E}\left[\overline{f(X_0)} g(X_t)\right]$$

where $\mathbb{E}[\cdots] = \int \Phi_{\rm p}^2(x)dx \int dP^x \cdots$.

$$(1\!\!1, e^{-TH} 1\!\!1)_{\mathscr{H}} = \mathbb{E}\left[e^{\int_0^T dt \int_0^T ds W(X_t, X_s, |t-s|)}\right]$$

$$W = W(X, Y, |t|) = \frac{1}{2}(\rho(\cdot - X), \omega^{-1} e^{-|t|\omega} \rho(\cdot - Y))$$

$$\gamma = \lim_{T\to\infty} \frac{(1\!\!1, e^{-TH} 1\!\!1)^2}{(1\!\!1, e^{-2TH} 1\!\!1)}$$

- Denominator:

$$(1\!\!1, e^{-2TH} 1\!\!1) = \mathbb{E}\left[e^{\int_0^{2T}\int_0^{2T} W}\right] = \mathbb{E}\left[e^{\int_{-T}^{T}\int_{-T}^{T} W}\right]$$

- Numerator:

$$(1\!\!1, e^{-TH} 1\!\!1)^2 \le \mathbb{E}\left[e^{\int_{-T}^{T}\int_{-T}^{T} -2\int_{-T}^{0}\int_0^T W}\right]$$

$$\gamma \le \lim_{T\to\infty} \frac{\mathbb{E}\left[e^{\int_{-T}^{T}\int_{-T}^{T} -2\int_{-T}^{0}\int_0^T W}\right]}{\mathbb{E}\left[e^{\int_{-T}^{T}\int_{-T}^{T} W}\right]} = \lim_{T\to\infty} \mathbb{E}_{\mu_T}\left[e^{-2\int_{-T}^{0}\int_0^T W}\right]$$

Proof of THEOREM:

$$\mathbb{E}_{\mu_T}\left[e^{-2\int_{-T}^{0}\int_0^T W}\right] = \mathbb{E}_{\mu_T}\left[1\!\!1_{A_T}\cdots\right] + \mathbb{E}_{\mu_T}\left[1\!\!1_{A_T^c}\cdots\right]$$

where

- $A_T = \{(x,w) | \sup_{|s|\le T} |X_s(w)| \le T^\lambda, X_0(w) = x\}$
- $\frac{1}{1+\delta} < \lambda < 1$ ($\delta > 0$ **is needed.**)

$$\mathbb{E}_{\mu_T}\left[1\!\!1_{A_T} e^{-\int_{-T}^0\int_0^T W}\right]$$

$$W(X,Y,|t|) = \frac{1}{2}(\rho(\cdot - X), \omega^{-1}e^{-|t|\omega}\rho(\cdot - Y))$$

Lemma (GHPS)

Suppose $m(x) \le a\langle x\rangle^{-1-\varepsilon}$. Then
$C_1 e^{-C_2 t\omega_\infty^2}(x,y) \le e^{-t\omega^2}(x,y) \le C_3 e^{-C_4 t\omega_\infty^2}(x,y)$ where $\omega_\infty^2 = -\Delta$

- $C_1 W_\infty(x,y,C_2|t|) \le W(x,t,|t|) \le C_3 W_\infty(x,y,C_4|t|)$
- $W_\infty(X,Y,|t|) = \frac{1}{4\pi^2}\int \frac{\rho(x)\rho(y)}{|x-y+X-Y|^2+t^2}dxdy$

$\Longrightarrow 1\!\!1_{A_T}\int_{-T}^0\int_0^T W \ge$
$1\!\!1_{A_T}\text{cons.}\int\int dxdy\rho(x)\rho(y)\log\left\{\frac{8T^{2\lambda}+2|x-y|^2+cT^2}{8T^{2\lambda}+2|x-y|^2}\right\} \to \infty$ as
$T \to \infty$.

$$\mathbb{E}_{\mu_T}\left[1\!\!1_{A_T^c} e^{-\int_{-T}^0 \int_0^T W}\right]$$

- $\mathbb{E}_{\mu_T}\left[1\!\!1_{A_T^c} e^{-\int_{-T}^0 \int_0^T W}\right] \leq Ce^{TC}\mathbb{E}\left[A_T^c\right]$
- Exponential decay $\Phi_{\rm p}(x) \leq C_0 e^{-C_1|x|^{\delta+1}}$

Lemma (Kipnis-Varadhan)

$\mathbb{E}\left[A_T^c\right] \leq T^{-\lambda}(a+bT)^{1/2}e^{-T^{\lambda(\delta+1)}}$

- $\lambda(\delta+1) > 1 \Longrightarrow \mathbb{E}_{\mu_T}\left[1\!\!1_{A_T^c}\cdots\right] \to 0$ ($T \to \infty$).

UV problem

Standard Nelson model Let

- (UV cutoff) $\hat{\rho}_\Lambda(k) = \begin{cases} (2\pi)^{-3/2} & |k| \leq \Lambda \\ 0 & |k| > \Lambda \end{cases}$
- (Renormalization) $E_\Lambda = -\frac{1}{2}(2\pi)^{-3} \int \frac{|\mathbb{1}_{|k|<\Lambda}}{|k|(|k|^2/2 + |k|)} dk$
- $\lim_{\Lambda\to\infty} \hat{\rho}_\Lambda(k) = (2\pi)^{-3/2}$

THEOREM (E. Nelson 1964) Removal of UV cutoff

There exits H_∞ st $s - \lim_{\Lambda\to\infty} e^{-t(H_\Lambda - E_\Lambda)} = e^{-tH_\infty}$

Nelson model on Lorentzian mfd

- (UV cutoff) $\rho_\Lambda(\cdot) = \Lambda^3 \rho(\Lambda \cdot)$
- (Renormalization) $E_\Lambda(X) =$
$$-\frac{1}{2}(2\pi)^{-3} \int (h_0(X,\xi)+1)^{-1/2} \frac{K(X,\xi)}{(K(X,\xi)+1)^2} |\hat{\rho}(\xi/\Lambda)^2| d\xi$$
- **Symbols**

$$h_0(X,\xi) = \sum \xi_i a^{ij}(X)\xi_j \quad K(X,\xi) = \sum \xi_i A^{ij}(X)\xi_j$$

- $\rho_\Lambda(x-X) \to \delta(x-X) \int \rho(y)dy$

THEOREM (GHPS) Removal of UV cutoff

There exists a self-adjoint operator $H_{\rm ren}$ bounded from below such that $s - \lim_{\Lambda\to\infty} e^{-t(H_\Lambda - E_\Lambda(X))} \to e^{-tH_{\rm ren}}$.

Concluding Remarks

- Critical ratio $\langle x\rangle^{-1}$

$$\begin{array}{lllll} a\langle x\rangle^{-1} \leq & m(x) & & & H \text{ has a ground state} \\ & m(x) & \leq a\langle x\rangle^{-1-\varepsilon} & & H \text{ has no ground state.} \end{array}$$

- Condition $V(x) \geq \langle x\rangle^{2\delta} - \varepsilon$ can be changed to **binding condition** due to Griesemer-Lieb-Loss (InvMath 01), which include Coulomb potentials.
- The standard Nelson model without UV cutoff also has a ground state (Hirokawa-H.-Spohn, AdvMath 05). **However it is unknown the uniqueness of the ground state.** (Gubinelli-H.-Lőrinczi 2011).
- **Geometric characterization** of the existence and the absence of ground state of QFT model.

Thank You !