The lowest eigenvalue of Rabi model and non-commutative harmonic oscillator

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Workshop on "Random analytic functions, random matrices, and determinantal point processes" on Nov. 29,2013

1. Rabi model
2. Non-Commutative Harmonic Oscillator
3. Decomposition of Q
4. Simplicity of the lowest eigenvalue and no crossing

2level atom+ harmonic oscillator=Rabi model

■$\omega > 0$, $|\Delta| < 1$

■2level atom+ harmonic oscillator

$$H_0 = \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix} + \omega a^* a$$

■$a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx})$, $a^* = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$, $[a, a^*] = 1$, $Spec(a^*a) = \{n\}_{n=0}^{\infty}$

■$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$H = \Delta\sigma_z + \omega a^* a + g\sigma_x(a + a^*) \cong -\Delta\sigma_x + \omega a^* a + g\sigma_z(a + a^*)$$

■Nobel prize for Physics 2012 was awarded to S. Haroche and D. Wineland!!, who observed the Rabi model in experiment.

■$Spec(H) = \{E_n(g)\}_{n=0}^{\infty}$

■spectral curves: $g \mapsto E_n(g)$

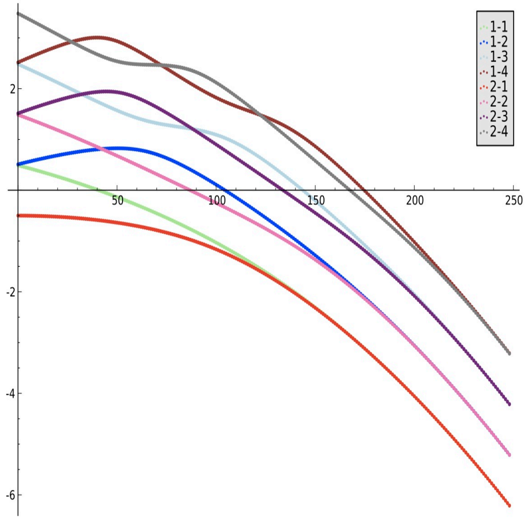


Figure: $E_n(g)$, $\omega = 1$, $\Delta = 1/2$

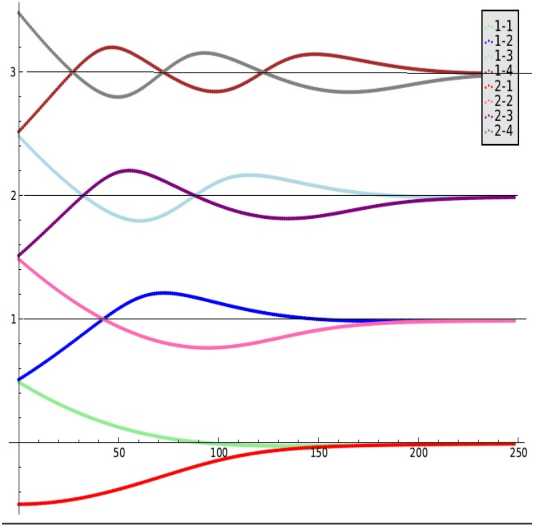


Figure: $E_n(g)+g^2$, $\omega=1$, $\Delta=1/2$

We can expect that
(degeneracy) $E_n(g)$ is degenerate $\Longrightarrow E_n(g)+g^2 \in \mathbb{N}$
(corssing) $E_{2m+1}(g)$ and $E_{2m}(g)$ crosses m times, or
$m = E(g)+g^2$ for some $g_1, ..., g_m$
(lowest e.v.) $E_0(g)$ is simple.

Bargmann representation

■$H_B = L^2(\mathbb{C}, e^{-|z|^2}/\pi)$,
■$a \cong d/dz$, $a^* \cong z$
■$\psi \in L^2(\mathbb{C}) \Longrightarrow \Psi(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n$ an entier analytic function
■

$$\left(\Delta\sigma_x + wz\frac{d}{dz} + g\sigma_z(z + \frac{d}{dz})\right)\Psi(z) = E\Psi(z)$$

■$\Psi(z) = \sum c_n z^n$ **Poincare difference equation**(Poincare 1885)

$$c_{n+1} + \alpha_n c_n + \beta_n c_{n-1} = 0$$

■$\alpha_n \to \alpha$, $\beta_n \to \beta$ characteristic equation $t^2 + \alpha t + \beta = 0$
■$t = s_1, s_2 \in \mathbb{C}$ are solutions. Then $\exists\{c_n^j\}$ st $\lim c_{n+1}^j / c_n^j = s_j$
■$\exists s = 0 \Longrightarrow \Psi(z) = \sum c_n z^n$ is entire.

$E(g)$ is degenerate $\Longrightarrow E(g)+g^2=n$

■Bogoliubov transformation ($\omega=1$) $U=e^{-g(a^*-a)}$
■$UHU^{-1}\Phi=E\Phi$
■$\sigma_x v_\pm=\pm v_\pm$, $\Phi=\Phi_+v_++\Phi_-v_-$, $x=E(g)+g^2$
■$(x-a^*a)\Phi_\pm=\Delta U\Phi_\mp$

$$\left(x+(z\mp g)(\frac{d}{dz}\mp g)\right)\left(x-z\frac{d}{dz}\right)\Phi_\pm=\Delta^2\Phi_\pm$$

■$\Phi_+(z)=\sum c_nz^n$, $(n-x)c_n=b_n$,

$$b_{n+1}-\frac{1}{n+1}\left(\frac{1}{g}\left((n-x)-\frac{\Delta^2}{n-x}+g\right)\right)b_n+\frac{1}{n+1}b_{n-1}=0$$

■$\exists K_n$ st $K_{n+1}/K_n\to 0$ and $\Phi_+(z)=\frac{K_n}{n-x}z^n$
■$E(g)$ is degenerate$\Longrightarrow n-x=0$ with some $n\Longrightarrow E(g)+g^2=n$

$E(g)+g^2=n \Longrightarrow E(g)$ is degenerate

■$(\Delta=0)$ $H(g)\Phi_n^\pm=E_n\Phi^\pm$, $H(-g)\Psi_n^\pm=E_n\Psi^\pm$,

$$\Phi_n^+=\begin{bmatrix}(z+g)^n e^{-gz}\\(-1)^n(z-g)^n e^{gz}\end{bmatrix}, \Phi_n^-=\begin{bmatrix}(z+g)^n e^{-gz}\\(-1)^{n+1}(z-g)^n e^{gz}\end{bmatrix}$$

$$\Psi_n^+=\begin{bmatrix}(-1)^n(z-g)^n e^{gz}\\(z+g)^n e^{-gz}\end{bmatrix}, \Psi_n^-=\begin{bmatrix}(-1)^{n+1}(z-g)^n e^{gz}\\(z+g)^n e^{-gz}\end{bmatrix}$$

■$P_0(x,y)=1$, $P_1(x,y)=x+y-1,\ldots,$

$P_k(x,y)=(kx+y-k^2)P_{k-1}-k(k-1)(n-k+1)xP_{k-2}$.

Theorem (Kuś(1984))

1)Suppose that g satisfies that $P_n((2g)^2,\Delta^2)=0$. Set

$$\psi_n^\pm=(2g)^n\Phi^\pm+\sum_{l=1}^{n-1}\frac{(2g)^{n-l}}{l!}P_{l-n}((2g)^2,\Delta^2)(\Delta\Phi_{n-l}^\pm+l\Psi_{n-l}^\pm)$$

Then $H\psi_n^\pm=E(g)\psi_n^\pm$ $\quad E(g)=n-g^2$

2) There are n roots, $g=g_1,\ldots,g_n$, of $P_n((2g)^2,\Delta^2)=0$.

Simplicity of $E_0(g)$

■$\mathbb{C}^2 \otimes L^2(\mathbb{R}) \cong L^2(\mathbb{R} \times \{-1,1\})$.
■$(X_t)_{t\geq 0}$ OU process, $(\sigma_t)_{t\geq 0} = ((-1)^{N_t})_{t\geq 0}$ Poisson process
■Let $\sum_{\sigma\in\mathbb{Z}_2} \int \pi^{-1} e^{-|x|^2} dx \mathbb{E}_{P^x} \mathbb{E}_\nu [\cdots] = \mathbb{E}[\cdots]$.

Theorem (Hirokawa and H.(2012))

$$(\Delta > 0) \quad (f, e^{-tH} g) = e^t \mathbb{E}[\overline{f(X_0, \sigma_0)} g(X_t, \sigma_t) e^{-g\sqrt{2\omega}\int_0^t \sigma_s X_s ds} \Delta^{N_t}]$$

$\mathscr{P} = \{f \in L^2(\mathbb{R} \times \{-1,1\}) | f \geq 0\}$, $\mathscr{P}_+ = \{f | f > 0\}$
■$e^{-tH} \mathscr{P} \setminus \{0\} \subset \mathscr{P}_+ \Longrightarrow$ ground state is strictly positive $\Longrightarrow E_0$ is simple.

NCHO

■**Non-Commutative Harmonic Oscillator(NCHO)** is introduced by A. Parmeggiani and M. Wakayama as a non-commutative extension of harmonic oscillators.

$$Q = Q(\alpha,\beta) = A \otimes \left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2\right) + J \otimes \left(x\frac{d}{dx} + \frac{1}{2}\right)$$

■$\mathscr{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R})$

■$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\alpha > 0$, $\beta > 0$ and $\alpha\beta > 1$

■Q is self-adjoint on $D(Q) = \mathbb{C}^2 \otimes (D(d^2/dx^2) \cap D(x^2))$

■Q has purely discrete spectrum $E_0 \le E_1 \le E_2 \le \cdots \nearrow \infty$.

$(\alpha=\beta)$

■Let $\alpha=\beta$.

$$Q(\alpha,\alpha)\cong\begin{pmatrix} -\frac{1}{2}\frac{d^2}{dx^2}+\frac{\alpha^2-1}{2}x^2 & 0 \\ 0 & -\frac{1}{2}\frac{d^2}{dx^2}+\frac{\alpha^2-1}{2}x^2 \end{pmatrix}$$

■$E_n=\sqrt{\alpha^2-1}(n+\frac{1}{2})$, $n\geq 0$.
■E_n is two fold degenerate.
■Spectral zeta function:

$$\zeta_Q(s)=\sum_{j=0}^{\infty}\frac{1}{E_j^s}$$

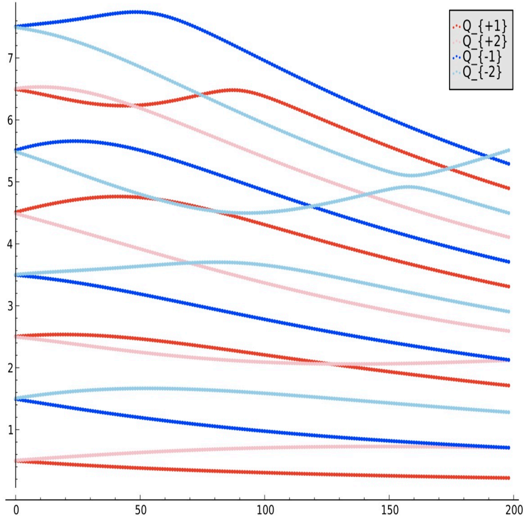
■Analytic continuation due to Ichinose and Wakayama (2005)
■$\alpha=\beta\Longrightarrow\zeta_Q(s)=c(s)\times\zeta(s)$

$(\alpha \neq \beta)$

■We set $E = E_0$ and its multiplicity by M.
■A long-standing problem is to determine the multiplicity of eigenvalues of Q explicitly.
■The map

$$c_n : (\alpha, \beta) \mapsto E_n(\alpha, \beta) \in \mathbb{R}$$

is called an **eigenvalue-curve**.
■To consider the multiplicity of eigenvalues is reduced to studying **crossing** or **no crossing** of eigenvalue-curves.


Q_{+1}
Q_{+2}
Q_{-1}
Q_{-2}
1
2
3
4
5
6
7
0
50
100
150
200

History of the multiplicity of eigenvalues of Q

■Parmeggiani and Wakayama (2003) shows that the multiplicity of any eigenvalues is ≤ 3.

■Nagato-Nakao-Wakayama (2002) found that eigenvalue-curves cross at some points except for $E = E_0$ in a numerical level.

■Ichinose-Wakayama (2007) derived

$$\left(n - \tfrac{1}{2}\right) \min\{\alpha, \beta\} \sqrt{\frac{\alpha\beta - 1}{\alpha\beta}} \leq E_{2n-1} \leq E_{2n} \leq \left(n - \tfrac{1}{2}\right) \max\{\alpha, \beta\} \sqrt{\frac{\alpha\beta - 1}{\alpha\beta}}$$

From this $M \leq 2$ if $\beta < 3\alpha$ or $\alpha < 3\beta$.

■Parmeggiani (2004) shows that $M = 1$ for sufficiently large $\alpha\beta$.

■FH. and Sasaki (2012) proven that $M \leq 2$ and **all the eigenvectors are even** for $(\alpha,\beta) \in D_{\sqrt{2}} = \{(\alpha,\beta) | \alpha,\beta > \sqrt{2}\}$
■FH+Sasaki (2012) also shown that $M = 1$ for $(\alpha,\beta) \in \exists D \subset D_{\sqrt{2}}$.

Theorem (Wakayama(2012))

Assume that (1) $\alpha \neq \beta$*; (2)* $\ker(Q - E) \subset \mathscr{H}_+$*. Then* $M = 1$*.*

■Combining this with HF+Sasaki, $M = 1$ for $(\alpha,\beta) \in D_{\sqrt{2}}$.
■**Wakayama (2013) gives relationship between Rabi and NCHO.**

Decomposition of $Q(\alpha,\beta)$

■Q is represented as

$$Q = A(a^*a + \frac{1}{2}) + \frac{J}{2}(aa - a^*a^*).$$

■$\mathscr{H}_+$ (resp. $\mathscr{H}_-$): the set of even (resp. odd) functions in $\mathscr{H}$
■P_+ (resp. P_-): the orthogonal projection onto $\mathscr{H}_+$ (resp. $\mathscr{H}_-$)
■$|n\rangle = \frac{1}{\sqrt{n!}}(a^*)^n|0\rangle$ with $|0\rangle = \pi^{-1/4}e^{-x^2/2}$. Then $a^*a|n\rangle = n|n\rangle$
■$L^2(\mathbb{R}) = \bigoplus_{n=0}^{\infty}\mathbb{C}|n\rangle$
■The total Hilbert space is

$$\mathscr{H} \cong \left\{ \begin{bmatrix} X \\ Y \end{bmatrix} \,\middle|\, X, Y \in \bigoplus_{n=0}^{\infty}\mathbb{C}|n\rangle \right\} \cong \bigoplus_{n=0}^{\infty}\mathscr{H}_n, \quad \mathscr{H}_n = \begin{bmatrix} \mathbb{C}|n\rangle \\ \mathbb{C}|n\rangle \end{bmatrix}.$$

■ $aa : \mathscr{H}_n \to \mathscr{H}_{n-2}$, $a^*a^* : \mathscr{H}_n \to \mathscr{H}_{n+2}$, $a^*a : \mathscr{H}_n \to \mathscr{H}_n$

■ $Q : \mathscr{H}_n \to \mathscr{H}_{n-2} \oplus \mathscr{H}_n \oplus \mathscr{H}_{n+2}$.

■ From these observation we can find invariant domains of Q.

■ $P_\uparrow(n) = \begin{pmatrix} |n\rangle\langle n| & 0 \\ 0 & 0 \end{pmatrix}$, $P_\downarrow(n) = \begin{pmatrix} 0 & 0 \\ 0 & |n\rangle\langle n| \end{pmatrix}$

■ In order to decompose Q, we define

$$T_{+1} = \sum_{n=0}^{\infty} (P_\uparrow(4n) + P_\downarrow(4n+2)), \qquad T_{+2} = \sum_{n=0}^{\infty} (P_\downarrow(4n) + P_\uparrow(4n+2)),$$

$$T_{-1} = \sum_{n=0}^{\infty} (P_\uparrow(4n+1) + P_\downarrow(4n+3)), \quad T_{-2} = \sum_{n=0}^{\infty} (P_\downarrow(4n+1) + P_\uparrow(4n+3)).$$

■ $\mathscr{H}_{\sigma \mathrm{p}} = \mathrm{Ran}(T_{\sigma \mathrm{p}})$ and $Q_{\sigma \mathrm{p}} = Q\lceil_{\mathscr{H}_{\sigma \mathrm{p}}}$.

■ $\mathscr{H}$ is decomposed as $\mathscr{H} = \bigoplus_{\sigma=\pm, \mathrm{p}=1,2} \mathscr{H}_{\sigma \mathrm{p}}$.

Theorem

Q is reduced by $\mathscr{H}_{\sigma \mathrm{p}}$, i.e., $Q = \oplus_{\sigma=\pm,\mathrm{p}=1,2} Q_{\sigma \mathrm{p}}$

Simplicity of E and no crossing

Let $Q = Q_+ \oplus Q_-$. Let $E_\sigma = \inf Spec(Q_\sigma)$.

Lemma

It follows that $E_+ < E_-$.

Theorem

Assume that $\alpha \neq \beta$. Then $M = 1$ and the ground state is even.

■$Spec(Q) = \bigcup_{\sigma=\pm, \mathrm{p}=1,2} Spec(Q_{\mathrm{p}\sigma})$

Theorem

■*Each eigenvalue of $Q_{\sigma \mathrm{p}}$ is simple.*
■*Assume that $\sqrt{\alpha\beta} > 1 + \frac{1}{1600000000}$. Then $\widehat{Q}_{-1} - \widehat{Q}_{+1} \geq \Delta(\alpha, \beta)$ and $\widehat{Q}_{-2} - \widehat{Q}_{+2} \geq \Delta(\alpha, \beta)$.*

Thank you!