



## § 1 Topology & topological spaces

- ルーチニ 1904年 関数空間 (= metric)
- ハウスドルフ 1914年 開集合の抽象的定義  
(20世紀)

Def 1.1  $X(\neq \emptyset)$  set,  $\mathcal{U} : X$  の subsets の族

(1)  $\emptyset, X \in \mathcal{U}$  (2)  $A_1, A_2 \in \mathcal{U} \Rightarrow A_1 \cap A_2 \in \mathcal{U}$

(3) 任意の集合族  $\{A_\lambda\}_{\lambda \in \Lambda} = \text{def} \cup_{\lambda \in \Lambda} A_\lambda \in \mathcal{U}$

$\mathcal{U}$  を  $X$  上の 位相 (topology) を定める いふ

$(X, \mathcal{U})$  topological space

$A \in \mathcal{U}$  open set,  $\mathcal{U}$  system of open sets  
(開集合系)

Def 1.2  $(X, \mathcal{U})$  topological space.

$F \subset X$  且  $F^c = \{x \in X ; x \notin F\} \in \mathcal{U}$  とき

$F$  closed set いふ。



$(X, \mathcal{U}), (X, \mathcal{V})$  topological spaces

$\mathcal{U} \subset \mathcal{V} \Rightarrow \mathcal{U}$  は  $\mathcal{V}$  より弱い topology

$\mathcal{U} \supset \mathcal{V} \Rightarrow \mathcal{U}$  は  $\mathcal{V}$  より強い topology

$\mathcal{U} = \mathcal{V} \Rightarrow \mathcal{U}$  と  $\mathcal{V}$  は同じ topology.

Prop 1.3  $\{U_\lambda; \lambda \in \Lambda\}$  は  $X$  上の top の定義

$\Rightarrow \mathcal{U} = \bigcap \{U_\lambda; \lambda \in \Lambda\}$  は  $X$  上の top.

①  $\emptyset, X, \emptyset \in \mathcal{U}$  ok

②  $A_1, A_2 \in \mathcal{U} \therefore A_1, A_2 \in {}^\Delta U_\lambda$

$A_1 \cap A_2 \in {}^\Delta U_\lambda \therefore A_1 \cap A_2 \in \mathcal{U}$

③  $A_m, m \in M \in \mathcal{U} \therefore A_m, m \in M \in {}^\Delta U_\lambda$

$\bigcup_{m \in M} A_m \in {}^\Delta U_\lambda \therefore \bigcup_{m \in M} A_m \in \mathcal{U}$

$P(X) : X$  のべき集合 (部分集合全体)

記号で書けば  $P(X) = \boxed{\bigcup_{A \subset X} A} \{B; B \subset X\}$

$P(X)$  は topology になる: 自由.

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Prop 1.4  $\theta \in P(x) \ni U, \theta \in \sum_{\text{Kyushu Univ}}^f$  より含む最小の top

$$\therefore U = \bigcap \{ V : \theta \subset V \text{ 且 } V \text{ は } X \text{ 上の top} \}$$

$U$  は topology 1, 2, 3 ( $U \neq \emptyset$ )

$\theta \in U' \quad U'$  topology 2, 3 と  $U' \subset U$

$\therefore U$  は 最小の topology.

## §2 topological space or 1511

①  $(X, P(X))$  は topological sp.

discrete top (离散的位相)

②  $(X, \{\emptyset, X\})$  は topological sp

trivial top  
(密着位相)

③ metric space

Def 2.1  $d : X \times X \rightarrow \mathbb{R}$  metric funct.

$$(1) d(x, y) \geq 0, d(x, y) = 0 \Leftrightarrow x = y$$

$$(2) d(x, y) = d(y, x)$$

$$(3) d(x, z) \leq d(x, y) + d(y, z)$$

$(X, d)$  metric space

1511  $X = \mathbb{R}^d$ ,  $d_1(x, y) = \|x - y\|$

$$d_2(x, y) = \max |x_i - y_i|$$

$$d_3(x, y) = \sum |x_i - y_i|$$

$$X = C([0, 1]) \quad \|f - g\|_p \quad 1 \leq p \leq \infty.$$



$(X, d)$  metric space,  $x \in U, \varepsilon > 0$

$$U(x, \varepsilon) = \{y \in X; d(y, x) < \varepsilon\}$$

$$\bar{U}(x, \varepsilon) = \{y \in X; d(y, x) \leq \varepsilon\}$$

Def 2.2  $(X, d)$  metric space

$$U_d \ni A \Leftrightarrow \forall x \in A, \exists \varepsilon_x > 0 \text{ s.t. } U(x, \varepsilon_x) \subset A$$

Prop 2.3  $(X, d)$  metric space

$(X, U_d)$  topological space

$$\textcircled{1} \quad \textcircled{1} \quad x, \emptyset \in U_d$$

$$\textcircled{2} \quad A_1, A_2 \in U_d \quad A_1 \cap A_2$$

$$\therefore A_1 \cap A_2 \ni x \quad \exists U(x, \varepsilon_1) \ni x \quad \exists U(x, \varepsilon_2) \ni x$$

$$U(x, \varepsilon_1) \cap U(x, \varepsilon_2) \subset A_1 \cap A_2$$

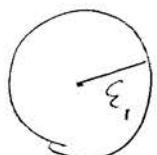
$$\varepsilon_1 < \varepsilon_2 \Rightarrow U(x, \varepsilon_1) \subset U(x, \varepsilon_2)$$

$$\therefore A_1 \cap A_2 \in U_d$$

$$\textcircled{3} \quad A_\lambda, \lambda \in \Lambda, \in U_d$$

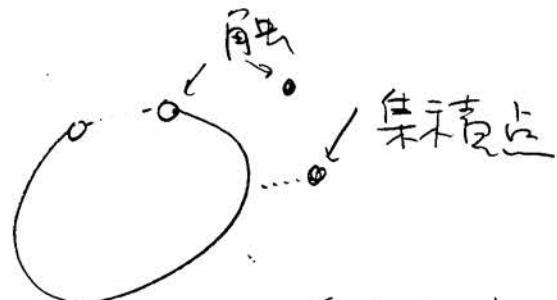
$$A = \bigcup_{\lambda} A_\lambda \ni x \quad \therefore x \in A_\lambda \quad U(x, \varepsilon) \subset A_\lambda$$

$$\therefore U(x, \varepsilon) \subset A$$



$(X, U_d)$  : metric space がもとで induce される top sp.

- $x \in X$  が 角点 (point)  $\Leftrightarrow U(x, \varepsilon) \cap A \neq \emptyset$
- $x \in X$  が 内点 (interior point)  $\Leftrightarrow U(x, \varepsilon) \subset A$
- $x \in X$  が 集積点 (accumulation point)  $\Leftrightarrow A \setminus \{x\}$  の 角点.



$$A^{\circ} = \{x \in A; \text{ 内点}\}, \quad \bar{A} = \{x; \text{ 角点}\}$$

$$\bar{A} \subset A \subset \bar{A}$$

Prop 2.4  $\bar{A}$  は  $A$  を 含むる 最大の open set

$\bar{A}$  は  $A$  を 含むる 最小の closed set

∴  $\bar{A}$  は open で ok  $\quad \bar{A} \in U_d$  :  $y \in \bar{A} \quad U(y, \varepsilon) \subset \bar{A}$

$U \subset A$  open とする.  $\therefore \forall x \in U$  は  $A$  の 内点.

$\therefore x \in \bar{A} \quad \therefore U \subset \bar{A}$

∴



$(\bar{A})^c \ni x \quad \therefore U(x, \varepsilon) \cap A = \emptyset$

$\therefore U(x, \varepsilon) \cap \bar{A} = \emptyset$   $\therefore (\bar{A})^c \cap \bar{A} = \emptyset$   $\therefore U(x, \varepsilon) \cap \bar{A} = \emptyset \quad \therefore \bar{A}$  は closed

$B \supset A$  closed とする.  $\forall x \in B^c$  は  $A$  の 角点

∴  $B^c \subset (\bar{A})^c$   $\therefore B \supset \bar{A}$

# §3 基 (base)

(2)-1

Def 3.1  $(X, \mathcal{U})$  top sp.  $\mathcal{U} \subset \mathcal{U}$  の base

$\Leftrightarrow \forall A \in \mathcal{U}$  は  $A = \bigcup_{\lambda} B_{\lambda}, B_{\lambda} \in \mathcal{U}$  と表さう。

Example (重要)  $(\mathbb{R}, \mathcal{U})$  Euclidean space

$\mathcal{U} = \{(a, b); a, b \in \mathbb{R}\}$  は base

$\circledcirc A \in \mathcal{U} \Leftrightarrow \forall x \in A, \exists \varepsilon > 0 \text{ で } x \in (x-\varepsilon, x+\varepsilon) \subset A$

$$(x-\varepsilon, x+\varepsilon) \in \mathcal{U} \quad \therefore \bigcup_{x \in A} U_x \subset A \subset \bigcup_{x \in A} U_x$$

Prop 3.2  $(X, \mathcal{U})$  top sp.  $\mathcal{U} \subset \mathcal{U}$

$\forall A \in \mathcal{U}, \forall x \in A \exists \varepsilon > 0 \exists B \in \mathcal{U} \text{ で } x \in B \subset A$

$\Leftrightarrow \mathcal{U}$  は base

$\circledcirc \Rightarrow \forall A \in \mathcal{U} \forall x \in A, \exists B \in \mathcal{U} \text{ で } x \in B \subset A \quad \because \bigcup_{x \in A} B_x = A$

$$\Leftarrow A = \bigcup_{\lambda} B_{\lambda}, \forall x \in A \exists \varepsilon > 0 \exists B_x \in \mathcal{U} \text{ で } x \in B_x \subset A$$

※証明

Prop 3.3  $(X, \mathcal{B})$   $X$  の  $\mathcal{B}$  subset の族で  $A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B}$   $\Rightarrow A \cap B \in \mathcal{B}$

$\mathcal{U} = \{\bigcup_{\lambda} B_{\lambda}; B_{\lambda} \in \mathcal{B}\}$  は top  $\mathcal{C}$  の base

$\circledcirc$  ①  $x, \phi \in \mathcal{U}$  ok

②  $A, B \in \mathcal{U} \Rightarrow A = \bigcup_{\lambda} B_{\lambda}, B = \bigcup_{\mu} C_{\mu} \Rightarrow A \cap B = \bigcup_{\lambda, \mu} B_{\lambda} \cap C_{\mu} \in \mathcal{U}$

③  $A_{\lambda} \in \mathcal{U} \Rightarrow A_{\lambda} = \bigcup_{\mu} B_{\lambda \mu} \Rightarrow \bigcup_{\lambda} A_{\lambda} = \bigcup_{\lambda, \mu} B_{\lambda \mu} \in \mathcal{U}$

$\mathcal{B}$  は  $\mathcal{U}$  の base にならぬ。

Prop 3.4  $X$  の  $\{U_{\lambda}; \lambda \in \Lambda\}$  top の族

$\{U_{\lambda}; \lambda \in \Lambda\}$  を含む最も弱い top  $\mathcal{U}$  とする。

$\Leftrightarrow \{A_{\lambda_1} \cap \dots \cap A_{\lambda_n}; A_{\lambda_j} \in U_{\lambda_j}, \lambda_1, \dots, \lambda_n \in \Lambda, n \geq 0\}$

は  $\mathcal{U}$  の base。

$$B = \left\{ \bigcup_{\lambda} B_{\lambda}; \quad B_{\lambda} = \bigcap_{m \in \lambda} A_m \right\} \quad (2)-2$$

有限個

は topology は  $T_2, T_3$  ① は Prop 3.3 と同じ。

さて  $B = U$

②  $B \subset U \Leftrightarrow B_{\lambda} \in U$  なので  $\bigcup_{\lambda} B_{\lambda} \in U$

$B \supset U \Leftrightarrow B \supset U_{\alpha} \Leftrightarrow B \supset \bigcup_{\lambda} U_{\lambda}$   
最小性より  $B \supset U$

$\therefore \forall A \in U$  は  $A = \bigcup_{\lambda} B_{\lambda}$  なので prop が従う。

Topology の作り方

base = H

①  $B$  を含む最も  $\supset$  top. ② Prop 3.3 のように  $\subset$

③  $(X, U)$  top sp  $X \subset X$   $U_Y = \{Y \cap A; A \in U\}$

$= \emptyset \subset U_Y$  は top は  $T_2, T_3$  各自確かめよ

$(Y, U_Y) \in (X, U)$  の 1 つ相空間  $\&$  部分空間

Ex  $(\mathbb{R}, U)$  Euclidean space

$(a, b] \notin U$

$([a, b], U_{[a, b]})$   $(a, b] \in U_{[a, b]}$

$\therefore (a, b] = (a, \infty) \bigcap [a, b]$ .  
↑ open

②-3

## §4 近傍系 (System of neighborhood)

開集合系  $\Leftrightarrow$  近傍系 を示す。

Def 4.1  $(X, \mathcal{U})$  top sp.  $x \in X, A \subset X$ .

①  $W \subset X$  が  $x$  の近傍  $\Leftrightarrow \exists A \in \mathcal{U}$  st  $x \in A \subset W$

②  $W \subset X$  が  $A$  の近傍  $\Leftrightarrow A \subset U$  st  $A \subset B \subset W$

直観的には  内点  $= \rightarrow$  集合

これは  $W$  近傍  $W \in \mathcal{U}$  が open nbh

$W^c \in \mathcal{U}$  が closed nbh

$N(x) = \{W \in X; W$  は  $x$  の近傍  $\} \text{ a system of nbh.}$

(註) Top がつくる  $N(x)$  が def 2 通り

$N(m)$  の algebraic な性質



Prop 4.3 ①  $V \in N(x) \Rightarrow x \in V$  ②  $V \in N(x), V \subset W \Rightarrow W \in N(x)$

③  $V_1, V_2 \in N(x) \Rightarrow V_1 \cap V_2 \in N(x)$



④  $\forall V \in N(x), \exists W \in N(x) \text{ st } \forall y \in W \exists z \in V \in N(y)$



① ①自明 ②  $x \in A \subset V \subset W$  が  $\rightarrow$  通り



③  $x \in A_1 \subset V_1 \Rightarrow x \in A_1 \cap A_2 \subset V_1 \cap V_2$   
 $\in A_2 \subset V_2$

④  $x \in A \subset V \in N(x) \text{ A=W とする } W \in N(x)$

$y \in W \subset V$  とする  $V \in N(y)$

Nbh :=  $\{x \in X \mid \exists U \text{ open s.t. } x \in U \subseteq A\}$  (2)-4

Prop 4.2  $(X, \mathcal{U})$  top sp.

$A \in \mathcal{U} \Leftrightarrow \forall x \in A \exists U \in \mathcal{U} \text{ s.t. } A \subseteq U$

i.e. open ref =  $\{U \in \mathcal{U} \mid \forall x \in A \exists U \in \mathcal{U} \text{ s.t. } x \in U\}$  集合の定義

$\circlearrowleft (\Rightarrow) \forall x \in A \Rightarrow x \in A \subseteq A \subseteq N(x)$

$(\Leftarrow) x \in \bigcup_{A \in \mathcal{U}} A \subseteq A \therefore \bigcup_{x \in A} A_x = A \in \mathcal{U}$

Prop 4.4 (今日の main proposition)

$X$  st  $\forall x \in X \exists U \in \mathcal{U} \text{ s.t. } (1) \sim (4)$  of Prop 4.3

$\Rightarrow \exists \mathcal{U} \text{ top of } X \text{ s.t. } \bigcup_{x \in X} N(x) = \mathcal{U}$

System of nbh ( $= \{N(x) \mid x \in X\}$ ) =  $\mathcal{U}$

I.e. topology  $\Rightarrow$  System of nbh  $\Rightarrow$  Prop 4.3 (1)  $\sim$  (4)

↑ 逆方向

$\circlearrowleft \mathcal{U} = \{\emptyset, A ; \forall x \in A \exists U \in \mathcal{U} \text{ s.t. } x \in U\}$  自然な設定 (Prop 4.2)

It topology  $\Rightarrow$  topology

$\circlearrowleft \text{① } \emptyset \in \mathcal{U} \text{ ok } x \in \emptyset \therefore \forall x \in X \exists U \in \mathcal{U} \text{ s.t. } x \in U$  ( $\emptyset \in \mathcal{U}$ )

$\therefore x \in \emptyset \text{ (②) } \emptyset \in \mathcal{U}$

$\circlearrowleft \text{② } A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$

$\therefore x \in A \cap B \Rightarrow \begin{cases} x \in A & \exists U \in \mathcal{U} \text{ s.t. } x \in U \\ x \in B & \exists V \in \mathcal{U} \text{ s.t. } x \in V \end{cases} \Rightarrow A \cap B \in \mathcal{U}$  (3)

$\circlearrowleft \text{③ } A_\lambda \in \mathcal{U} \Rightarrow \bigcup_{\lambda} A_\lambda \in \mathcal{U}$

$\therefore x \in \bigcup_{\lambda} A_\lambda \Rightarrow \exists \lambda \in \Lambda \text{ s.t. } x \in A_\lambda \Rightarrow \bigcup_{\lambda} A_\lambda \in \mathcal{U}$  (3)

$\mathcal{S}(n) = N(n) \cap \mathbb{C}$

(2)-5

①  $\mathcal{S}(n) \subset N(n)$

$V \in \mathcal{S}(n) \Rightarrow x \in \bigcup_{A \in \mathcal{U}} A \subset V \Rightarrow A \in N(n) \Rightarrow V \in N(x)$

②  $\mathcal{S}(n) \supset N(n)$  の裏付け

$V \in N(n) \Rightarrow x \in \bigcup_{A \in \mathcal{U}} A \subset V$  の裏付け

$V$  が直線に沿った点全部で構成

$A = \{y \in X; V \in N(y)\} \quad (\text{左端 } v \text{ 内部})$

$x \in A \subset V$

∴  $x \in A$  はOK

$A \subset V \because y \in A \Rightarrow V \in N(y) \Rightarrow y \in V$

$A \in \mathcal{U}$

③  $y \in A \Rightarrow V \in N(y) \rightarrow \exists W \in N(y) \text{ s.t. } \underbrace{\forall z \in W, V \in N(z)}_{W \subset A \cap \mathbb{C}}$

$\therefore W \in N(y) \text{ かつ } W \subset A \because A \in N(y) \quad (2)$

$\therefore A \in \mathcal{U}$

(最後 1つ - 意味)  $\exists \mathcal{U}'$

$\mathcal{U}' \ni A \Leftrightarrow \forall x \in A, A \in N(x) \Leftrightarrow A \in \mathcal{U}$

prop 4.2

def



## §5 基本近傍系

Def 5.1  $(X, \mathcal{U})$  top. sp  $\underline{N(x)}$  sys of nbh.

$V(x) \subset N(x)$  が 基本近傍系

$\Leftrightarrow \forall A \in N(x), \exists B \subset V(x) \text{ s.t. } B \subset A$

Ex.  $V(x) = \{A \in \mathcal{U}; x \in A\}$  は 基本近傍系

$\therefore \forall A \in N(x) \Leftrightarrow x \in B \subset N(x), B \in \mathcal{U}$ ,

Ex.  $B \in (X, \mathcal{U})$  の 基本近傍系

$B(x) = \{A \in B; x \in A\}$  は 基本近傍系

$\therefore \forall A \in N(x) \Leftrightarrow x \in B \subset N(x), B \in \mathcal{U}$

$$B = \bigcup_{\lambda} A_{\lambda}, A_{\lambda} \in \mathcal{B}$$

$$\Rightarrow A_m \text{ s.t. } x \in A_m$$

$$\therefore x \in A_m \subset N(x),$$

open set のみ がるる 基本近傍系を 基本開近傍系とす。



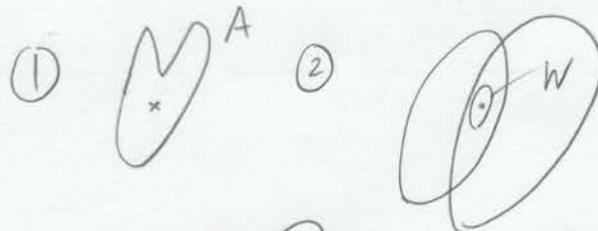
## 代数的な関係式

Prop 5.2  $(X, \mathcal{U})$  top sp.  $\mathcal{V}^{(n)}$  基本近傍系

$$\textcircled{1} A \in \mathcal{V}^{(n)} \Rightarrow x \in A$$

$$\textcircled{2} A, B \in \mathcal{V}^{(n)} \Rightarrow \exists W \in \mathcal{V}^{(m)} \text{ s.t. } A \cap B \supset W$$

$$\textcircled{3} A \in \mathcal{V}^{(n)} \Rightarrow \exists W \in \mathcal{V}^{(m)} \text{ s.t. } \forall y \in W \\ \exists A_y \subset A, A_y \in \mathcal{V}^{(n)}$$



$$\therefore \textcircled{1} \text{ trivial} \quad \textcircled{2} A \cap B \in \mathcal{N}^{(n)} : \exists W \in \mathcal{V}^{(n)} \\ \text{s.t. } W \subset A \cap B \quad (\text{ほぼ trivial に近い})$$

③ 性質 ④ を書く下では てな

$$A \in \mathcal{N}^{(n)} \Rightarrow \exists W \in \mathcal{V}^{(n)}, W \subset A \text{ s.t. } \forall y \in W, A \in \mathcal{N}^{(n)}$$

$$\text{ここで } \exists \tilde{W} \subset W, \tilde{W} \in \mathcal{V}^{(n)} \quad (+\mathcal{V}^{(n)} \text{ の def})$$

$$\forall y \in \tilde{W} \Rightarrow A \in \mathcal{N}^{(n)} \text{ すなはち } \exists A_y \subset A, A_y \in \mathcal{V}^{(n)} \quad (\mathcal{V}^{(n)} \text{ の def})$$

$\tilde{W}$  と  $W$  とは わは よい。



Prop 5.3  $X$  かつ  $V(x)$  は Prop 5.2 の ①~③ を満たす  
 ことを  $V(x)$  を 基本近傍系とする  $\text{Top}^{\exists} \mathcal{U}$ .

$$\textcircled{1} N(x) = \left\{ A \subset X ; \exists B \in V(x) \text{ 且 } B \subset A \right\}$$

は Prop 4.3(1)~(4) を満たす。

$\therefore N(x)$  を 近傍系 に して  $\mathcal{U}$  (Prop 4.4)

$\therefore V(x)$  は 基本近傍系 である。

$$\textcircled{1} A \in N(x) \rightarrow \exists_{x \in B \subset A} \quad : x \in A$$

$$\textcircled{2} A, B \in N(x) \rightarrow \begin{array}{l} \exists_{A' \subset A, B' \subset B, A' \cap B' \in V(x)} \\ A' \cap B' \subset A \cap B \quad : \exists_{W \subset A' \cap B' \subset A \cap B} \\ A \cap B \in N(x) \end{array}$$

$$\textcircled{3} A \in N(x), A \subset B \Rightarrow \exists_{A' \subset A \subset B} \quad : \begin{array}{l} A' \subset A \subset B \\ \exists_{B \in N(x)} \end{array}$$

$$\textcircled{4} A \in N(x) \rightarrow \exists_{B \in V(x) \text{ 且 } B \subset A} \quad : \begin{array}{l} B \in V(x) \\ \exists_{y \in B} \quad : \exists_{B_y \subset B, B_y \in V(y)} \end{array}$$

$\therefore W \subset B \subset A \Rightarrow W \subset A$

$\forall y \in W \rightarrow A \in N(y) \quad : \quad B_y \subset A, B_y \in V(y),$

④が得られる。

Def 5.4.  $(X, \mathcal{U})$  top sp.  $\forall x \in X$  が 高々可算個の元からなる  $V(x)$  をもつと、第一可算公理をみたす。

• 高々可算個の元からなる基をもつと、第二可算公理をみたす。

例  $\mathbb{R}$  Euclidean space

$$\mathcal{B} = \{(a, b); a, b \in \mathbb{R}\}$$

基

$$\mathcal{B} = \{(a, b); a, b \in \mathbb{Q}\}$$

これは 第二可算公理をみたす。

$V(x) = \{A \in \mathcal{B}; x \in A\}$  は 基本近傍系

Σ

$$\tilde{\mathcal{B}} = \{(a, b); a, b \in \mathbb{Q}\}$$

基

$\tilde{V}(x) = \{A \in \tilde{\mathcal{B}}; x \in A\}$  は 基本近傍系

第一可算公理をみたす。

$$(X, \mathcal{U}) \text{ top sp } \frac{A \subset X}{}$$

$$\overset{\circ}{A} = \bigcup \{B \in \mathcal{U}; A \subset B\} (\subset \mathcal{U})$$

$$\bar{A} = \bigcap \{F \subset X; F^c \in \mathcal{U}, F \supset A\}$$

$$\overset{\circ}{A} \subset A \subset \bar{A}$$

$\overset{\circ}{A}$  - A の 内部  
 $\bar{A}$  - A の 閉包

•  $A \ni x$  内点

•  $A \ni x$  角点

•  $\overline{A \setminus \{x\}} \ni x$  集積点

•  $A \setminus \text{集積点} \ni x$  孤立点

$\exists S \subset X$  st

$S = X$  かつ

$S$  が可算個のとき

$(X, \mathcal{U})$  は 可分

# §6 連続写像



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Def 6.1  $(X, \mathcal{U}), (Y, \mathcal{V})$  top. sp.

$f: X \rightarrow Y$  "cont."  $\Leftrightarrow f^{-1}(B) \in \mathcal{U} \quad \forall B \in \mathcal{V}$

(注)  $f^{-1}(B) = \{x \in X; f(x) \in B\}$

( $\frac{1}{2}$ )  $(Y, \mathcal{V}) = \text{Euclidean space or } \mathbb{R}$

$f$   $\in \mathbb{R}$ -valued cont. function

Def 6.2  $(X, \mathcal{U}), (Y, \mathcal{V})$  top sp

$f: X \rightarrow Y$  "  $x \in X$  は おなじ cont."

$\Leftrightarrow \forall B \in \mathcal{V} (f^{-1}(B) \text{ は } N_x(z) \subset f^{-1}(B))$

( $N_Y, N_X$  は 還従系 sys of nbh  $\in \mathcal{V}, \mathcal{U}$ )

(注) open set おなじ nbh  $\cup$  def 3

Prop 6.3  $(X, \mathcal{U}), (Y, \mathcal{V})$  top sp.

$f: X \rightarrow Y$  cont  $\Leftrightarrow f \circ h$   $\forall x \in X$  cont.

Def 6.4 (Homomorphism)
 $(X, \mathcal{U}) (Y, \mathcal{V}) \quad f: X \rightarrow Y \text{ bijection}$ 

$f, f^{-1}$  ともに cont.

$\Rightarrow X \approx Y$  は (3) 不同, 1-1 相同型, 同値不同  
homomorphic even

$X \cong Y$  と書く /  $f$  は 1-1 子像, 逆像集合子/が  
homomorphism は 1-1 保形

Def 6.5  $(X, \mathcal{U}) (Y, \mathcal{V})$  top sp 同値関係

$f: X \rightarrow Y$  且  $f(A) \in \mathcal{V} \quad \forall A \in \mathcal{U}$

$\Leftrightarrow f$  is open mapping

$f: X \rightarrow Y$  且  $f(A) \in \mathcal{V}^c \quad \forall A \in \mathcal{U}^c$

$\Leftrightarrow f$  is closed mapping

$(\mathcal{V}^c, \mathcal{U}^c$  は closed sets かつ)

Prop 6.6  $(X, \mathcal{U}) (Y, \mathcal{V})$  top. sp.

$f: X \rightarrow Y$  hom.

$\Rightarrow f$  is open & closed.

① bijective  $\Rightarrow (f^{-1})^{-1} = f \quad \forall A \in \mathcal{U}$

$\therefore f(A) = (f^{-1})^{-1}(A) \in \mathcal{V} \quad (f^{-1} \text{ is cont.})$

$\forall B \in \mathcal{U}^c \quad f(B) = f(X \setminus B^c) = Y \setminus f(B^c) \in \mathcal{V}^c$



①(⇒)  $\forall B \in N_Y(f(x))$ , すなはち  $B \in \mathcal{U}$  1=2712

$f^{-1}(B) \subset U$  かつ  $f^{-1}(B) \ni x \therefore f^{-1}(B) \in N_x(x)$

( $x \in \text{含む open set} \in N_x(x)$ )

一方で  $\exists B \in N_Y(f(x))$  1=2712

$f(x) \in B_0 \subset B, B_0 \in \mathcal{U} \therefore \underset{x \in B_0}{\underset{\cap}{\underset{\mathcal{U}}{\underset{\cap}{f^{-1}(B)}}} \subset f^{-1}(B)$

$\therefore f^{-1}(B) \in N_x(x)$

(⇐)  $B \in \mathcal{U}$  かつ  $f^{-1}(B) = \emptyset \Rightarrow f^{-1}(B) \in \mathcal{U}$

$f^{-1}(B) \neq \emptyset \therefore f^{-1}(B) \ni x$  かつ

$\therefore f(x) \in B \therefore B \in N_Y(f(x)) \Rightarrow B \in \mathcal{U}$

$\therefore f^{-1}(B) \in N_x(x)$

$\therefore x \in A \subset f^{-1}(B) \quad A \in \mathcal{U}$

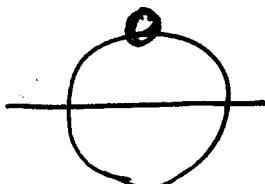
$\therefore \exists A_x \in \mathcal{U}$

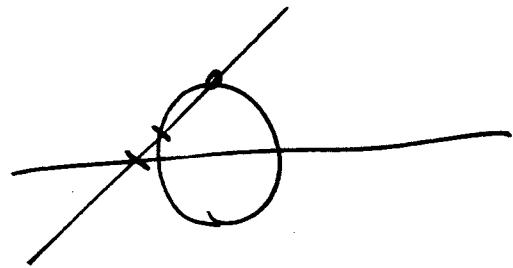
$\therefore f^{-1}(B) = \bigcup_x A_x$  1=2712-2713

$\therefore f^{-1}(B) \in \mathcal{U}$

Example

$$S^1 = \{(x, y) \in \mathbb{H}^2 ; x^2 + y^2 = 1\}$$

$$S^1 \setminus \{(0, 1)\} =$$




$$\exists f: S^1 \setminus \{(0, 1)\} \rightarrow \mathbb{R} \quad \text{homomorphism}$$

$$\therefore S^1 \setminus \{(0, 1)\} \cong \mathbb{R}$$

$$\exists g: \mathbb{R} \rightarrow (0, 1) \quad \text{homomorphism}$$

$$\therefore S^1 \setminus \{(0, 1)\} \cong \mathbb{H}^2 \cong (0, 1)$$

$$\cong \text{---} \cong \text{---}$$


Examples:

$\mathbb{R}$  = Euclidean sp

$$S^1 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$$

$\mathbb{R}^2$  (Euclidean space)  $\cong$  relative top  $\mathbb{E}^{n+3}$

$$f: \mathbb{R} \rightarrow S^1$$

$$x \mapsto \begin{matrix} \downarrow \\ \cos 2\pi x + i \sin 2\pi x \\ (\cos 2\pi x, \sin 2\pi x) \end{matrix} \quad (\mathbb{R}, \mathcal{U}), (S^1, \gamma)$$

①  $f$  is cont.

$$\mathbb{R}^2 \text{ a base } \{(a_1, b_1) \times (a_2, b_2); a_j, b_i \in \mathbb{R}\}$$

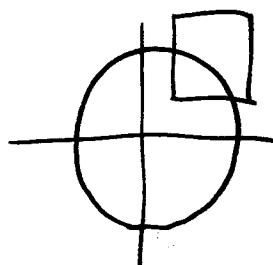
$$\forall B \in \gamma \quad \exists A \in \mathcal{U}$$



$$B = A \cap S^1 \quad A \in \mathcal{U}$$

$$= \left( \bigcup_{\lambda} I_{\lambda} \right) \cap S^1 \quad I_{\lambda} \text{ is open intervals}$$

$$= \bigcup_{\lambda} (I_{\lambda} \cap S^1)$$



$$\therefore f^{-1}(B) = \bigcup_{\lambda} f^{-1}(I_{\lambda} \cap S^1)$$

$$f^{-1}(I_{\lambda} \cup S^1) = \underbrace{\text{open set}}_{\infty \text{ sets}} \quad \text{open set}$$

$$\therefore f^{-1}(B) \in \mathcal{U}.$$

② f is open mapping:

$$\forall A \in \mathcal{U} \quad A = \bigcup_{\lambda} I_{\lambda} \quad \left( \left\{ (a, b) ; a, b \in \mathbb{R} \right\} \text{ base} \right)$$

$I_{\lambda}$  is open interval

$$\therefore f(A) = \bigcup_{\lambda} f(I_{\lambda})$$

$$f(I_{\lambda}) = \boxed{\text{intervals}} \cap S. \quad \text{由表可得} \quad \therefore f(I_{\lambda}) \text{ is open}$$

↑  
intervale

$$\therefore f(A) \in \mathcal{U}.$$

③ f is closed mapping てのとる

$$\sqrt{2}\mathbb{Z} = \left\{ \sqrt{2}z ; z \in \mathbb{Z} \right\} \text{ is closed set in } \mathbb{R}$$

$$f(\sqrt{2}\mathbb{Z}) = \bigcirc \quad \underline{\text{dense}} \quad (\text{既正則かつ密})$$

$$S' = f(\overline{\sqrt{2}\mathbb{Z}}) \neq f(\sqrt{2}\mathbb{Z}) \quad \therefore f(\sqrt{2}\mathbb{Z}) \text{ is closed in } S'$$

特に  $\mathbb{R} \not\subset S'$

Def 6.7  $(X, \mathcal{U}) (Y, \mathcal{V})$  top. spaces.

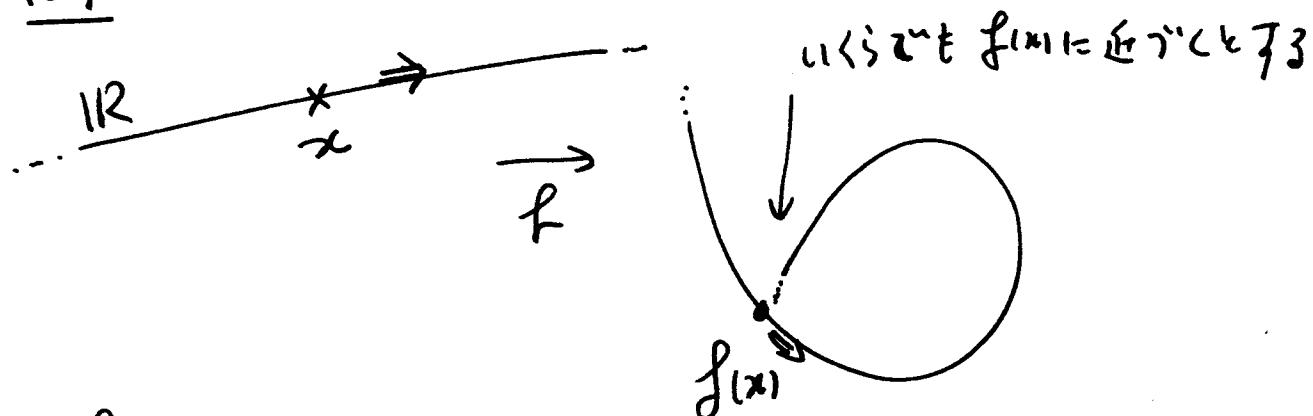
$A \subset Y$   $(A, \mathcal{V}_A)$  relative topology

$\exists h: (X, \mathcal{U}) \rightarrow (A, \mathcal{V}_A)$  homo.

$f: X \rightarrow Y$  s.t.  $f(x_n) = h(x_n)$   $x_n \in X$  & 実数

$f$  が 標の込み,  $f(x) \in$  標の込みの  $\cap_{n=1}^{\infty} f^{-1}(U_n)$ .

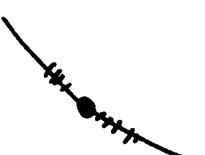
1511:



$f: 1:1, \text{onto}$   
bijective

$f: \text{cont.}$

$f': \text{cont. in } \mathbb{R} \text{ etc. etc.}$

$f(U(x, \epsilon)) =$   は  $f(x)$  の nbh で "1+2"。

$f(x)$  の nbh は 

## §7 誘導位相



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$f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$  は cont ならば top  $\mathcal{U}$

の構成法

Lem 7.1  $f^{-1}(\mathcal{V}) = \{f^{-1}(A); A \in \mathcal{V}\}$  は top  
( $f^{-1}(\emptyset) = \emptyset$  など)

① (1)  $f^{-1}(\mathcal{V}) \ni \emptyset, X$  は ok

(2)  $B_j \in f^{-1}(\mathcal{V})$  かつ  $B_1 \cap B_2 = f^{-1}(A_1) \cap f^{-1}(A_2)$   
 $= f^{-1}(A_1 \cap A_2)$

(3)  $B_j \in f^{-1}(\mathcal{V})$  かつ  $\bigcup B_j = \bigcup f^{-1}(A_j)$

Cor 7.2  $f: (X, f^{-1}(\mathcal{V})) \rightarrow (Y, \mathcal{V})$   $f^{-1}(\mathcal{V})$

は cont かつ  $f^{-1}(\mathcal{V})$  は  $f$  が cont ならば 最強の top

② cont. かつ ok

$f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$  cont ならば.

$\therefore f^{-1}(A) \in \mathcal{U} \forall A \in \mathcal{V} \therefore \mathcal{U} \supset f^{-1}(\mathcal{V})$ ,

Lem 7.3  $\mathcal{V} = \{A \in 2^Y; f^{-1}(A) \in \mathcal{U}\}$   
( $f^{-1}(\emptyset) = \emptyset$  など)

( $\mathcal{V}$  は top) は

③ (1)  $\emptyset \rightarrow \emptyset, X$  は ok.

(2)  $A_j \in \mathcal{V}$  かつ  $f^{-1}(A_1 \cap A_2) = f^{-1}(A_1) \cap f^{-1}(A_2) \in \mathcal{U}$

(3)  $A_j \in \mathcal{V}$  かつ  $f^{-1}(\bigcup A_j) = \bigcup f^{-1}(A_j) \in \mathcal{U}$ .

Cor 7.4  $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$   $\mathcal{V}$  は Lem 7.3

は cont かつ  $\mathcal{V}$  は  $f$  が cont ならば 最強の top

④ cont かつ ok.  $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V}')$  cont ならば

$\exists a \in f^{-1}(A') \in \mathcal{U} \forall A' \in \mathcal{V}' \therefore \mathcal{V}' \subset \mathcal{V}$  "

Def 7.5

(1)  $f: X \rightarrow (Y, \tau)$   $\forall v \in Y \exists x \in f^{-1}(v) \in$

$f^{-1}(v)$  は  $y$  の  $\tau$  の下で開拓

(2)  $f: (X, \tau) \rightarrow Y$   $\forall v \in Y \exists$

$x \in f^{-1}(v)$   $\tau$  の下で開拓

① top spaces  $\Rightarrow$  cont map.

② cont map  $\Rightarrow$  topology

直積位相と直積空間

商位相と商空間

Lem 7.6 Base (= 2<sup>n+2</sup>)

$f: X \rightarrow (Y, \tau_\lambda)$   $\lambda \in \Lambda \subset$  全て cont

$\exists \{f_\lambda^{-1}(U_\lambda)\}_{\lambda \in \Lambda}$   $\tau_\lambda$  の基底,  $U_\lambda$  は  $\tau$  の基底

$\left\{ f_{\lambda_1}^{-1}(A_{\lambda_1}) \cap \dots \cap f_{\lambda_n}^{-1}(A_{\lambda_n}); A_{\lambda_n} \in \mathcal{U}_{\lambda_n}, \lambda_n \in \Lambda, n \in \mathbb{N} \right\}$

定義

∴  $\{f_\lambda(u_\lambda), \lambda \in \Lambda\}$  を全て含む最弱の

$\exists$  top  $U$  使得ばし  $f: (X, U) \rightarrow (Y_\lambda, U_\lambda)$

で  $f$  は  $\forall \lambda \in \Lambda$  cont である

Prop 3.4 により

$$\left\{ f_{\lambda_1}^{-1}(A_{\lambda_1}) \cap \dots \cap f_{\lambda_n}^{-1}(A_{\lambda_n}); \lambda_1, \dots, \lambda_n \in \Lambda, A_{\lambda_1} \in \mathcal{U}_{\lambda_1} \right\}$$

$n \geq 1$

は  $U$  の base ( $\vdash$  す)

Def 7.7 直積位相と直積空間

$$\{(X_\lambda, \mathcal{U}_\lambda)\}_{\lambda \in \Lambda} \text{ top SP } \lambda \in \Lambda.$$

$$X = \prod_{\lambda} X_\lambda = \left\{ \{x_\lambda\}_{\lambda \in \Lambda}; x_\lambda \in X_\lambda \right\}$$

$$P_\lambda: X \rightarrow X_\lambda \text{ projection } P_\lambda(\{x_\lambda\}) = x_\lambda$$

$\forall P_\lambda \in \text{cont} \vdash$  最弱な位相で

直積位相  $\vdash$   $(X, U) = \prod_{\lambda} (X_\lambda, \mathcal{U}_\lambda)$  を表す。

Rem  $(X, U)$  の base  $B$  は

for  $\lambda \neq \lambda_j$  ときは

$$P_{\lambda_1}^{-1}(A_{\lambda_1}) \cap \dots \cap P_{\lambda_n}^{-1}(A_{\lambda_n}) = \prod_{\lambda \in \Lambda} A_\lambda \left( \begin{array}{l} A_\lambda = X_\lambda \\ \lambda = \lambda_j \end{array} \right)$$

となる  $\lambda \neq \lambda_j$  のとき  $A_\lambda = X_\lambda$



15'1  $(\mathbb{R}^1, \mathcal{U})$  Euclidean space

$$(\mathbb{R}^n, \mathcal{U}^n) = \prod^n (\mathbb{R}^1, \mathcal{U}^1)$$

- $\mathcal{U}^n$  a base on  $\prod_{j=1}^n (a_j, b_j)$   $t^n \in \mathbb{N}_0$ .
- $\rightarrow$  To show a top space a base exist

$$\prod^n A_j \quad A_j \in \mathcal{U}^1 \quad t^n \in \mathbb{N}_0$$

$$A_j = \bigcup_{\lambda} (a_j^\lambda, b_j^\lambda) \text{ 表示}.$$

$$\therefore \prod_j \bigcup_{\lambda} (a_j^\lambda, b_j^\lambda) = \bigcup_{\lambda} \prod_j (a_j^\lambda, b_j^\lambda)$$

$\prod_j (a_j, b_j)$  is To show a top a base ( $\approx \mathbb{N}_0$ )

$$\therefore (\mathbb{R}^n, \mathcal{U}^n) = \prod^n (\mathbb{R}^1, \mathcal{U}^1).$$

# 商位相と商空間

Defn.  $X$  set,  $\sim$  同値関係  $X/\sim = Y$

$\pi: (X, \mathcal{U}) \rightarrow Y$  が cont ならば 最強の位相

$X/\sim$  上の商位相  $(quotient\ topology)$   $\mathcal{U}_{\sim}$

$(X/\sim, \mathcal{U}_{\sim})$  商空間

I.e.  
 $A \in \mathcal{U}_{\sim} \Leftrightarrow \bar{\pi}(A) \in \mathcal{U}$   
 $\bar{\pi}(A) = \{x \in X; \pi(x) \in A\}$

Prop 7.9  $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$  cont.

かつ  $x \sim x' \Rightarrow f(x) = f(x')$

二番目  $\bar{f}: X/\sim \rightarrow Y$  if well-def  
 $[x] \mapsto f(x)$  かつ cont

$$(X, \mathcal{U}) \xrightarrow{f} (Y, \mathcal{V})$$

$$\begin{array}{ccc} & \downarrow \pi & \\ & G & \nearrow \bar{f} \\ (X/\sim, \mathcal{U}_{\sim}) & & \end{array} \quad \therefore \bar{f} \circ \pi = f$$

$$\textcircled{1} \quad \bar{f}([x]) = f(x)$$

$$\bar{f}([y]) = f(y)$$

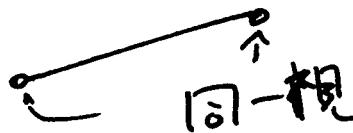
$$[x] = [y] \Leftrightarrow x \sim y \therefore f(x) = f(y)$$

$\exists$  a cont.  $\bar{f}$  s.t.  $\bar{f}^{-1}(A) \in U_n$  となる  $n$

$$\bar{f}^{-1}(A) = (\underline{f \circ \pi})^{-1}(A) = \pi^{-1} \circ \underbrace{\bar{f}^{-1}(A)}_{\in U_n} \in U_n,$$

15')  $I = [0, 1] \subset \mathbb{R}$  relative top.

$$x \sim y = \begin{cases} x = y, & x, y \in (0, 1) \\ |(x, y)| = (0, 1) \text{ or } (1, 0) \end{cases}$$



$$\begin{array}{ccc} I & \xrightarrow{f} & S^1 \\ \pi \downarrow & \nearrow \bar{f} & \\ I/\sim & & \end{array}$$

$$f(z) = \frac{\cos 2\pi z + i \sin 2\pi z}{2}$$

$$\frac{\bar{f} \circ \pi^{-1}}{\text{① cont.}} : x \sim x' \Rightarrow f(x) = f(x')$$

②  $\bar{f}$  is 1:1, onto,

$$\bar{f}([x]) = \bar{f}([y]) \therefore x \sim y$$

$$\bar{f}(x) \quad \bar{f}(y) \therefore [x] = [y]$$

③  $\bar{f}$  is open mapping

$\bar{f}$  is open mapping  $\Leftrightarrow \text{①}$

$B \in \mathcal{U}_\sim \Rightarrow \bar{f}(B) \subset S^1$  the open unit disk

$\therefore \bar{f}$  is open mapping  $\Leftrightarrow \bar{f}^{-1}$  is cont.

$B \in \mathcal{U}_\sim \Leftrightarrow A = \pi^*(B) \subset I$  the open

$$A = A' \cup A''$$

$$A'' \subset (0, 1), A' \ni 0 \text{ or } 1 \quad (= \text{not ctg.})$$

$$\pi^*(A' \cup A'') = \pi^*(A') \cup \pi^*(A'')$$

$$\bar{f}(\pi(A' \cup A'')) = \bar{f}(A') \cup \bar{f}(A'')$$

①  $\bar{f}(A'')$  is  $S^1$  a open set  $\Leftrightarrow$   $A'' \subset (0, 1)$

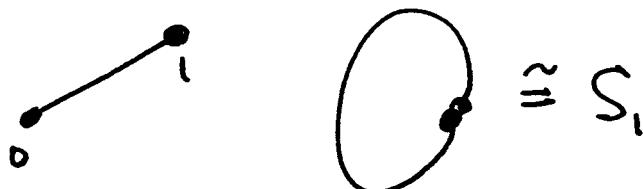
②  $A' \ni 0 \text{ or } 1 \Leftrightarrow A' \ni \{0, 1\} \text{ is not } 3$

$$A' = [0, \varepsilon_1] \cup (\varepsilon_2, 1] \cup I \leftarrow \text{break } (a_1, b_1) \text{ at } I$$

$$\pi(A') = \pi([0, \varepsilon_1]) \cup \pi((\varepsilon_2, 1]) \cup \pi(I)$$

$$\bar{f}(\pi(A')) = f([0, \varepsilon_1]) \cup f((\varepsilon_2, 1)) \cup f(I)$$

is open //



$(X, \mathcal{U})$  top space,  $A \subset X$  subset ⑦-1

$x \sim y \Leftrightarrow x = y \text{ or } x, y \in A$

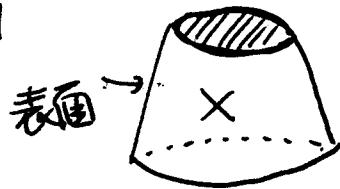
$X/\sim \in X/A$  と書く  $A \in -\text{点} \Rightarrow 3\text{の}\infty$

いわゆる商空間といふ。

例題 a (5)  $I = [0, 1]$ ,  $\sim I/\sim \cong S^1$

$\{\cdot\}$   $= A$  と思えば  $I/\sim = I/A \cong S^1$

例題



$$O = A$$

$$X/A \cong \Delta$$

例題  $\mathbb{R}^2 \setminus \{x_1 \leq 1\} = // O - S^1 = X$

$$X/S^1 \cong \mathbb{R}^2$$

例題  $[0, 1]/(0, 1) \cong \{0, 1\} = D$

$$(0, 1) \xrightarrow{f} D \quad f(0) = 0, \quad f(x) = 1 \quad (x \in (0, 1])$$

cont  $\mathbb{Z} \times \mathbb{Z}_m!$

$$\pi \downarrow \quad \bar{f}$$

$$[0, 1]/(0, 1) \quad \bar{f}$$

$$\bar{f}(\pi(x)) = f(x)$$

① onto

② 1:1



③ cont

$$\bar{f}(\phi) = \bar{\phi}$$

$$\bar{f}^{-1}(D) = [0, 1]/(0, 1)$$

$$\left( \bar{f}^{-1}(1) = \pi(1) \text{ open} \right)$$

$$\left( \bar{f}^{-1}(0) = \pi(0) \text{ open} \right)$$

$$\text{④ } \bar{f}(\pi(1)) = 1 \quad \bar{f}(\pi(0)) = 0 \leftarrow \text{open}$$

$\therefore \bar{f} \notin \text{open}$

# §8 分離公理

⑦-2

(Axiom of separations)

Open set の 2つ の 分離

2つの点 or 集合を open set で 分離できることか?

Def 8.1 ( $X, \mathcal{U}$  top. sp.)

open nbh で

$T_0$ :  $\forall x, y \in X$  は  $x \neq y$  に  $\exists$  open nbh  $U$  で  $x \in U, y \notin U$   
1つだけを含まないのが存在する



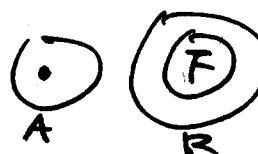
$T_1$ :  $\forall x, y \in X$  は  $x \in$  open nbh  $U$  で  $y \notin U$   
存在する,  $y \in$  open nbh  $V$  で  $x \notin V$  が存在する  
となる存在する



$T_2$ :  $\forall x, y \in X$  は  $\exists A, B \in \mathcal{U}$  で  
 $x \in A, y \in B$  且  $A \cap B = \emptyset$

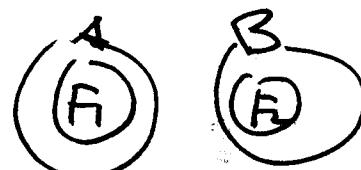


$T_3$ :  $\forall x \in X, \forall F$  closed set は  $\exists A, B \in \mathcal{U}$  で  
 $A \ni x, B \supset F$  且  $A \cap B = \emptyset$



$T_4$ :  $\forall F_1, F_2$  closed set

$\exists A, B \in \mathcal{U}$  で  $A \supset F_1, B \supset F_2$  且  $A \cap B = \emptyset$



$T_2$ : Hausdorff space

$T_1 + T_3$  regular space

$T_1 + T_4$  normal space

⑥ 高度分离性, つまり  $(X, \mathcal{U})$  は  $T_0 \sim T_4$  である.

⑦ normal  $\rightarrow$  regular  $\rightarrow$  Hausdorff  $\rightarrow T_1 \rightarrow T_0$

Prop 8.1  $(X, \mathcal{U})$  top space

#K32 ⑦-3

$T_1 \Leftrightarrow \forall x \exists \lambda \in \mathcal{U} \{x\}$  is closed

$\Leftrightarrow F = \{x, y \in F^c \text{ s.t. } y \in A_y, A_y \subset F^c\}$

$\therefore F^c = \bigcup_y A_y \in \mathcal{U}$

$\Leftrightarrow x, y \in X \quad \{x\}^F \text{ closed} \quad : F^c \text{ open s.t. } F^c \ni y$

Prop 8.2  $(X, \mathcal{U})$  top sp

$T_2$  (Hausdorff)  $\Leftrightarrow \bigcap F = \{x\} \Leftrightarrow X \times X \supset A = \{(x, x)\}$   
 $F: x \text{ a closed nbh}$   $\Leftrightarrow X \times X \text{ a closed}$   
 $\Rightarrow$  Hausdorff

$\therefore \Rightarrow$

$\overline{\cap F} = \cap \overline{F} = \cap F \therefore \underline{\text{closed}}, \underline{x \in \cap F}$

$y \in \cap F \Rightarrow y \neq x \text{ s.t.}$

$A^c$  is closed  $\Rightarrow A^c \ni x$



$\Rightarrow A^c \ni y \therefore A^c \supset \cap F$   $\Rightarrow A^c \supset \{y\}$   $\Rightarrow A^c \supset \{y\}$   
 $\therefore A^c \supset \{y\}$

$\Leftrightarrow x, y \in X, x \in \cap F \therefore \underline{x \in F \text{ s.t. } y \notin F}$

$F$  is nbh  $\Leftrightarrow x \in A^c \subset F$   $\therefore x \in A, y \in F^c, A \cap F^c = \emptyset$

( $\Leftarrow$ )  $\Delta^c = \{(x, y) \in X \times X \mid x \neq y\}$  は  $T_3$ . (7)-4

$x \in A, y \in B, A \cap B = \emptyset$  は  $T_3$ .

$A \times B$  は  $X \times X$  の open set.  
( $\neq$  不整数の基)



$(A \times B)^c = C_{xy} \cap \Delta = \emptyset \quad \therefore \exists t \in \Delta \setminus \{(x, y)\} \text{ すなはち } A \cap B = \emptyset$

$\exists U$  open set 使得する  $\exists z \in U$ .

$C_{xy} \in U$  open set 使得する  $\Delta^c = \bigcup_{(x,y)} C_{xy}$  は open.

$\therefore \Delta$  は closed

( $\Leftarrow$ )  $x, y \in X, x \neq y$  i.e.  $(x, y) \in \Delta$

$\therefore (x, y) \in \Delta^c$  open set 積分位相の基

$\{A \times B; A \in U, B \in U\}$  使得する  $\exists z \in U$

$(x, y) \in A \times B$

i.e.  $x \in A, y \in B \Rightarrow A \cap B = \emptyset \quad \therefore A \cap B = \emptyset$

Prop 8.3

Normal  $\rightarrow$  Regular  $\rightarrow$  Hausdorff  $\rightarrow T_1 \rightarrow T_0$

$\therefore$  Prop 8.3

Prop 8.4  $(X, d)$  metric space  $\Leftrightarrow$   $\exists \varepsilon_0 > 0$   $\forall x \in X$   $\exists r > 0$   $\forall y \in X$   $d(x, y) < r \Rightarrow d(x, y) < \varepsilon_0$

$\Leftrightarrow X \setminus \{x\} = \bigcup_{y \in X} U_{\varepsilon_0}(y)$  & closed

$\therefore X \setminus \{x\}$  is closed

$\therefore X \setminus \{x\} = \bigcup_y U_{\varepsilon_0}(y)$  is open

$\therefore \{x\}$  is closed

$\therefore X$  is  $T_1$ -space

$\geq R \in F, G \subset X$  & closed spaces  $\Leftrightarrow (F \cap G = \emptyset)$

$d(x, F) := \inf \{d(x, a); a \in F\}$  &  $\forall n \in \mathbb{N}$

①  $d(x, F) : x \mapsto d(x, F)$  (is cont)

②  $d(x, F) = 0 \Leftrightarrow x \in F$

$X \ni x \mapsto f(x) = \frac{d(x, F)}{d(x, F) + d(x, G)}$   $\in [0, 1]$  (is

① cont ②  $0 \leq f(x) \leq 1$  ③  $f(x) = \begin{cases} 1 & x \in F \\ 0 & x \in G \end{cases}$

$A = f^{-1}([0, \frac{1}{2}]), B = f^{-1}([\frac{1}{2}, 1])$

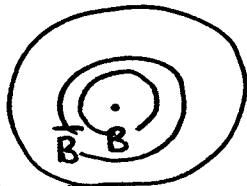
④ ⑤

$f : X \rightarrow [0, 1] \subset \mathbb{R}$

①  $A, B$  open ②  $A \cap B = \emptyset$  ③  $A \supset F, B \supset G$

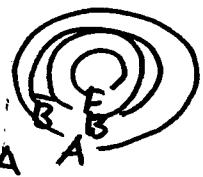
⑥

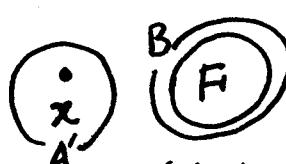
$\therefore X$  is  $T_4$ -space

$T_0 \sim T_4$ 
 $T_1 \equiv \{x\} \text{ closed}, T_2 \text{ is Hausdorff}$ 
 $T_3 \quad \textcircled{O} \quad T_4 \quad \textcircled{O} \quad \textcircled{O}$ 
Lemma 8.5  $(X, \mathcal{U})$  top. sp.


$\neg \textcircled{1} T_3 \Leftrightarrow \forall x \in X, \forall A \in \mathcal{U} \text{ st } x \in A$   
 $\quad \quad \quad (= \exists x \in X \exists B \in \mathcal{U} \text{ st } x \in B \subset \overline{B} \subset A)$

$\textcircled{2} T_4 \Leftrightarrow \forall F \subset X \text{ closed } \forall A \in \mathcal{U} \text{ st } F \subset A$   
 $\quad \quad \quad (= \exists x \in X \exists B \in \mathcal{U} \text{ st } F \subset B \subset \overline{B} \subset A)$


 $\therefore \textcircled{1} (\Rightarrow) x \notin X/A = F \text{ closed}$ 

  $\exists A' \ni B = \emptyset \therefore x \in A' \subset \overline{A'} \subset X/B \subset A$

$(\Leftarrow) x \in X, F \subset X \text{ closed } x \notin F$

$A = \boxed{X \setminus F} \ni x, \text{ open } \exists B \in \mathcal{U} \text{ st }$

$x \in B \subset \overline{B} \subset A \therefore F \subset \overline{B}, x \in B \cap \overline{B} = \emptyset$

$\textcircled{2} \wedge \textcircled{1} \text{ は } \pm \tau^{\prime \prime} x \rightarrow F \text{ closed} \vdash \text{がえりはん} \dots$

$\oplus \quad \overline{A}' \subset X/B \text{ の部分の } \oplus$

$A' \cap B = \emptyset \Rightarrow \overline{A}' \cap B = \emptyset$

$\therefore \overline{A}' \cap B \ni x \text{ ならば } x \text{ は } \overline{A}' \cap B \text{ に} \in B \text{ である}$

$x \in A' \cap B \text{ ならば } x \in C \subset \mathcal{U} \text{ は } x \in C$

$A' \cap C \neq \emptyset \quad \text{なら} \quad \vdash A' \cap B = \emptyset \text{ が矛盾}$

# §9 Urysohn's lemma



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Metric  $\rightarrow$  normal  $\rightarrow$  regular  $\rightarrow$  Hausdorff  $\rightarrow T_1 \rightarrow T_0$   
normal + ?  $\rightarrow$  metric

$(X, d)$  metric space  $F, G$  closed

$$f(x) = \frac{d(x, F)}{d(x, F) + d(x, G)} = \begin{cases} 1 & x \in F \\ \alpha & x \notin F \cup G \quad 0 < \alpha < 1 \\ 0 & x \in G \end{cases}$$

cont.  $0 \leq f \leq 1$ ,  $x=1 \Leftrightarrow x \in F$   
 $x=0 \Leftrightarrow x \in G$

Theorem 9.1 (Urysohn's lemma)  $(X, \mathcal{U})$  normal

$F, G \subset X$  closed,  $F \cap G = \emptyset$

$\exists \alpha_i \in \mathbb{R} : x \mapsto 1/2^i$  s.t.

(0)  $f$  is cont, (1)  $0 \leq f(x) \leq 1$ , (2)  $x \in F \rightarrow f(x)=0$   
(2')  $x \in G \rightarrow f(x)=1$

$\therefore A_1 = X \setminus G$  open

$$\bar{F} \subset A_1 \Rightarrow \bar{F} \subset A_0 \subset \bar{A}_0 \subset A_1$$

$$\Rightarrow \bar{F} \subset A_0 \subset \bar{A}_0 \subset A_{\frac{1}{2}} \subset \bar{A}_{\frac{1}{2}} \subset A_1$$

$$\Rightarrow \bar{F} \subset A_0 \subset \bar{A}_0 \subset \underline{A_{\frac{1}{4}}} \subset \bar{A}_{\frac{1}{4}} \subset \underline{A_{\frac{1}{2}}} \subset \bar{A}_{\frac{1}{2}} \subset \bar{A}_{\frac{3}{4}} \subset \bar{A}_{\frac{3}{4}} \subset A_1$$

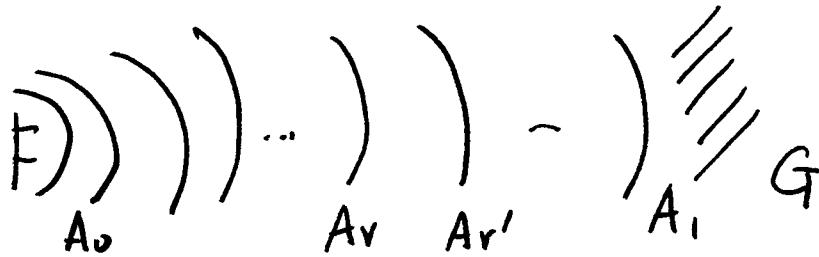
$$r = \frac{n}{2^m} \in \mathbb{Q} \quad A_r \in \mathcal{U}$$

$n=0, \dots, 2^m$

$$\bar{A}_{\frac{1}{2}} \subset A_{\frac{3}{4}} \subset \bar{A}_{\frac{3}{4}} \subset A_1$$

$$A_r \subset \bar{A}_r \subset A_{r'}, \quad (r < r')$$

$$f(x) = \begin{cases} 1 & x \notin A_1 \\ \inf \left\{ r = \frac{n}{2^m} : x \in A_r \right\}, & x \in A_1 \end{cases}$$



①  $\forall \alpha, 0 < f(\alpha) < 1$  は OK

④  $x \in G \rightarrow f(x) = 1$

$x \in F \rightarrow f(x) = 0$

③  $f$  は cont.  $\therefore f$  は 各  $x \in X$  で "cont" (Prop 6.3)

$$N_{\mathbb{R}}(f(x)) \xrightarrow{\exists} B \text{ は } \exists r \text{ で } f^{-1}(B) \in N_x(x)$$

$\mathbb{R}$  は Euclidean space  $U_\varepsilon(f(x))$ :  $f(x)$  が  $\varepsilon$  の開域には 基本 wbh. で  $f^{-1}(U_\varepsilon(f(x))) \in N_x(x)$  は "cont"

∴  $\exists B \in \mathcal{U}$  で  $f^{-1}(U_\varepsilon(f(x))) \supset B \ni x$  "cont"

(1)  $f(x) = 0 \wedge \exists \varepsilon \quad U_\varepsilon(f(x)) = U_\varepsilon$  表す

$$r = \frac{n}{2^m} < \varepsilon \text{ で } y \in A_r \rightarrow |f(x) - f(y)| = |f(y)| \leq r < \varepsilon$$

$$\therefore f(A_r) \subset U_\varepsilon \quad \therefore A_r \subset f^{-1}(U_\varepsilon)$$

$$x \in A_r \subset f^{-1}(U_\varepsilon) \quad //$$

(2)  $f(x) = 1 \wedge \exists \varepsilon$

$$r = \frac{n}{2^m} < \varepsilon \text{ で } y \in X / A_{1-r} = B_r \text{ で } f(y) = 1$$

$$f(y) \geq 1-r \quad \therefore |f(x) - f(y)| = |1 - f(y)| \leq 1 - (1-r) = r < \varepsilon$$

$$\therefore f(B_r) \subset U_\varepsilon \quad \therefore B_r \in f^{-1}(U_\varepsilon) \quad \therefore x \in B_r \subset f^{-1}(U_\varepsilon),$$



$$0 < f(x) \text{ かつ } x \in f^{-1}(\{y\})$$

$$f(x)-\varepsilon < r < f(y) < r' < f(x)+\varepsilon$$

$$r = \frac{n}{2^m}$$

$$r' = \frac{n'}{2^{m'}}$$

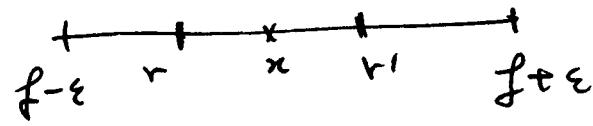
( . )

$$y \in A_{r'} \setminus \bar{A}_r = C_{r'} \text{ open}$$

$$A_{r'} \quad A_r$$

$$r < f(y) \leq r'$$

$$\therefore |f(x) - f(y)| < \varepsilon$$



$$(\because -\varepsilon \leq r - f(x) \leq f(y) - f(x) \leq r' - f(x) < \varepsilon)$$

$$(\varepsilon \geq f(x)-r \geq f(x)-f(y) \geq f(x)-r' \geq -\varepsilon)$$

$$f(C_r) \subset U_\varepsilon \therefore C_{r'} \subset f^{-1}(U_\varepsilon)$$

$$x \in C_{r'} \subset f^{-1}(U_\varepsilon) \quad //$$

§10 距離付可度空間  $\sum_{\text{Kyushu Univ}}$   
(metrizable)

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$(X, \mathcal{U})$  top sp.  $\exists d$  metric on  $X$  st

$(X, d) \in (X, \mathcal{U})$  are  $(X, \mathcal{U})$  is metrizable  
e.g.

$d$  は  $\mathcal{U}$  の top  $\mathcal{U}_d \rightarrow A$

$\Leftrightarrow \forall x \in A \exists \varepsilon > 0$  st  $V_\varepsilon(x) \subset A$ . |  $\mathcal{U}_d = \mathcal{U}$   
 $V_\varepsilon(x) = \{y \in X; d(x, y) < \varepsilon\}$  | e.g. 3:2.

Lemma 10.1  $(X_u, d_u)$  可度の metric space

$\Rightarrow \exists \varepsilon \in \mathbb{R} (X_u, d_u)$  is metrizable.

①  $d'_u(x, y) = \min\{d(x, y), 1\}$  は metric は?

は  $\mathcal{U}_{d'_u} = \mathcal{U}_{d_u}$  ( $\vdash$  check TS)

$\Rightarrow d'_u \rightarrow d_u$  は?

$$d(x, y) = \sum_n \frac{1}{2^n} d_n(x, y) \quad \begin{array}{l} x = (x_1, \dots) \\ y = (y_1, \dots) \end{array}$$

metric は?  $\vdash$  check TS

$\prod_n (x_n, d_n) = (\prod_n X_n, \mathcal{U})$   $\mathcal{U}$  product top

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n) \text{ for } x = (x_n) \in \prod_n X_n$$

(f metric l: 73)

$$(\prod_n X_n, d) \rightarrow (\prod_n X_n, \mathcal{U}_d)$$

statement (f)  $\mathcal{U} = \mathcal{U}_d$

$$\begin{aligned} ① \mathcal{U}_d &\subset \mathcal{U} \quad \text{base } \left\{ U_{\varepsilon}(x); x \in X, \varepsilon > 0 \right\} \\ &\text{base } \left\{ \prod_n A_n; \text{有限}\{n \in \mathbb{N} \mid A_n \neq \emptyset\} \cup \{A_n = X_n\} \right\} \\ &\quad A_{n_j} = U_{\varepsilon_j}(x_j) \text{ 有限} \end{aligned}$$

$$U_{\varepsilon}(x) \in \mathcal{U}_d \in \mathcal{U}, \text{ (3.)}$$

$$A = U_{\frac{1}{2}\varepsilon}(x_1) \times \dots \times U_{\frac{1}{2}\varepsilon}(x_m) \times \bigcap_{n>m+1} \prod_n X_n \quad (m \gg 1)$$

$$A \ni y \Leftrightarrow \tilde{x} \geq y$$

$$\begin{aligned} d(x, y) &= \sum_{n=1}^m \frac{1}{2^n} d_n(x_n, y_n) + \sum_{n=m+1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n) \\ &\leq \frac{1}{2} \varepsilon \sum_{n=1}^m \frac{1}{2^n} + \sum_{n=m+1}^{\infty} \frac{1}{2^n} \leq \frac{1}{2} \varepsilon + \frac{1}{2^m} < \varepsilon \quad (m \gg 1) \end{aligned}$$

$$\therefore A \subset U_{\varepsilon}(x) \quad \text{Therefore } \Rightarrow A \in \mathcal{U}$$

$$\therefore \bigcup_x A = U_{\varepsilon}(x) \in \mathcal{U},$$



②  $U_d \supset U$

① 同様に base  $A = U_{\varepsilon_{i_1}}(x_{i_1}) \times \dots \times U_{\varepsilon_{i_k}}(x_{i_k}) \times \prod_{n \neq i_1, \dots, i_k} X_n$

$\in \mathcal{U}$

$$\max\{i_1, \dots, i_k\} = m, \quad \min\{\varepsilon_{i_1}, \dots, \varepsilon_{i_k}\} = \varepsilon$$

$B = U_{\frac{\varepsilon}{2}}(x_1) \times \dots \times U_{\frac{\varepsilon}{2}}(x_m) \times \prod_{n > m+1} X_n$  とする

~~( $x_1 + \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2}, \dots, x_m + \frac{\varepsilon}{2}, x_{m+1}, \dots, x_n$ )~~

~~$x_1 + \frac{\varepsilon}{2} + p \in B$~~

~~$B \subset U_d$~~

$B \subset A \Rightarrow B \in \mathcal{U}_d \therefore U_B = \mathbb{H} \in \mathcal{U}_d$

Lemma 10.2


$(X, \mathcal{U})$  top sp       $(X, \mathcal{U}) \cong (Y, \mathcal{U}_d)$   
 $(Y, d)$  metric space      home

$\therefore \exists \varepsilon \in (X, \mathcal{U})$  is metrizable

$\oplus f: X \rightarrow Y$  homeo

$d_f(x, y) = d(f(x), f(y))$  is  $X$  a metric

$U_\varepsilon(y)$  is  $Y$  a  $d$   $\tau$ ;  $\exists r_1, r_2$  open ball

$\tilde{U}_\varepsilon(x)$  is  $X$  a  $d_f$   $\tau$ ;  $\exists r_1, r_2$  open ball

$$\text{すこし} f^{-1}(U_\varepsilon(y)) = \tilde{U}_\varepsilon(f^{-1}(y))$$

$$f(\tilde{U}_\varepsilon(x)) = U_\varepsilon(f(x)) \quad \text{すこしあと}$$

$\mathcal{U}_{df} = \mathcal{U} \oplus \mathcal{U}_{df} \supset \mathcal{U} : \mathcal{U} \ni A \Rightarrow x \in A$

$x = f^{-1}(y) \left( \begin{array}{l} y \in Y \\ y = f(x) \end{array} \right)$  は  $f(A) = B \in \mathcal{U}_d$

$$U_\varepsilon(f(x)) \subset B \quad \therefore f(\tilde{U}_\varepsilon(x)) \subset B \quad \therefore \tilde{U}_\varepsilon(x) \subset A$$

$\therefore A \in \mathcal{U}_{df} / \mathcal{U}_{df} \subset \mathcal{U} : \mathcal{U}_{df} \ni A$

$\therefore \forall x \in A = \bigcup_{x \in A} \tilde{U}_\varepsilon(x) \quad \text{と書く。}$

$$= \bigcup_{y \in f(A)} \tilde{U}_{\varepsilon(f^{-1}(y))}^{(f^{-1}(y))} = \bigcup_{y \in f(A)} f^{-1}\left(U_{\varepsilon(-)}^{(y)}\right) = \text{はるかに開集合}$$

Theorem (Urysohn's metrizable theorem)

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$(X, \mathcal{U})$  normal かつ 2nd countable axiom

$\Rightarrow (X, \mathcal{U})$  is metrizable

$\Leftarrow$  方針  $I = [0, 1] \times_{12} (I^{\mathbb{N}}, d)$  metric sp  
(metric sp)

~~以下は略す~~  $F : (X, \mathcal{U}) \rightarrow (I^{\mathbb{N}}, d)$

理の込みを定めます。

$\therefore (F(x), d) \cong (X, \mathcal{U}) \quad \therefore (X, \mathcal{U})$  is metrizable  
metric sp  $\because$  countable ならば  $\exists$

$\bar{A} \subset B$  - (\*) pair  $i = \bar{A} \bar{B}$

Urysohn's lemma  $\exists f_{AB} : X \rightarrow [0, 1]$

$$\text{s.t. } f_{AB}(x) = \begin{cases} 0 & x \in \bar{A} \\ \leq 1 & \text{ow} \\ 1 & x \in X \setminus B \end{cases} \quad \therefore \exists A, B \in \mathcal{B}$$

$(A, B)$  は ~~既存~~ 番号化可能とします

$f_j \quad j=1, 2, 3, \dots$   $\tau$  表す。

$$F : X \rightarrow I^{\mathbb{N}}$$

$$\psi \quad \varphi \\ x \mapsto (f_1(x), f_2(x), \dots) \quad \forall x$$

$$X \hat{=} F(x) \subset I^N$$

左端・右

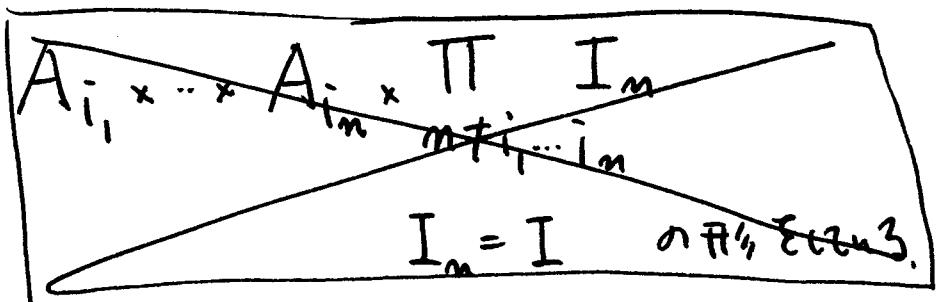
① onto SK

② cont. +

$I^N$  a base

$$A = \prod_{j=1}^{\infty} A_j \subset I^{\infty}$$

$j = i_1 \dots i_m$  (有理数) の除数  $\therefore f_j = I$



$$F^{-1}(A) = f_{i_1}^{-1}(A_{i_1}) \cap \dots \cap f_{i_m}^{-1}(A_{i_m}) \text{ is open}$$

$$\therefore \forall A \in \mathcal{U} \subset \mathcal{D}_1, A = \bigcup_{\lambda} A_{\lambda} \quad \leftarrow \text{base}$$

$$F^{-1}(\bigcup_{\lambda} A_{\lambda}) = \bigcup_{\lambda} F^{-1}(A_{\lambda}) \in \mathcal{U} \quad \therefore F \text{ is cont}$$

③ injective  $x \neq y \in X \Leftrightarrow$

$\{x\}, \{y\}$  of closed sets  $\begin{cases} x \\ y \end{cases} \subset B$

$$\begin{aligned} T_1 & \quad \{x\} \subset A \subset \bar{A} \subset B \quad \text{closed} \\ & \quad \text{and } \{y\} \subset B \end{aligned}$$

$$= \sigma(A, B) \Rightarrow \text{closed}$$

$$f_j(x) = 1, f_j(y) = 0$$

$$\therefore F(x) = (f_1(x), \dots, f_j(x), \dots) \quad \therefore F(x) \neq F(y)$$

$$F(y) = (f_1(y), \dots, f_j(y), \dots)$$

④  $F^{-1}$  a cont. +

$$\bigcup_{j \in J} A_j \xrightarrow{\bigcup_{j \in J} f_j} \bar{A}_0 \quad \downarrow \text{trivial}$$

$\{x\} \subset B \Rightarrow \{x\} \subset A' \subset \bar{A}' \subset B \Rightarrow A' \in \mathcal{U}$

base  $\in \mathcal{D}_1$   $\{x\} \subset A_j \subset \bar{A}_j \subset B$  かつ  $A_j \in \mathcal{U}$

$F$  が "open mapping"  $\Leftrightarrow \sum_{\text{Kyushu Univ.}}$

$A \in \mathcal{U} \Rightarrow F(A) \in \mathcal{U}_d \in \mathbb{E}^n$ .

$x \in C \subset A \quad c \in B \in \mathbb{E}^3$ .

③  $\exists$   $\overline{B} \subset \mathbb{R}^3$  で

$x \in B \subset \overline{B} \subset C \subset A \quad \forall B, C \in \mathbb{E}^3$

$f_{BC} = f_j$  すなはち  $j = j(x)$  が存在する

$O_j = \{y \in \mathbb{I}^N ; y_j \in [0, 1]\} \in \mathcal{U}_d$

$= I \times \cdots \times [0, 1] \times \cdots I = \mathbb{E}^n$

$\bigcup_{x \in A} O_{f(x)} \cap F(x) = F(A)$  すなはち

$\therefore \bigcup_{x \in A} O_j \cap F(x) \ni y \in F(y) = (f_j(y), \dots)$

$\therefore y \in A \Rightarrow y \notin A \Rightarrow f_j(y) = 1, \dots, f_j(y) \in [0, 1]$ .

$\therefore F(y) \in F(A)$

$\therefore F(A) \ni F(y) = (f_1(y), \dots)$

$\therefore \exists j \text{ で } f_j(y) \in [0, 1] \Rightarrow f_j(y) = 1$

$\therefore F(y) \in O_j \cap F(x)$ ,

$y \notin A$

理由  $y \in A \Rightarrow$

$O_j \cap F(x) \subset F(x) \cap \mathbb{I}^N$

$y \in B \subset \overline{B} \subset C \subset A$

$\cap$  relative top  $\Rightarrow$  open で  $F(A)$  が open,

$\exists f_{BC}(y) = 0 < 1$

# 10-1

## 2nd countable axiom & separability

可算個の base の  $\overline{\beta}$

6T 算個の  $\overline{\beta}$   
dense set  
 $\cap \overline{\beta}$  在

Lemma <sup>10.4</sup>  $(X, \mathcal{U})$  は 2nd countable axiom &

かつ  $\overline{\beta}$  は separable.

○ separable  $\Leftrightarrow \exists D \subset X \quad \forall A \in \mathcal{U} \quad \exists x_A \in D \cap A \quad (A \neq \emptyset)$

$B \subset \mathcal{U}, \quad \#B = \text{可算} \quad \text{base } \overline{\beta}$

$B \ni A \ni x_A \quad (A \neq \emptyset) \quad \overline{\beta}$

$D := \{x_A\}_{\substack{A \in B \\ A \neq \emptyset}}$ .

$\exists D \subset X \quad \forall A \in \mathcal{U} \quad \exists x_A \in D \cap A \quad A = \bigcup_{\lambda} A_{\lambda} \quad A_{\lambda} \in B$

$\therefore A \ni A_{\lambda} \ni x_{A_{\lambda}} \quad \therefore A \cap D \ni x_{A_{\lambda}}$

Lemma <sup>10.5</sup>  $(X, d)$  metric space

separable  $\Leftrightarrow$  2nd countable axiom

○ ( $\Leftarrow$ ) は Lemma 2" check (C2).

( $\Rightarrow$ )  $D = \{x_1, x_2, \dots\}$

$$\left\{ U_r(x_j) ; r \in \mathbb{Q}, x_j \in D \right\} = B$$

と呼ばれる。

§

## 11 Connected topological spaces (連続空間)

Def 11.1  $(X, \mathcal{U})$  top space

$X \supset A$  が closed & open  $\Rightarrow A = X \text{ or } \emptyset$

$\mathcal{U}$  は  $(X, \mathcal{U})$  が connected なら.

Prop 11.2  $(X, \mathcal{U})$  connected top sp.

$\Leftrightarrow A \cup B = X, A \cap B = \emptyset, A, B \in \mathcal{U} \Rightarrow A = \emptyset \text{ or } X$

① ( $\Rightarrow$ )  $B^c = A^c \therefore A$  は open & closed  $\therefore A = \emptyset \text{ or } X$

( $\Leftarrow$ )  $X \supset A$  open & closed  $\therefore X = A \cup A^c, A \cap A^c$   
 $A, A^c \in \mathcal{U}$

$\therefore A = \emptyset \text{ or } X$

( $\exists$ ) Prop 11.2 より  $A, B \in \mathcal{U}$  は connected.

Def 11.3  $(X, \mathcal{U})$  top sp.  $A \subset X$

$A$  が  $X$  の connected subset

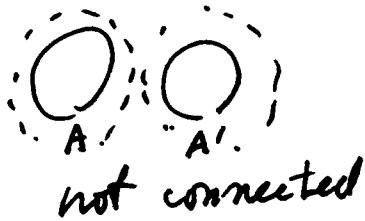
$\Leftrightarrow (A, \mathcal{U}_A)$  が connected

Prop 11.4  $(X, \mathcal{U})$  top sp  $A \subset X$  connected

$\Leftrightarrow A \subset B \cup C, A \cap B \cap C = \emptyset, B, C \in \mathcal{U}$

$\Rightarrow A \subset B$  or  $A \subset C$

② Prop 11.2 より  $(X, \mathcal{U}) \rightarrow (A, \mathcal{U}_A)$  が connected か?



Prop 11.5  $(X, \mathcal{U}) \cong (Y, \mathcal{V})$  とする  
 $(X, \mathcal{U})$  connected  $\Leftrightarrow (Y, \mathcal{V})$  connected

$\therefore (\Rightarrow) \exists f: X \rightarrow Y$  homeo.

$Y \supset A$  open & closed  $\Leftrightarrow$

$f^{-1}(A)$  is open & closed in  $X$

$\therefore f^{-1}(A) = X \text{ or } \emptyset$

(1)  $f^{-1}(A) = X$  かつ  $A = f(X) = Y$

(2)  $f^{-1}(A) = \emptyset$  かつ  $A = \emptyset$

$(\Leftarrow) \exists g: Y \rightarrow X$  homeo 後は同じく。

同様に像で不变性もしくは連續性の性質。  
e.g. metrizable, connected, compact.

Prop 11.6  $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$  cont.

$(X, \mathcal{U})$  connected  $\Leftrightarrow (f(X), \mathcal{V})$  connected

$\therefore f(x) \subset A \cup B$ ,  $f(x) \cap A \cap B = \emptyset$ ,  $A, B \in \mathcal{V}$  とする

$f^{-1}(A \cup B) = X = f^{-1}(A) \cup f^{-1}(B)$

$\therefore f^{-1}(A) \cap f^{-1}(B) = \emptyset \therefore f^{-1}(A) = X \text{ or } \emptyset$

(1)  $f^{-1}(A) = X$  かつ  $f(x) \subset A$

(2)  $f^{-1}(A) = \emptyset$  かつ  $f(x) \cap A = \emptyset \therefore f(x) \subset B$

cont map の像は connected.

Prmrs 11.7  $(X, \mathcal{U})$  top sp,  $A$  connected

$\Rightarrow A \cap \bar{B} \subset \bar{A}$   $\Leftrightarrow B$  is connected

$\because B$  is connected  $\Leftrightarrow$  2nd cond:

$$B \subset \bigcup_{j=1,2} G_j \cup C_2, \quad B \cap G_j \cap C_2 = \emptyset, \quad G_j \cap B \neq \emptyset$$

$$\therefore A \subset G_1 \cup C_2, \quad A \cap G_1 \cap C_2 = \emptyset$$

$$\text{But } A \cap G_1 = \emptyset \rightarrow \bar{A} \cap G_1 = \emptyset \quad (\text{check it})$$

$$\therefore B \cap G_1 = \emptyset \text{ 由 } \bar{A} \cap G_1 = \emptyset \therefore A \cap G_1 \neq \emptyset$$

(2) 不滿足  $A \cap C_2 \neq \emptyset$ .  $\therefore A \subset G_1$  or  $A \subset C_2$   
 有 1 份  $y_n$  在  $G_1$  與  $C_2$  之間  $\Rightarrow A$  is connected  
 $\Leftrightarrow$  1, 2, 3 滿足.

Example 1  $\mathbb{R}$  is connected

$\because \mathbb{R} = A \cup B, A \cap B = \emptyset, A, B$  ~~closed~~  
 は假定す:  $A \neq \emptyset, B \neq \emptyset$  is closed

$a \in A, b \in B$   
 $(a < b)$        $\vdots \quad \vdots \quad \vdots \quad \vdots$   
 $a \quad c \quad x \quad b$

$$A \cap (-\infty, b) = H \ni a \quad \exists x \in H \neq \emptyset$$

$$c = \sup H \quad \Rightarrow \quad c \in \overline{H} \subset \overline{A} = A \quad \therefore c < b$$

$$\therefore c < x < b \Rightarrow x \notin A \quad \therefore x \in B \quad \therefore C \text{ is } B \text{ の}$$

触点は假定す.  $\therefore c \in \overline{B} = B$

$$\therefore c \in A \cap B = \text{矛盾} \quad \text{TL } x \in A \cap B \text{ は矛盾}$$

$$x \in A \cap (-\infty, b)$$

$$\text{P.g. } x \in H, \underset{\substack{\text{は } H \text{ の} \\ \text{触点}}}{\underset{\substack{\text{は } H \text{ の} \\ \text{触点}}}{c = \sup H}}$$

Example 2  $\mathbb{R} \cong (a, b) \therefore (a, b)$  is not connected

Example 3.  $[a, b], (a, b], [a, b]$  is not connected

$$(a, b) \subset [a, b]$$

$$(a, b) \subset \frac{[a, b)}{(a, b]} \subset [a, b] \quad \text{は矛盾},$$

Example 4  $A \subset \mathbb{R}$  connected は假定す

$$A = (a, b), [a, b), (a, b], [a, b] \text{ の } k \text{ つある!}$$

$\therefore C$ : connected sets 全体

$\cup$ : 区間全体

$C \supset \cup$  は既定.  $C \subset \cup$  は既定.

③  $\forall A \in \mathcal{C}$  esz:  $a = \inf A, b = \sup A$

①  $a = b$  orz  $A = \{a\}$  : mir  $\{a\} \in \mathcal{Q}$

②  $a < b$  orz  $(a, b) \ni^* c$  zwz

$c \in A \Rightarrow c \notin A$  zwz  $(-\infty, c) \cap A = G_1$   
 $(c, \infty) \cap A = G_2$

esz zwz  $A \subset G_1 \cup G_2, A \cap G_1 \cap G_2 = \emptyset$

$A \cap G_1 \ni a, A \cap G_2 \ni b$  zwz

$A$  connected zwz  $\frac{1}{15}$ .

$\therefore (a, b) \subset A$

$A \setminus (a, b) \Rightarrow x$  esz zwz  $x, b$  or def F'

$x = a$  or  $b$

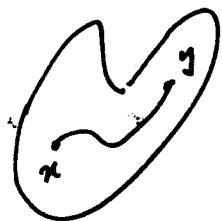
$\therefore A = (a, b), [a, b], [a, b], (a, b] \dots$

## §12 Arcwise connected & connected component

Def 12.1  $(X, \mathcal{U})$  top sp.  $\forall x, y \in X$  zwz zwz

$\exists w: [0, 1] \rightarrow X$  cont s.t.  $w(0) = x, w(1) = y$

esz zwz  $(X, \mathcal{U})$  is arcwise connected zwz



not arcwise conn.



not arcwise conn.

12.2  
Prop 12.9 Arcwise conn  $\Leftrightarrow$  conn

$(X, \mathcal{U})$  top space arcwise conn.

$X = A \cup B$ ,  $A \cap B = \emptyset$ ,  $A, B \in \mathcal{U}$  且  $\exists$

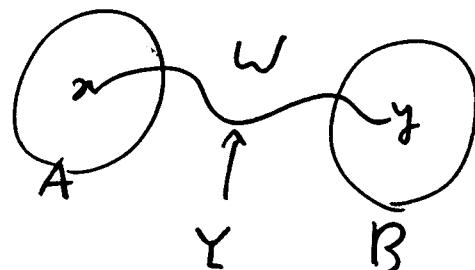
$\exists \omega$  s.t.  $\omega(0) = {}^3x \in A$   
 $\omega(1) = {}^3y \in B$

cont map.

$x \in A$   
 $y \in B$

- $\omega([0, 1]) \subset X$  且 connected  
 (conn a cont map)  
 $\omega([0, 1]) = Y$  且  $Y \neq \emptyset$ .

$$\begin{aligned} Y &\subset A \cup B \\ Y \cap A \cap B &= \emptyset \\ Y &\neq \emptyset \\ Y &\subset A \text{ or } Y \subset B \end{aligned}$$



$Y \subset A \cup B$   $w(1) = y \in B$   $\Rightarrow$  否.

同様  $Y \subset B$  と矛盾

I.e.,  $A \cup B$  の要素だけ  $\emptyset$  //

$(X, \mathcal{U})$  top sp.

$x \sim y \Leftrightarrow \exists A \text{ connected st } x, y \in A$

$X/\mathcal{U} \ni T(x) \cdots x \in \text{同一連続構成の} \in \mathcal{U}$

(\*)  $T(x)$  は connected subset  
 $\therefore T(x) \subset A \cup B$ ,  $A \cap B \cap T(x) = \emptyset$   
 $A \neq \emptyset, B \neq \emptyset \Rightarrow A \cap B = \emptyset$   
 $\exists C \ni x, y = n + c \in T(x)$

Prop 12.3 ~~H.10~~  $(X, \mathcal{U})$  top space  $X/\sim \xrightarrow{\pi} \mathbb{E}_X^{\text{top}}$   
 $x \in \text{含む最も}\overset{\text{sub}}{\text{connected set}} \Rightarrow \text{closed}$

① ~~证明:  $\forall A \in \mathcal{U}, A$  connected set  $\Leftrightarrow A/\sim$  closed~~

(connected)  $\pi(x) \subset A \cup B$   $A, B \in \mathcal{U}$

$\pi(x) \cap A \neq \emptyset, \pi(x) \cap B \neq \emptyset \quad (\Leftrightarrow A \cap B = \emptyset)$   
 ~~$\pi(x) \cap A \neq \emptyset, \pi(x) \cap B \neq \emptyset$~~   $\exists z \in \pi(x) \cap A, \pi(x) \cap B = \emptyset$

$\exists c \ni \frac{y, z}{x, y}$  connected  $\Leftrightarrow \frac{x \sim y}{x \sim z} \Leftrightarrow y \sim z$

$= \pi \subset \pi \cap \pi(x) \subset \pi \neq \emptyset$ .

$\therefore C \subset A \cup B, C \cap A \cap B = \emptyset$

$\checkmark$   $C \cap A \neq \emptyset, C \cap B \neq \emptyset$   $\Rightarrow$   $C$  connected  $\Leftrightarrow$   $C \subset \pi(x)$ .

$(\frac{1}{12} \text{下}) V = \bigcup \{A; x \in \text{含む connected subset}\}$

上記論理

$\pi(x) \subset \bigcup \{A; \dots\} \text{ はOK}$

$\pi(y) \subset \bigcup \{A; \dots\} \in \pi$

$\exists b \ni y \in \pi(y) \in A \text{ 且 } x, y \in A, \text{ connected}$

$y \in \pi(y) \quad \text{は}\checkmark$

$\pi(x) = V \quad \forall A \text{ connected } \pi(A) \subset \pi \text{ 且 } \pi \text{ connected set.}$

( $\frac{1}{12}$ )  $A$  connected  $\bigcap_{\lambda} A_{\lambda} \overset{*}{\not\in} T_3$  且  $\bigcup_{\lambda} A_{\lambda}$  if

connected. 且  $\bigcup \{A; x \in \text{含む connected subset}\}$   
 は connected.

(用意)

11-3

$\overline{\pi(n)}$  is connected  $\therefore \overline{\pi(n)}$  is connected

$$\therefore \overline{\pi(n)} = \pi(n) \quad (\text{由上是连通的})$$

C.R.  $(x_1, u_1)$   
 $\cap A \neq \emptyset$   
 $\forall x \in A$  connected  
 $\exists a \in A$   
 $x \sim a \rightsquigarrow x \sim y$   
 $x, y \in \pi^{-1}(x)$

Prop 12.4  $(X_\lambda, U_\lambda)$  connected  $\Rightarrow \forall \lambda \in \Lambda \Rightarrow \bigcup_{\lambda \in \Lambda} X_\lambda$  connected

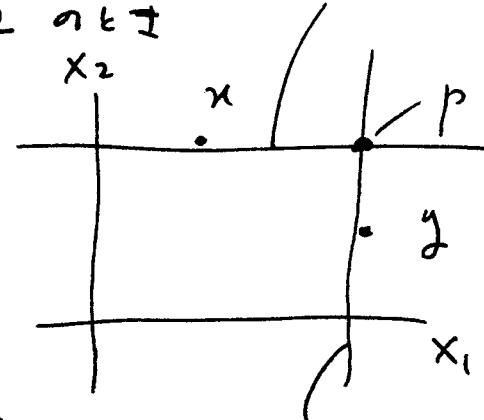
$\prod_{\lambda \in \Lambda} (X_\lambda, U_\lambda)$  is connected.

① 希尔伯特空间  $\#(\lambda) = 2$  时

$$(X_1, U_1) \times (X_2, U_2)$$

$$n = (x_1, x_2) \in X_1 \times X_2$$

$$y = (y_1, y_2) \in X_1 \times X_2$$



(\*)  $- X_1 \times \{x_2\} \cup \{y_1\} \times X_2$  is connected  $\{y_1\} \times X_2$

$$\begin{aligned} \therefore X_1 \times \{x_2\} &\cong X_1 \\ \{y_1\} \times X_2 &\cong X_2 \end{aligned}$$

$f: X_1 \times \{x_2\} \rightarrow X_1$   
 $\downarrow$   
 $(x_1, x_2) \rightarrow x_1$

各自 check it

$$\text{这样 } X_1 \times \{x_2\} \cap \{y_1\} \times X_2 \ni (y_1, x_2)$$

$\therefore$  不是 connected.

$\therefore x_1 \sim y_1 \therefore X_1 \times X_2$  is connected

$\#(\lambda) < \infty$  时  $(X_\lambda, U_\lambda)$  is connected

$\#(\lambda) = \infty$  时不是

$\#(\lambda) = \infty$  のとき

$\pi X_\lambda \rightarrow a = (a_\lambda) \quad \varepsilon - \text{近似}.$

$$A = \left\{ (x_\lambda); \text{ 希望の固形除く } x_\lambda = a_\lambda \right\}$$

$$= \bigcup \left\{ (x_\lambda); \begin{array}{l} \lambda_1, \dots, \lambda_k \quad x_{\lambda_j} \in X_{\lambda_j}, \\ \lambda \neq \lambda_1, \dots, \lambda_k \Rightarrow a_\lambda = x_\lambda \end{array} \right\}$$

$$= X_{\lambda_1} \times \cdots \times X_{\lambda_k} \times \prod_{\lambda \neq \lambda_1, \dots, \lambda_k} \{a_\lambda\} \cong X_{\lambda_1} \times \cdots \times X_{\lambda_k}$$

は connected.

$$\text{また } \bar{A} = \pi X_\lambda$$

$\therefore \forall x \in \pi X_\lambda$  にえり

$n$  の 基本近傍系と

  $V = \bigcap_\lambda V_\lambda, \quad V_\lambda \text{ は } x_\lambda \text{ の open nbhd.}$   
 $\text{希望の } a_\lambda \text{ を除く } V_\lambda = X_\lambda$

例えば  $\lambda_1, \dots, \lambda_k$  を除く  $V_\lambda = X_\lambda$  とする

$$\phi \neq \bigcap_\lambda V_\lambda \cap \left\{ (x_\lambda); \begin{array}{l} \lambda_1, \dots, \lambda_k, x_{\lambda_j} \in X_{\lambda_j} \\ \lambda \neq \lambda_1, \dots, \lambda_k \Rightarrow a_\lambda = x_\lambda \end{array} \right\}$$

なぜ  $A \cap V \neq \emptyset \therefore x$  は  $A$  の触点

$$\therefore \bar{A} = \bigcap_\lambda \pi X_\lambda.$$

## Ex. 1 中間値の定理

$(X, \tau)$  connected top sp.

$f: X \rightarrow \mathbb{R}$  cont  $\exists x, y \in X$  s.t.  $f(x) = a < b = f(y)$

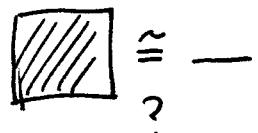
$a < c < b$   $\exists z \in \mathbb{R}$   $\exists z \in X$  s.t.  $f(z) = c$

$\therefore f(X) \subset \mathbb{R}$  is connected

$f(x) \geq a, b \therefore f(X) \supset [a, b]$  "

$\therefore \mathbb{R}$  is connected if  $[\alpha, \beta], (\alpha, \beta), [\alpha, \beta], (\alpha, \beta)$ .

Ex. 2  $I = [0, 1]$  connected



$[0, 1] \times [0, 1] \stackrel{\cong}{=} [0, 1] ?$

$\exists f: [0, 1] \rightarrow [0, 1] \times [0, 1]$  homeo

$$f(I \setminus \{\frac{1}{2}\}) = [0, 1] \times [0, 1] \setminus \{x\}$$

$\uparrow$                                $\uparrow$   
 connected region                  connected

$\therefore f^{-1}([0, 1] \times [0, 1] \setminus \{x\})$  is connected

$\therefore f^{-1}([0, 1] \times [0, 1])$

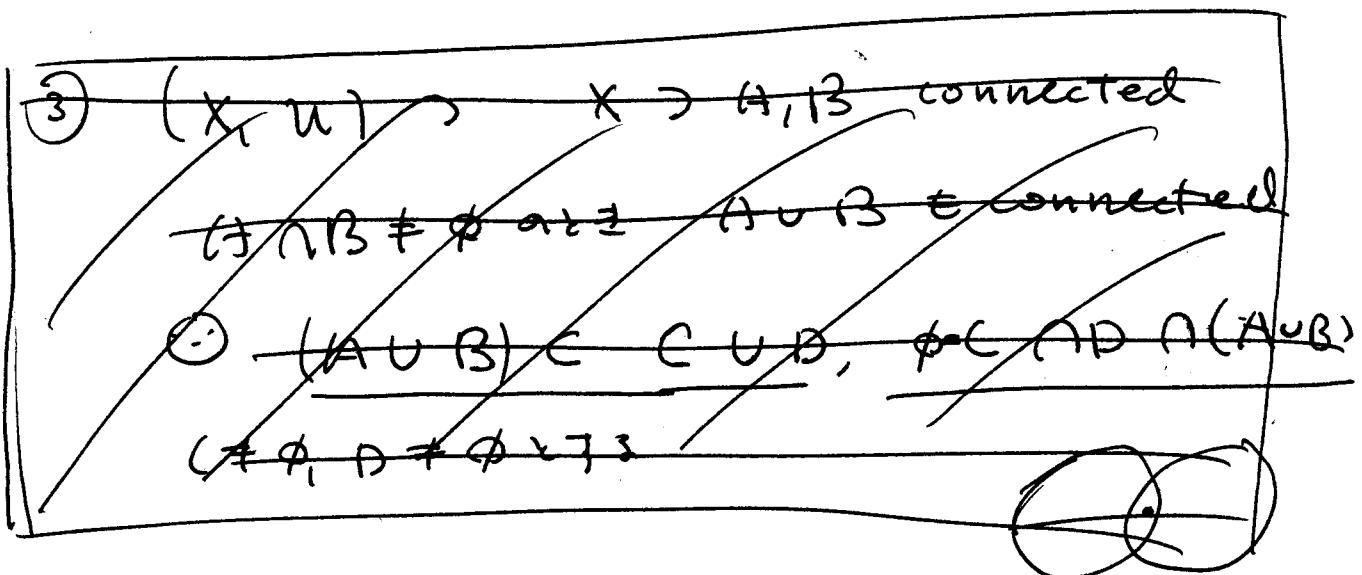
Prop 12.4 の (2) で 必要な事実

- ①  $x$  が  $A$  の 角点  $\Leftrightarrow U \cap A \neq \emptyset \quad \forall U \in N(x)$
- ②  $(x_\lambda, u_\lambda) \quad i=1, 2, \dots$

$\pi|A_\lambda$  有限個で除る  $A_\lambda = x_1$

$\begin{array}{c} (\exists \lambda_j \in N_j(x) \quad \lambda_1, \dots, \lambda_k) \\ \uparrow \\ \text{閉集合} \end{array}$   
 有限個

(は  $\pi(x_\lambda, u_\lambda)$  の 基本近傍系に在る)



$x \in \bigcap_{F \supset A} F \Leftrightarrow U \cap A \neq \emptyset, \forall U \in N(x)$

$\text{closed} \quad (\Rightarrow) U \cap A \neq \emptyset \Leftrightarrow \exists V \text{ open s.t. } x \in V \subset U$

$V \cap A = \emptyset \quad \therefore V^c \supset A \quad \therefore x \in V^c \supset A$

$(\Leftarrow) x \in \bigcap F \Leftrightarrow \text{trivial}$

# §13 Compactness

12-1

Def 13.1  $(X, \mathcal{U})$  top sp.  $\{A_\lambda\}$ ,  $A_\lambda \in \mathcal{U}$

s.t.  $X = \bigcup_\lambda A_\lambda$  となれば  $\{A_\lambda\}$  open covering となる  
(開被覆)

$\{B_\lambda\}$ ,  $B_\lambda$  は closed set

s.t.  $X = \bigcup_\lambda B_\lambda$  となれば  $\{B_\lambda\}$  closed covering となる.

Def 13.2 (1)  $(X, \mathcal{U})$  top sp.  $\{A_\lambda\}$  open cov.

~~はくせん~~  $\Leftrightarrow \exists \lambda_1, \dots, \lambda_k$  で  $X = \bigcup_{j=1}^k A_{\lambda_j}$ .  $\forall \varepsilon \in (X, \mathcal{U})$  cpt sp に

(2)  $A \subset X$  で  $(A, \mathcal{U}_A)$  が cpt sp かつ  $A \in$  cpt set となる. I.e.,  $A \subset \bigcup B_\lambda \Leftrightarrow A \subset B_\lambda$ .

Prop 13.3 (cpt は 伝不脱性の性質)

$(X, \mathcal{U}), (Y, \mathcal{V}), X \cong Y$

$X$  が cpt  $\Leftrightarrow Y$  が cpt.

有限個

$\Leftrightarrow$   $f: X \rightarrow Y$  homeo.  $Y = \bigcup_{\lambda \in \Lambda} B_\lambda$   
 $f^{-1}(B_\lambda) \in \mathcal{U}$  かつ  $\bigcup f^{-1}(B_\lambda) = X \therefore \bigcup_j f^{-1}(B_\lambda) = X$   
 $\therefore X = f^{-1}(\bigcup_j B_\lambda) \therefore Y = f(X) = \bigcup B_\lambda //$

Lemma  $(X, \mathcal{U})$  cpt sp  $(Y, \mathcal{V})$  top sp

$\Rightarrow \exists \varepsilon \in \mathcal{V}$  ~~はくせん~~ は cpt sp  $f: X \rightarrow Y$

$\textcircled{1} f(x) \in \bigcup B_\lambda \therefore x \in \bigcup f^{-1}(B_\lambda) \therefore x = \bigcup f^{-1}(B_\lambda)$

$\therefore X = \bigcup f^{-1}(B_\lambda) \therefore f(X) = \bigcup f(f^{-1}(B_\lambda)) \subset \bigcup B_\lambda //$

Lemma 13.5  $(X, \mathcal{U})$  cpt sp,  $A \subset X$  closed 12-2

$\Rightarrow A$  は cpt set

①  $(A, \mathcal{U}_A)$  が cpt set を示す

$A \in \bigcup_{\lambda} A_{\lambda}$ ,  ~~$\{A_{\lambda}\}_{\lambda}$  は  $X$  の open cov.~~

$A^c \in \mathcal{U} \therefore \{A_{\lambda}, A^c\}$  は  $X$  の open cov.

$\therefore \{A_{\lambda_j}^c, A^c\}$  は  $X$  の open cov. ~~は  $X$  の open cov.~~

有限個

$X = \bigcup_{\lambda} A_{\lambda_j}^c \cup A^c \therefore A \subset \bigcup_{\lambda} A_{\lambda_j}^c$

Lemma 13.6  $(X, \mathcal{U})$  Hausdorff sp,  $A \subset X$  cpt

$\Rightarrow A$  は closed

②  $A^c$  open  $\Leftrightarrow \exists$  ...

$y \in A^c, x \in A$

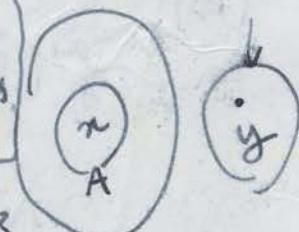


$B_y \ni y, B_x \ni x, B_y \cap B_x = \emptyset, B_y \in \mathcal{U}$

$A \subset \bigcup_x B_x$

$\therefore A \subset \bigcup_y B_{2y}$   
 $y \in B$   
 $\therefore B \cap \bigcup_y B_{2y} = \emptyset$

~~有限個だから~~



~~$A \subset \bigcup_{x \in A} B_{2x} \Rightarrow A \subset \bigcup_{x \in A} B = A$~~

$\therefore y \in B \subset A^c$

$B = B_y$  が成り立つ

$\bigcup_y B_y = A^c$  が成り立つ

$A^c$  は open!

Lemma 13.7  $(X, \mathcal{U})$ ,  $(Y, \mathcal{V})$  cpt sp  $\Rightarrow_{\text{def}} \exists$  12-3

$(X \times Y, \mathcal{U} \times \mathcal{V})$  is cpt set.

~~開被覆~~

$$\textcircled{1} \quad X \times Y = \bigcup_{x \in X} \mathcal{M}_x \quad \mathcal{M} = \{\mathcal{M}_x\}$$

$\mathcal{O}_1 = \{A \in \mathcal{U}; A \times Y \text{ 有 } \mathcal{M}_x \text{ 有 } \beta \text{ 個 open sets 覆之}\}$

$\bigcup \mathcal{O}_1 = X$  -  $\textcircled{1}$  を示す。

$$\text{証明} \quad X \times Y = \bigcup_j (A_j \times Y) \text{ は } (\mathcal{M}_x \text{ 有 } \beta \text{ 個 open sets 覆之}}$$

$$X = \bigcup_j A_j \quad (\text{cpt は } \textcircled{2}) \quad \text{で 覆之}$$

$\textcircled{2} \quad \forall x \in X \exists \mathcal{M}_x$

$$\mathcal{O}_{(x)} = \left\{ B \in \mathcal{V}; \exists A \in \mathcal{U} \text{ s.t. } A \times B \subset \mathcal{M}_x \right\}$$

$$\bigcup \mathcal{O}_{(x)} = Y$$

$\therefore \bigcup \mathcal{O}_{(x)} \subset Y$  (trivial)

$$\bigcup \mathcal{O}_{(x)} \supset Y \in \mathcal{V},$$

$$\forall y \in Y, (x, y) \in \mathcal{M}_x$$

$$\begin{aligned} & (A \times B \text{ の } \textcircled{1}, \text{ は ban } T_2 \text{ の } 3) \quad (x, y) \in A \times B \in \mathcal{M}_x \\ & A \in \mathcal{U}, B \in \mathcal{V} \end{aligned}$$

$$\therefore B \in \mathcal{V} \Rightarrow y \in B$$

$$\therefore y \in \bigcup \mathcal{O}_{(x)} //$$

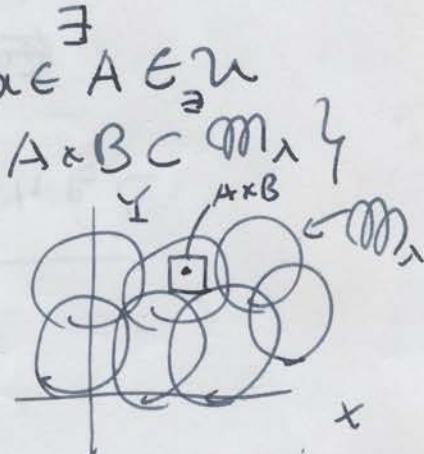
$Y \text{ は cpt は } \textcircled{3}$

$$\bigcup_j B_j = Y, B_j \in \mathcal{O}_{(x)}.$$

$$\therefore \exists A_j \times B_j \in \mathcal{M}_x, x \in A_j \quad \therefore \emptyset = \bigcap A_j \text{ (open)}$$

$$O \times Y = \bigcup_i (\bigcap_j A_j \times B_i) \subset \bigcup_i (A_i \times B_i) \subset \bigcup \mathcal{M}_x;$$

$$\therefore O \subset O \quad \therefore X \subset \bigcup \mathcal{O}_{(x)}$$



12-4

Cor 13.8  $(X, \mathcal{U})(Y, \mathcal{V})$  top sp

$A \subset X, B \subset Y$  cpt set

$\Rightarrow \alpha \in A \times B$  is  $X \times Y$  a cpt

$\therefore (A \times B, \mathcal{U}_A \times \mathcal{V}_B)$  is cpt (Prop 13.7)

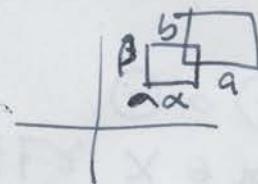
$(A \times B, (\mathcal{U} \times \mathcal{V})_{A \times B})$  is cpt ??

$$\mathcal{U}_A \times \mathcal{V}_B = (\mathcal{U} \times \mathcal{V})_{A \times B}$$

$\therefore \{\alpha \cap A \times \beta \cap B\}$  is a base.

$\{( \alpha \times \beta ) \cap (A \times B)\}$  is a base

~~$\{\alpha \cap A \times \beta \cap B\}$~~



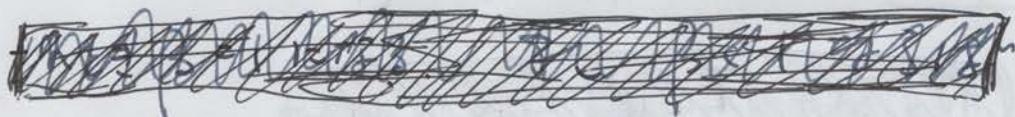
$\Rightarrow$  is a base for topology.

## Example 1

$[a, b] \subset \mathbb{R}$  ( $\mathbb{R}$  a cpt set)  $A = \{A_\alpha\}$

$$\therefore [a, b] = \bigcup_{\alpha} A_\alpha \quad A_\alpha \text{ open sets}$$

$\beta = \sup\{\alpha; [0, \alpha) \text{ is } A_\alpha \text{ 有限で覆える}\}$



$[a, \beta]$  は 有限で 覆える  $\Rightarrow \sup \text{ ndef}$

もし  $\forall \varepsilon > 0 \exists \alpha \in \mathbb{N} \ni \forall \alpha > \beta - \varepsilon \text{ 且 } [0, \alpha)$  は  
有限で 覆える. i.e.  $\beta \in {}^3A_\lambda$  すなはち  $(-\varepsilon + \beta, \beta + \varepsilon) \cap A_\lambda$

$\therefore [0, \beta - \frac{\varepsilon}{2})$  は 有限で 覆える

$[0, \beta)$  も 有限で 覆える.

実は  $\beta = b$  になる. 上の  $\varepsilon = 0$  と 同不等式

$\beta < b$  ならば  $[0, \beta + \varepsilon) (\beta + \varepsilon < b)$  が  
有限で 覆える  $\beta$  の def に反する

$\therefore [a, b)$  は 有限で 覆える.  $b \in {}^3A_\lambda$  すなはち

$[a, b]$  は 有限で 覆える.

12-6

Example 2.  $[a_j, b_j] \subset \mathbb{R}$  のとき

$\prod_j [a_j, b_j] \subset \mathbb{R}^n$  は cpt で sm

Example 3  $K \subset \mathbb{R}^n$  が cpt  $\Leftrightarrow K$  は有界閉

(⇒)

~~( $\mathbb{R}^n$  は Hausdorff で closed で cpt)~~

$\mathbb{R}^n$  は Hausdorff で closed で cpt

$$K \subset \bigcup_{x_i \in K} U_\varepsilon(x_i) \quad \therefore K \subset \bigcup_{x_j} \underline{U_\varepsilon(x_j)}$$

このとき有界



(⇐)

~~( $\mathbb{R}^n$  は Hausdorff で closed で cpt)~~

$$\therefore K \subset \prod_j [a_j, b_j] = A \text{ cpt}$$

$(A, \mathcal{U}_A)$  は cpt sp.  $K \subset A$

$\mathcal{U}_A$  は top で  $K$  は closed

$\therefore K$  は cpt.

Example 4  $\mathbb{R}, (a, b)$  は cpt で sm

Example 5  $(X, \mathcal{U}_\infty)$  が cpt  $\Leftrightarrow X$  は有界集合

## § 9.14 Compactness II

① Tychonoff の定理

② compact set は cont ft

③ Hausdorff + compact

④ 1点 cpt 化

① Thm<sup>14</sup> 9.1 (Tychonoff の定理)

$(X_\lambda, \mathcal{U}_\lambda)$   $\lambda \in \Lambda$  cpt sp.

$\prod(X_\lambda, \mathcal{U}_\lambda)$  が cpt  $\Leftrightarrow (X_\lambda, \mathcal{U}_\lambda)_{\lambda \in \Lambda}$  が cpt.

② ( $\Rightarrow$ ) easy contr

$P_\lambda : \prod X_\lambda \rightarrow X_\lambda$  projection が cpt にはならぬ。

最弱な top.  $\therefore P_\lambda(\prod(X_\lambda)) = X_\lambda$  は cpt.

( $\Leftarrow$ ) 略

Thm<sup>14</sup> 9.2

②  $(X, \mathcal{U})$  cpt sp  $f : X \rightarrow \mathbb{R}$  cont

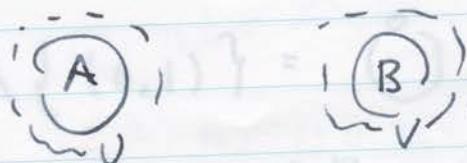
このとき  $f(x)$  は min & max を含む。

③  $f(x) : \mathbb{R} \rightarrow \mathbb{R}$  は cpt.  $\therefore f(x)$  は 有界 関  
min & max を含む。

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③ Thm #3  $(X, \mathcal{U})$  は cpt の Hausdorff  
space  $(X, \mathcal{U})$  は 正規

$\because A, B \subset X$  closed

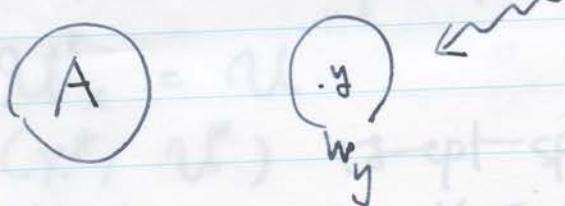


$$U \cap V = \emptyset \text{ が成り立つ。}$$

$A$  は cpt ok  $y \in B$  を fix する。

Lemmas 3.6 の ② によると  $\exists U \in \mathcal{U}$  で

$U \supset A$ ,  $\bar{U} \not\ni y$   $\therefore W_y$   
 $\therefore X \setminus \bar{U} \ni y$  つまり  $X \setminus \bar{U}$  は  $y$  の open nbh.  
 $\therefore \overline{W_y} \cap A = \emptyset$



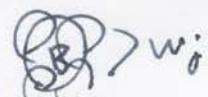
$B = \{W \in \mathcal{U}; \bar{W} \cap A = \emptyset\} \Rightarrow W_y$  など

$B$  は  $\mathcal{B}$  の open cov. である  $B$  が cpt 性質

$B \subset \bigcup_j W_j$ ,  $W_j \in B$ ,  $\overline{W_j} \cap A = \emptyset$

$\therefore \bigcup_j W_j$ ,  $X \setminus \overline{\bigcup_j W_j}$  は open  $\therefore C \cap D = \emptyset$

$\exists C, D \subset X$ ,  $A \subset C \cap D$  なる  $C, D$  が存在する。



~~§9~~ 1. 点 cpt は

$$\mathbb{R}^2 \ni S' = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$$

は bounded & closed  $\therefore$  cpt

$$S' \setminus \{(0, 1)\} = \overset{\circ}{\bigcirc} \cong \mathbb{R}$$

-  $S'$  は  $\mathbb{R}^2$  の 1 点 cpt は など

Thm 9.1 (左の定理の証明) 1 点 cpt は

$$(X, U) \text{ top sp. } X^* = X \cup \{\infty\}$$

$$U^* = \{X^* \setminus K; K \uparrow \text{closed + cpt}\} \cup \{\infty\}$$

( $\infty$  は cpt な点)

①  $(x^*, U^*)$  top sp

$$\text{② } U_x^* = U$$

③  $(x^*, U^*)$  は cpt sp

④  $X \neq \text{cpt} \Rightarrow X \subset X^*$  ( $\Rightarrow$  dense in  $U^*$ )

⑤  $X^*$  Hausdorff  $\Leftrightarrow X$  locally cpt + Hausdorff.

（証） $(X, U)$  top sp.  $A \subset X$

①  $\bar{A}$  cpt  $\Rightarrow A$  relatively cpt は

②  $\forall x \in X$  が relatively cpt は open neighborhood  $\epsilon + x \in U$  (locally cpt は)  $\therefore \exists x \in U, \bar{U}$  cpt open.



13-4

①  $U^* \ni x^*, \text{ すなはち } \exists U$

$U^* \ni A, B \text{ のとき } A \cap B \in U^*$

$\therefore A, B \in U \text{ のとき } \exists U$

$A = X \setminus K, B = X \setminus L \quad K, L \text{ cpt + closed}$

$\therefore A \cap B = X \setminus (K \cup L) \quad X \setminus K \cap B \in U$

$A = X \setminus K, B \in U \text{ のとき } X \setminus K \cap B \in U$

$U^* \ni A_\lambda \text{ のとき } \bigcup A_\lambda \in U^*$

$\therefore A_\lambda = X \setminus K_\lambda \in U$

$\cdot \bigcup A_\lambda = X \setminus \bigcap K_\lambda \in U \quad \text{closed cpt} \therefore \bigcap K_\lambda \subset K \text{ すなはち } \bigcap K_\lambda \text{ cpt}$

$\cdot \bigcup_{\lambda} X \setminus K_\lambda \cup A_\lambda = (X \setminus \bigcap K_\lambda) \cup A_\lambda \in U$

②  $A \in U_x^*$  i.e.  $A = \bigcup_{\lambda} A'_\lambda \cap X = \begin{cases} A' \in U \Rightarrow A \in U \\ A' = X \setminus K \Rightarrow X \setminus K \in U \end{cases}$

$\therefore U_x^* \subset U$

$\forall U \in U_x^* \subset U \in U$  すなはち  $\forall A \in U$  は  $A = A \cap X$   
 $\exists \lambda \in \omega$  で  $A \in U^*$  とみなせる

③  $X^* = \bigcup M_\lambda, M = \{M_\lambda\}$  とす。

$\omega$  の 開 集 合 の 族 は  $\bigcup M_\lambda \in U$  すなはち  $\bigcup M_\lambda = X^*$

$\omega \in M_\lambda = X \setminus K \quad \therefore X^* = (X \setminus K) \cup K$

$M'_\lambda = M_\lambda \setminus \{\omega\} = M_\lambda \cap X \in U_x^* = U$

$\therefore \bigcup_{\lambda \neq \lambda_0} M'_\lambda \supset K \quad \therefore \bigcup M'_\lambda \supset K$

$\therefore X^* = M_{\lambda_0} \cup M'_\lambda$

13-5

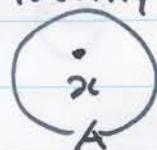
$$④ \bar{X} = X^* = X \cup \{\infty\}$$

$\bar{X} \setminus \{\infty\}$  է Հանսդորֆ սահմանական է և  $X^*$  է պահպան և  $X^*$  է առանց անշահագույն պահպան սահմանական է.

$\{\infty\} \in \text{open sets}$  ինչ թափանց է այս պահպան սահմանական է և  $\infty \in \text{closed sets}$ .

$x \neq \infty$  առանց  $x$  է  $X \setminus K \neq \emptyset$  ինչ էլ  $X \setminus K \cup \{\infty\} \neq \{x\}$  է առանց  $x$  է  $X \setminus K \cup \{\infty\} \cap x \neq \emptyset$  ի. բ.  $\{\infty\} \subseteq \bar{X}$ .

⑤ ( $\Rightarrow$ )  $X^*$  Հանսդորֆ  $\Rightarrow$   $\{x\}$  է ՏՅԱՀ և  $x \in \text{Hansdorff}$   
այս էլ locally cpt է այս է



$$A \cap B = \emptyset$$

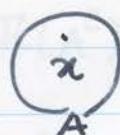
$$\therefore \bar{A} \cap B = \emptyset$$

$\bar{A}$  է cpt

$\therefore \bar{A} \subset X \Rightarrow \bar{A} \neq \infty$ .

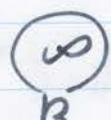
այս է  $\bar{A}$  է  $X^*$  է cpt

( $\Leftarrow$ )



$\bar{A}$  է cpt այս է այս է

$$B = (X \setminus \bar{A}) \cup \{\infty\}$$



$A \cap B = \emptyset$  է այս է այս է

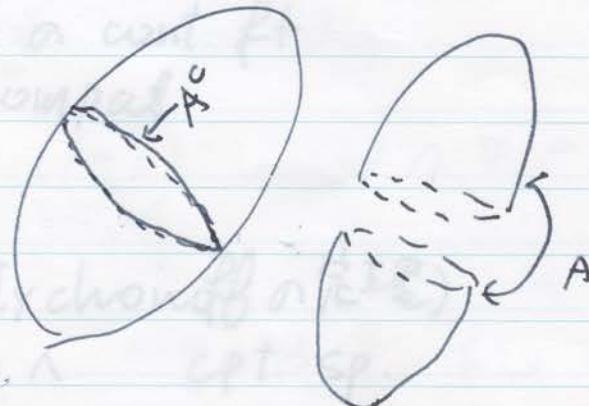
է պահպան և  $x, y \in X$  այս է այս է

$x$  է Hansdorff-ի է այս է ok.

(3-6)

Example.  $(X, \mathcal{U})$  cpt Hausdorff

$A \in \mathcal{U} \setminus \{\emptyset, X\}$



$$X/A^c = \text{closed set} \simeq A^* = \text{open set}$$

(-)

$$\pi : X \rightarrow X/A^c$$

$$f : A^* \rightarrow X/A^c$$

$$f(x) = \begin{cases} \pi(x), & x \in A \\ \pi(A^c), & x = \infty \end{cases}$$

∴  $\exists$   $\forall$   $B$   $f$  is bijective to  $X/A^c$

$\exists \exists B \subset X/A^c$  open set  $\forall \exists$

$$f^{-1}(B) = \{x \in A^* : f(x) \in B\} = \{x \in A^* : f(x) \in B\}$$

$\pi(A^c) \cap B = \emptyset$

$$= \pi^{-1}(B) \cup \left\{ \infty ; f(\infty) \in B \right\}$$

cont.

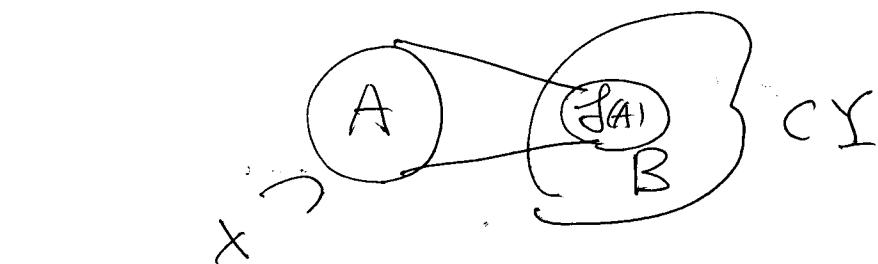
-自反角:  $(X, \mathcal{U}) \xrightarrow{\text{cpt}} (Y, \mathcal{V})$  Hausdorff  
 $f : X \rightarrow Y$  cont  $\Rightarrow$  closed mapping  $\} \therefore f$  is homeo. //

## §15 寫像空間

$(X, \mathcal{U}), (Y, \mathcal{V})$  top SP

$\text{Map}(X, Y) = \{f: X \rightarrow Y, \text{cont}\}$

$w(A, B) = \{f \in \text{Map}(X, Y); f(A) \subset B\}$



一般定義  $(X, \mathcal{U})$  top SP  $\mathcal{U}_0 \subset \mathcal{U}$

s.t.  $A_1, \dots, A_m \subset A$  全体が  $(X, \mathcal{U})$  の base

$H_j \in \mathcal{U}_0$

使得  $\mathcal{U}_0$  是 subbase or 基 ③

$w(K, A)$   $K \subset X$  cpt,  $A \in \mathcal{V}$

$\mathcal{M} = \left\{ \bigcap_{j=1}^m w(K_j, A_j); m \geq 1, K_j \subset X \text{ cpt}, A_j \in \mathcal{V} \right\}$

$\mathcal{M}$  是  $\cap$  ③ 的

Prop 3.3  $\mathcal{M} = \left\{ \bigcup_{\lambda} B_{\lambda}; B_{\lambda} \in \mathcal{M} \right\}$  是 topology ② ④

$\mathcal{M}$  是 base ② ③

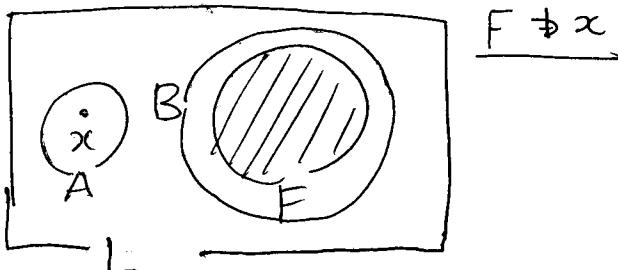
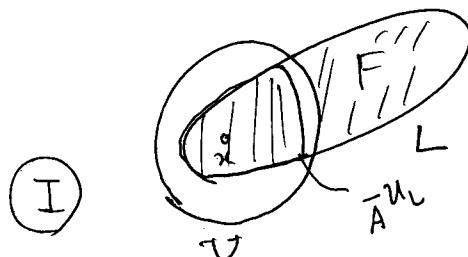
Def 15.1  $\text{Map}(X, Y)$  上 topology  $B \in$   
compact open topology ④

Lemma 15.  $\mathbb{R}^1$   $(X, \mathcal{U})$  locally cpt  $\Leftrightarrow$

- ①  $x \in X$  の 任意の open nbh  $U$  に  $\exists K$  cpt 使得し  $x \in K \subset U$

- ②  $(X, \mathcal{U})$  は 正則

○  $x \in L$ . cpt nbh  
 $\exists K \subset (L, \mathcal{U}_L)$  は cpt Hausdorff いはず  
 また  $K \subset (L, \mathcal{U}_L)$  は 正則 (check)  
 $F = L \cap U^c$  は  $(L, \mathcal{U}_L)$  で closed



$x \in A, B \supset F$   $A \cap B = \emptyset$ ,  $(A, B) \in \mathcal{U}_L$   
 (正則にはずれ)

$$\overline{A}^{\mathcal{U}_L} \cap F = \emptyset$$

$\Rightarrow \overline{A}^{\mathcal{U}_L}$  は  $X$  で cpt  $\therefore \overline{A}^{\mathcal{U}_L}$  は  $L$  で cpt

$\therefore \overline{A}^{\mathcal{U}_L} \subset \bigcup_x B_x$   $B_x \in \mathcal{U}$  いはずれ

$$\overline{A}^{\mathcal{U}_L} \subset \bigcup_x (B_x \cap L)$$

$$\therefore \overline{A}^{\mathcal{U}_L} \subset \bigcup_j (B_{x_j} \cap L) \subset \bigcup_j B_{x_j}$$

$\therefore \overline{A}^{\mathcal{U}_L}$  は  $X$  で cpt!

$\overline{A}^{\mathcal{U}_L} = K$  いはずれ  $x \in K \subset U$  (I) いはずれ

$x \in A \subset K \subset U \rightarrow$  いはずれ

14 3

$$(L, \mathcal{U}_L) \xrightarrow{\text{top}} \downarrow \quad \text{cpt} \\ x \in A \subset K \subset U$$

$$A \subset L \cap A \quad A \in \mathcal{U} \in \frac{1}{2}\text{reg}$$

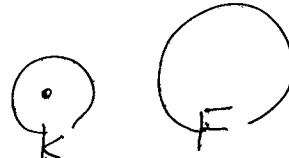
$$L \neq \text{cpt nbh} \quad \exists B \in \mathcal{U} \text{ s.t.} \\ x \in B \subset L$$

$$x \in \underline{A' \cap B} \subset A \subset K \subset U$$

これは閉じた

石塀かい =  $K \neq \text{cpt} \neq \text{近傍}$

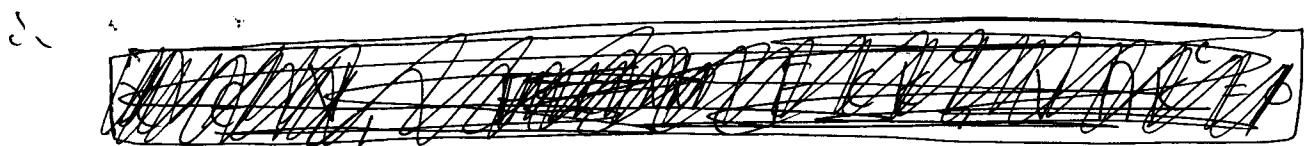
正則性の定義



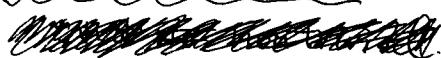
$$x \in X, F \subset X \text{ closed } x \notin F$$

$$x \in K \subset F^c \quad \nvdash \underline{\text{cpt 近傍}} \quad (\text{即ち開 Haudorff空間})$$

$$V = K^\circ \quad K^\circ \neq \text{open} \\ (\neq \emptyset)$$



$$x \in K^\circ, K^\circ \supset F, \quad \underline{K^\circ \cap K^\circ = \emptyset}$$



$$e \cdot \text{Map}(X, Y) \times X \rightarrow Y$$

$$(f, x) \mapsto f(x)$$

evaluation map

$$ex \quad \text{Map}(X, Y) \xrightarrow{f} f(x) \in Y$$

定理 15.2  $(X, \tau)$  locally cpt + Hausdorff  
 $(Y, \gamma)$  top sp

$$e \cdot \text{Map}(X, Y) \times X \rightarrow Y \text{ is cont.}$$

○  $\forall A \in \gamma$   $\exists$  open  $E(A)$

$$E(A) = \{(f, x) \mid f(x) \in A\}$$

$x \in f^{-1}(A)$  open

locally cpt sp  $a \in f^{-1}(A)$   $x \in \bigcap_{K \subset f^{-1}(A)} K$

~~cpt~~  $\Rightarrow$  近傍

$f(K) \subset A \therefore f \in W(K, A)$

$\therefore (f, x) \in W(K, A) \times K$

$$\in W(K, A) \times B = C_{f(x)}$$

open set

$\begin{array}{c} \cap \\ \cup \\ x \in B \subset K \end{array}$

近傍  
集合

$$E(A) = \bigcup_{(f, x) \in E(A)} C_{f(x)} \text{ is open}$$

## §16

閉集合空間

$$\text{Map}(X, \mathbb{R}) - C(X)$$

Lemma 16.1

$$X \text{ cpt} \Rightarrow d(f, g) = \max_{x \in X} |f(x) - g(x)| \quad (\exists)$$

 $C(X)$  上の metric  $d(f, g)$ .

$\therefore f: X \rightarrow \mathbb{R}$   
 $x \mapsto |f(x) - g(x)|$  (cont)  
 $f(x) := \max_{i \in \min}$  (check  $\forall$ )

①  $d(f, g) \geq 0$ ,  $d(f, g) = 0 \Leftrightarrow f = g$

②  $d(f, g) = d(g, f)$

③  $d(f, g) + d(g, h) \geq d(f, h)$  check  $\forall$

Theorem 16.2  $(X, \mathcal{W})$  cpt space

$$(C(X), d) \times (C(X), \mathcal{U}_{C_0}) \text{ 同構}$$

i.e.  $\mathcal{U}_d = \mathcal{U}_{C_0}$

○  $\mathcal{U}_d \supset \mathcal{U}_{C_0}$

$\mathcal{U}_{C_0} \ni W(K, U) = \{f; f(K) \subset U\} \ni f$   
 $\exists a \in U, f(K) \subset W(K, U) \subset \exists$   
 $(\because \exists W(K, U) \in \mathcal{U}_d \text{ は } \exists)$

$f(K) \subset U$   $f(K)$  cpt  $\therefore$  有界  $\checkmark$  check  $\forall$

$\exists \delta > 0$  s.t.  $f(K) \supset \bigcup_{x \in K} B_{\delta}(f(x)) = \bigcup_{x \in K} (x - \delta, x + \delta) \subset U$

$\therefore d(f, g) < \delta \text{ は } g(K) \subset U$

$U_\delta(f) \subset W(K, U) / \overline{\text{有限個}} \cdot f(g) - \delta < g(g) < f(g) + \delta$

$$\bar{f}(x \setminus A) = \{x ; f(x) \notin A\} \quad 14-6$$

$$= \{x ; f(x) \in A^c\}^c = (\bar{f}(A))^c$$

$U_d \subset U_{co}$

$U_\varepsilon(f) \in U_d$  if  $U_{co}$  is open in  $\mathbb{R}$ .

$$X = \bigcup_{x \in X} \bar{f}\left((f(x) - \frac{1}{3}\varepsilon, f(x) + \frac{1}{3}\varepsilon)\right)$$

$$= \bigcup_{j=1}^k \bar{f}\left((f(x_j) - \frac{1}{3}\varepsilon, f(x_j) + \frac{1}{3}\varepsilon)\right)$$

$K_i = f^{-1}\left([f(x_j) - \frac{1}{3}\varepsilon, f(x_j) + \frac{1}{3}\varepsilon]\right)$  if closed

$K_i$  is cpt  $\therefore X = \bigcup_{j=1}^k K_j$  開  $\rightarrow$  閉 is compact

$$W = \bigcap_{j=1}^k W\left(K_j, (f(x_j) - \frac{1}{2}\varepsilon, f(x_j) + \frac{1}{2}\varepsilon)\right) \ni f$$

: )  $f(K_j)$   
 $C(f(x_j) - \frac{1}{2}\varepsilon, f(x_j) + \frac{1}{2}\varepsilon)$

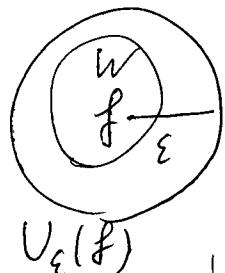
$\exists g \in W, \forall x \in \mathbb{R} \exists n \in \mathbb{Z} \forall i \in K_i$

$$|f(x) - g(x)| \leq |f(x) - f(x_i)| + |f(x_i) - g(x_i)| \leq \frac{\varepsilon}{2}$$

$$\max_n |f(x_i) - g(x_i)| < \varepsilon$$

$W \subset U_\varepsilon(f)$  ok

$W \subset U_\varepsilon(f) \subset W_f$



~~$U_\varepsilon(f)$  is basic~~  $\Rightarrow U_\varepsilon(f)$  is basic

$$U_\varepsilon(f) \ni g$$

$$V_\varepsilon(g)$$

$$V_\varepsilon(g) \subset U_\varepsilon(f) \subset U_\varepsilon(g)$$



$$U_\varepsilon(f) = U_\varepsilon(g)$$

$$V_\varepsilon(g) \subset U_\varepsilon(g) \subset U_\varepsilon(f)$$

$F \subset C(X)$  は  $\hookrightarrow$  compact ですか?  
(レーリーのままで問題)

Def 16.3  $(X, \mathcal{U})$  cpt sp.  $C(X)$

- ①  $F \subset C(X)$  は  $\forall f \in F \exists M > 0$  st  $\max_{x \in X} |f(x)| < M$   
すなはち  $F$  は 一様有界 である
- ②  $F \subset C(X)$  は  $\forall \epsilon > 0 \exists r > 0$  ある  $n$  の互傍  $A$   
st  $\forall f \in F \exists g \in A$  使得する  $|f(g) - f(x)| < \epsilon$   
同程度連續

Theorem 16.4 Ascoli Arzelà  
 $(X, \mathcal{U})$  cpt sp.  $F \subset C(X)$   $(C(X), d)$

$F$  cpt  $\Leftrightarrow F$  が 一様有界 + 同程度連續

Ex  $F = (f_n)$  関数列 が 一様有界、同程度連續  
 $(f_n)$  は 有界閉集合上 で 一様収束する部分列を持つ



Thm 16.4 も ( ) で前に metric space の cpt と  $\mathbb{R}$  は  
間の事実を示す。

Def 16.5  $(X, d)$  metric space

- ①  $A \subset X$ ,  $\forall \varepsilon > 0 \exists r > 0$  有限個の  $\varepsilon$  open ball  $\mathcal{B}$   
 $A$  が  $\mathbb{R}$  に  $\mathbb{R}$  全有界 (totally bounded) とする
- ②  $X$  が全有界  $\Rightarrow (X, d)$  は全有界とする

Thm 16.6  $(X, d)$  metric sp 次の 3 つは同値

- ①  $(X, d)$  が cpt
- ②  $(X, d)$  が 全有界かつ完備
- ③  $X$  は 点列上 cpt

二二互い Thm 16.4 と ( )

( $\Rightarrow$ )  $\bar{F}$  cpt, metric space  $\therefore F$  は全有界,  $F$  は有界  
同程度連続関数を示す

$$\bar{F} \subset \bigcup_{j=1}^k U(f_j, \frac{1}{3}\varepsilon) \quad (\forall \varepsilon > 0 \exists r > 0) \quad \text{と } \mathbb{R}$$

$f_j$  は cont と  $\forall x \in X \exists r > 0 \exists U_j$  st

$$y \in U_j \Rightarrow |f_j(x) - f_j(y)| < \frac{1}{3}\varepsilon \quad \text{ok}$$

$$U = \bigcap U_j \quad \therefore U \ni y \Rightarrow |f_j(x) - f_j(y)| < \frac{1}{3}\varepsilon \quad \forall j$$

$$\therefore \forall g \in F \exists r > 0 \exists U(f_j, \frac{1}{3}\varepsilon) \quad \text{251=}$$

$$y \in U \text{ ならば } |g(x) - g(y)| \leq |g(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - g(y)| < \varepsilon //$$

( $\Leftarrow$ )  $F$  は metric space たる  $f_i$   
・ 実数 + 全有界 ならば  $\cup U_n$

実数 - 後で示す  $\forall \epsilon > 0 \exists \delta > 0$

$$\exists U_n \ni x \ni \frac{|f(x) - f(y)| < \frac{1}{4}\epsilon}{\forall y \in U_n \forall f} \quad (同程度重系壳性)$$

Cpt. +  $\forall f$   
 $X = \bigcup_{j=1}^n U_{x_j}$  と  $\epsilon > 0$  とする

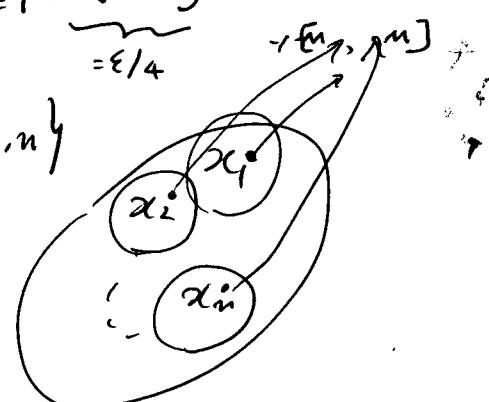
$$|f(x)| < M \quad (\text{図 様有界性 } + n)$$

$$f: X \rightarrow [-M, M] = \bigcup_{j=1}^s [a_j, b_j]$$

$$F(j_1, \dots, j_n) = \{f \in F; f(x_i) \in I_{j_i}, i=1, \dots, n\}$$

たとえば  $1 \leq j_i \leq s$ . 重複なし

$x_1, \dots, x_n$  の 間に先に  $M$  が  $\exists f = f_p$ .



$F(j_1, \dots, j_n) \neq \emptyset$  とし 集合を並べる

$F_1, \dots, F_m$   $F_k \in \mathcal{F} \rightarrow f_k \in F$  の出力 ( $F_k \supset f_k$ )

$$\forall f \in F \quad (= x_{j_1, \dots, j_n}) \stackrel{f_k}{\rightarrow} f$$

i.e.  $|f(x_i) - f_k(x_i)| \leq \frac{1}{4}\epsilon \quad \forall i$

$$\forall x \in X \quad \exists U_{x_i} \ni x$$

$$|f(x) - f_k(x)| \leq |f(x_1) - f(x_i)| + \underbrace{|f(x_1) - f_k(x_1)|}_{\leq \frac{\epsilon}{4}} + |f_k(x_i) - f_k(x_1)| < \frac{3}{4}\epsilon$$

$$\therefore d(f, f_k) \leq \frac{3}{4}\epsilon \quad F \subset \bigcup_{k=1}^m U(f_k, \epsilon) \quad //$$

完備性 Lemma 16.7  $X$  cpt  $C(X)$  は 完備

○  $\{f_n\}$  s.t.  $d(f_n, f_m) \rightarrow 0$  ( $n, m \rightarrow \infty$ )

$f_n(x) \in \mathbb{R}$  t Cauchy  $\exists \lim_n f_n(x) = f(x) \in \mathbb{R}$   
( $\mathbb{R}$  の 完備性)

$f: X \rightarrow \mathbb{R}$   $f \in C(X) \Leftrightarrow d(f, f_n) \rightarrow 0$  すなはち

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|$$

$$\because \exists \epsilon \quad d(f_n, f_N) < \frac{1}{3}\epsilon \quad (\forall n > N) \quad \epsilon > 0$$

$$\exists n \quad |f_n(x) - f_N(x)| < \frac{1}{3}\epsilon \quad \forall x$$

$$|f(x) - f_N(x)| < \frac{1}{3}\epsilon \quad \forall x \quad \text{④}$$

$$\exists U \quad f_N(x) \text{ は cont} \quad \exists V \text{ s.t. } |f_N(x) - f_N(y)| < \frac{1}{3}\epsilon \quad \forall y \in V$$

$$|f(x) - f(y)| \leq \epsilon \quad \forall y \in V \quad \therefore f \text{ は cont}$$

$$\therefore \text{④ かつ } d(f, f_N) < \frac{1}{3}\epsilon \quad \text{すなはち } f_N \rightarrow f \text{ は } \epsilon \text{ 附近で}$$