LOCALIZATION OF A RENORMALIZED HAMILTONIAN IN QFT BY A PATH MEASURE

FUMIO HIROSHIMA

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1. Renormalized Nelson Hamiltonian

In this talk we discuss the ground state of the renormalized Nelson Hamiltonian introduced by E. Nelson in 1964 [8] to consider the removal of cutoffs. The renormalized Nelson Hamiltonian describes a linear interaction between non-relativistic matters and spinless scalar mesons, where the non-relativistic matter is governed by Schrödinger operators. Firstly the Nelson Hamiltonian is defined by imposing ultraviolet cutoff functions and it can be realized as a self-adjoint operator acting in a Hilbert space given by

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F},$$

where $\mathcal{F}$ denotes a boson Fock space over $L^2(\mathbb{R}^3)$. Secondly subtracting a renormalization term from the Nelson Hamiltonian, we can define the renormalized Nelson Hamiltonian $H_{\rm ren}$, which has no cutoff functions. A crucial point is that $H_{\rm ren}$ is defined through a quadratic form and it can not be possible to see an explicit form of $H_{\rm ren}$ as an operator. Let

$$H_\Lambda = H_p \otimes \mathbb{1} + \mathbb{1} \otimes H_f + gH_i, \qquad g \in \mathbb{R},$$

where $H_p = -\frac{1}{2}\Delta + V$ denotes a Schrödinger operator in $L^2(\mathbb{R}^3)$, $H_f = \int \omega(k) a^\dagger(k) a(k) dk$ the free field Hamiltonian in $\mathcal{F}$ and H_i the interaction term defined by

$$H_i = \frac{1}{\sqrt{2}} \int \left(a^\dagger(k) e^{-ikx} \frac{\hat{\varphi}_{\kappa,\Lambda}(k)}{\sqrt{\omega(k)}} + a(k) e^{ikx} \frac{\hat{\varphi}_{\kappa,\Lambda}(k)}{\sqrt{\omega(k)}} \right) dk.$$

Here $\omega(k) = |k|$ denotes the dispersion relation and $\hat{\varphi}_{\kappa,\Lambda}$ the cutoff function given by

$$\hat{\varphi}_{\kappa,\Lambda}(k) = \begin{cases} 0 & |k| < \kappa \\ 1 & \kappa \le |k| \le \Lambda \\ 0 & |k| > \Lambda. \end{cases}$$

Here we fix $\kappa > 0$. We introduce assumptions on V. We assume that V is 3-dimensional Kato-decomposable potential, i.e., $V = V_+ + V_-$ and V_+ is local Kato-class and V_- Kato-class. We refer to see [6] for Kato-class. According to [8], we introduce the renormalization term defined by

$$E_\Lambda = -\frac{g^2}{2} \int_{\mathbb{R}^d} \frac{|\hat{\varphi}_{\kappa,\Lambda}(k)|^2}{\omega(k)} \beta(k) dk,$$

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where $\beta(k)$ describes a propagator given by $\beta(k) = \frac{1}{\omega(k)+\frac{|k|^2}{2}}$. We notice that $\lim_{\Lambda\to\infty} E_\Lambda = -\infty$. There exists a self-adjoint operator $H_{\rm ren}$ bounded below such that for any $T \geq 0$

$$\lim_{\kappa\to\infty} e^{-T(H_\Lambda - E_\Lambda)} = e^{-TH_{\rm ren}}. \tag{1.1}$$

Nelson [8] proved the convergence in (1.1) in the strong sense, it is however shown that this convergence is in the uniform sense in [1]. (1.1) is also shown in [3, 7, 2]. Let $(B_t)_{t\geq 0}$ be 3-dimensional Brownian motion on a probability space $(\Omega, \mathcal{B}, \mathcal{W}^x)$, where $\mathcal{W}^x(B_0 = x) = 1$. Let

$$U(k) = \int_0^T \frac{e^{-s\omega(k)}}{\sqrt{\omega(k)}} \mathbb{1}_{|k|\geq\kappa}\, e^{-ikB_s} ds, \quad \tilde{U}(k) = \int_0^T \frac{e^{-|T-s|\omega(k)}}{\sqrt{\omega(k)}} \mathbb{1}_{|k|\geq\kappa} e^{ikB_s} ds.$$

Both integrals are finite for arbitrary $\kappa \geq 0$ and $\mathbb{R}^d \ni k \neq 0$. Furthermore we can see that $U, \tilde{U} \in L^2(\mathbb{R}^3)$ almost surely. Hence both $a^\dagger(U)$ and $a(U)$ are well-defined closed operators almost surely. Let $A = e^{a^\dagger(U)} e^{-\frac{T}{2}H_f}$ and $\tilde{A} = e^{-\frac{T}{2}H_f} e^{a(\tilde{U})}$. Let $F, G \in \mathcal{H}$. Then we have

$$\left(F, e^{-TH_{\rm ren}} G\right)_{\mathcal{H}} = \int_{\mathbb{R}^d} dx \mathbb{E}^x \left[e^{-\int_0^T V(B_s)ds} e^{\frac{g^2}{2}S} \left(F(B_0), A\tilde{A}G(B_T)\right)_{\mathcal{F}} \right],$$

where phase factor S is given by

$$S = 2\int_0^T \left(\int_0^t \nabla\varphi_0(B_s - B_t, s-t) ds \right) dB_t - 2\int_0^T \varphi_0(B_s - B_T, s-T) ds$$

with $\varphi_0(X,t) = \int_{\mathbb{R}^d} \frac{e^{-ikX} e^{-|t|\omega(k)}}{2\omega(k)} \beta(k) dk$. This is shown by [3, 7].

2. Localization of the ground state

In the case of $\Lambda < \infty$ it is shown that the ground state of H_Λ exists and it is unique up to multiple constants. Using Feynman- Kac type formula mentioned above we can also show that the ground state of $H_{\rm ren}$ exists.

Theorem 2.1. *$H_{\rm ren}$ has the ground state and it is unique.*

Proof. Outline of a proof is as follows. The uniqueness is proven in [7]. The existence of the ground state is due to [4]. Let $G \subset \mathbb{R}$ be a bounded and open subset. Let $\tau_G(x) = \inf\{t > 0 | B_t + x \notin G\}$ be the exit time from G. In particular when $x \notin G, \tau_G(x) = 0$. Define the quadratic form $Q_t : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by

$$Q_t : \quad F \times G \mapsto \int_{\mathbb{R}^d} \mathbb{E}^x \left[\mathbb{1}_{\tau_G(x)\geq t}\, e^{\frac{g^2}{2}S}\, e^{-\int_0^t V(B_s)ds} \left(F(B_0), A\tilde{A}G(B_t)\right) \right] dx.$$

Thus it can be seen that there exists a self-adjoint operator H_G bounded from below such that $(F, e^{-tH_G}G) = Q_t(F,G)$. The self-adjoint operator H_G can be regarded as a self-adjoint operator on $L^2(G) \otimes \mathcal{F}$. Under the identifications $F \cong L^2(Q)$ and $L^2(G) \otimes \mathcal{F} \cong L^2(G \times Q)$, it can be checked that e^{-tH_G} is hypercontractive. Hence H_G must have the ground state in $L^2(G \times Q)$, since the measure of $G \times Q$ is finite and e^{-tH_∞} is hypercontractive. Let φ_G be the unique ground state of H_G and we extend φ_G to the vector on $L^2(\mathbb{R}^3) \otimes \mathcal{F}$ by zero-extension, i.e., $\tilde{\varphi}_G(x,\phi) = \begin{cases} \varphi_G(x,\phi) & (x,\phi) \in G \times Q \\ 0 & (x,\phi) \notin G \times Q \end{cases}$.

Let $\varphi_n = \tilde{\tilde{\varphi}}_{G_n}$ and $G_n \uparrow \mathbb{R}^d$. Then it can be seen that $\{\varphi_n\}$ is Cauchy and $\lim_{n\to\Lambda} \varphi_n$ exists for each $\Lambda < \infty$. The limit is denoted by φ_Λ. Suppose that $\Lambda \to \infty$. Hence it can be also shown that $\{\varphi_\Lambda\}$ is compact, which implies that $\{\varphi_\Lambda\}$ includes a convergent subsequence $\varphi_{\Lambda'}$. Then $\lim_{\Lambda'\to\infty} \varphi_{\Lambda'} = \varphi$ is the ground state of $H_{\rm ren}$. □

Let $\Phi_{\rm g}$ be the ground state.. We want to estimate $(\Phi_{\rm g}, O\Phi_{\rm g})$ for some self-adjoint operator O. Typical examples of O are $e^{+\beta N}$, $e^{+\beta\phi(f)^2}$.

Theorem 2.2. *We have*

(1): $\|e^{\beta N}\Phi_{\rm g}\| < \infty$ *for any* $\beta > 0$*;*
(2): $\|e^{\beta\phi(f)^2}\Phi_{\rm g}\| < \infty$ *for any* $\beta < \frac{1}{\|f\|^2}$*;*
(3): $\lim_{\beta\uparrow\frac{1}{\|f\|^2}} \|e^{\beta\phi(f)^2}\Phi_{\rm g}\| = \infty$.

Proof. This is due to [4]. □

Furthermore we estimate the spatial decay of $\|\Phi_{\rm g}(x)\|_{\mathcal{F}}$ as $|x| \to \infty$.

Theorem 2.3. *Assume that V obeys the lower bound*

$$V(x) \geq \frac{a^2}{2}|x|^{2n} - b, \quad |x| \geq R$$

for some $a, b, n > 0$ and $R > 0$ so that $a^2R^{2n}/2 - b > 0$. Then for all $\varepsilon > 0$, $\delta > 0$,

$$C_{\varepsilon,\delta,1}e^{-\sqrt{\triangle(1,\varepsilon)}\frac{a}{n+1}|x|^{n+1}} \leq \|\Phi_{\rm b}(x)\|_{\mathcal{F}} \leq c_\varepsilon e^{-(1-\varepsilon)\frac{a}{n+1}|x|^{n+1}}.$$

Here $\triangle(1,\varepsilon) > 1$ is a constant.

Proof. The lower bound follows from [5] and the upper bound from [4]. □

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KYUSHU UNIVERSITY, MOTOOKA744, NISHIKU, FUKUOKA, JAPAN
E-mail address: hiroshima@math.kyushu-u.ac.jp