Report problems

(Infinite dimensional analysis=無限解析大意=金 2 時限)

※ Solve at least 4 problems below and post it by 8/11 at the JIMUSHITSU.

※ Copys of problems are put in the JIMUSHITSU

1. Let $\mathscr{H}$ be a Hilbert space. Let $T : \mathscr{H} \to \mathscr{H}$ be a densely defined bounded operator. I.e., $D(T) \subset \mathscr{H}$ is dense and there exists c such that $\|Tf\| \leq c\|f\|$ for all $f \in D(T)$. Show that there exists unique extension $\bar{T} \supset T$ such that $\bar{T}$ is bounded and $\|\bar{T}\| = \|T\|$.

2. Let $T : \mathscr{H} \to \mathscr{H}$ be a densely defined operator.

   (a) Let T be a closed operator. Then show that $T^{**} = T$.

   (b) Let T be a closable operator. Then $(\bar{T})^* = T^*$

3. Let $T : \mathscr{H} \to \mathscr{H}$ be a bounded operator such that $\|T\| < 1$. Then show that $I - A$ is bijective, and $\sum_{n=0}^{\infty} A^n$ uniformly converges to $(I - A)^{-1}$.

4. Let $T : \mathscr{H} \to \mathscr{H}$ be a symmetric operator. Then show that T is closable and $\bar{T}$ is also symmetric. I.e., $\bar{T} \subset (\bar{T})^*$.

5. Let $T : \mathscr{H} \to \mathscr{H}$ be a closed operator.

   (a) Show that $\mathrm{Ker}T$ is a closed subspace.

   (b) Show that $\mathscr{H} = \mathrm{Ker}T \oplus \overline{\mathrm{Ran}(T^*)}$.

6. Let $F : \mathbb{R}^d \to \mathbb{R}$ be a Lebesgue measurable function such that $F \in L^2_{loc}(\mathbb{R}^d)$. Then show that M_F with the domain $D(M_F) = C_0^{\infty}(\mathbb{R}^d)$ is essentially self-adjoint on the Hilbert space $L^2(\mathbb{R}^d)$.

7. Let $F \in \mathscr{H}^*$ (the set of linear functionals $\mathscr{H} \to \mathbb{C}$). Then show that there exists $\Phi_F \in \mathscr{H}$ such that $F(\Phi) = (\Phi_F, \Phi)\ \forall \Phi \in \mathscr{H}$, $\|F\| = \|\Phi_F\|$ and it is unique. item Let $A : \mathscr{H} \to \mathscr{H}$ be self-adjoint and B be symmetric. Suppose that $D(A) \subset D(B)$ and $\|Bf\| \leq a\|Af\| + b\|f\|$ with $0 \leq a < 1$ and $b \geq 0$. Then show that $A + B$ is self-adjoint on $D(A)$.

8. Let $V(x) = -1/|x|$. It is well known that [1]

$$\int_{\mathbb{R}^d} |f(x)|^2/|x|^2 dx \leq 4/(d-2)^2 \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx.$$

Let $d = 3$. Using this inequality show that $D(V) \supset D(-\Delta)$ and for arbitrary $\varepsilon > 0$ there exists b_ε such that $\|Vf\| \leq \varepsilon\| - \Delta f\| + b_\varepsilon \|f\|$ for any $f \in D(-\Delta)$

9. Let $A : \mathscr{H} \to \mathscr{H}$ be a self-adjoint operator. Let $U_t = e^{itA}$, $t \in \mathbb{R}$.

   (a) Let $f \in D(A)$. Then show that $U_t f$ is strongly differentiable with respect to t.

   (b) Let $U_t f$ is strongly differentiable with respect to t. Then show that $f \in D(A)$.

10. Let $P, Q : \mathscr{H} \to \mathscr{H}$ be a self-adjoint operator satisfying the Weyl relation. Then there exists $\mathscr{H}_m$ and $U_m : \mathscr{H}_m \to L(\mathbb{R})$ such that $\mathscr{H} = \oplus_m \mathscr{H}_m$ and $U_m^{-1} Q U_m = \hat{x}$ and $U_m^{-1} P U_m = \hat{p}$.

11. Let $P_1 = \hat{p}_j - qA_j(x)$, $x = (x_1, x_2) \in \mathbb{R}^2$, where $A_1(x) = \dfrac{-1}{2\pi} \dfrac{x_2}{|x|^2} b$ and $A_2(x) = \dfrac{1}{2\pi} \dfrac{x_1}{|x|^2} b$.

    (a) Show that P_1 is essentially self-adjoint on $C_0^\infty(\mathbb{R}) \hat{\otimes} C_0^\infty(\mathbb{R} \setminus \{0\})$, and P_2 on $C_0^\infty(\mathbb{R} \setminus \{0\}) \hat{\otimes} C_0^\infty(\mathbb{R})$.

    (b) The closure of P_j is denoted by $\bar{P}_j$. Show that $e^{is\bar{P}_1} e^{it\bar{P}_2} = e^{-iq\Phi_{st}} e^{it\bar{P}_2} e^{is\bar{P}_1}$, where $\Phi_{st} = \displaystyle\int_{C_{st}} A(x) \cdot dx$. Here C_{st} is the rectangle in $\mathbb{R}^2$, which is defined in the lecture and $\int_{C_{st}} A(x) \cdot dx$ denotes the line integral on C_{st}

[1] Hardy's inequality