

①

1 Gaussian measures in Hilbert spaces

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space

Ω set, \mathcal{F} σ -field, \mathbb{P} prob. measure
 $\mathbb{P}(\Omega) = 1$

E complete metric space (= Polish space)

$(E, \mathcal{B}(E))$ $\mathcal{B}(E)$: Borel σ -field

$X: (\Omega, \mathcal{F}) \rightarrow (E, \mathcal{B}(E))$ meas.

$\Leftrightarrow X^{-1}(I) \in \mathcal{F} \quad \forall I \in \mathcal{B}(E).$

X : random variable \Leftrightarrow

X 's image measure

$$X_{\#}P(I) = P(X^{-1}(I)) \quad \forall I \in \mathcal{B}(E)$$

lem 1.1 $X_{\#}P$ is a prob meas on $(E, \mathcal{B}(E))$

☺ $\mathcal{B}(E) \ni I_n \quad I_n \cap I_m = \emptyset \quad (n \neq m)$

$$\begin{aligned} X_{\#}P\left(\bigcup_n I_n\right) &= P\left(X^{-1}\left(\bigcup_n I_n\right)\right) = P\left(\bigcup_n X^{-1}(I_n)\right) \\ &= \sum_n P(X^{-1}(I_n)) = \sum_n X_{\#}P(I_n) \quad // \end{aligned}$$

Prop 1.2 (change of variable)

$X: \Omega \rightarrow E$ v.v.

$\varphi: E \rightarrow \mathbb{R}$ b'dd Borel $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$$\int_{\Omega} \varphi(X(\omega)) P(d\omega) = \int_E \varphi(x) X_{\#} P(dx)$$

☺ step 1 $A \in \mathcal{B}(E)$ $\varphi(x) = \mathbb{1}_A(x)$ o.e.s

$$\int_{\Omega} \mathbb{1}_A(X(\omega)) P(d\omega) = \int_{\Omega} \mathbb{1}_{X^{-1}(A)}(\omega) P(d\omega)$$

$$= P(X^{-1}(A)) = X_{\#} P(A) = \int_E \mathbb{1}_A(x) X_{\#} P(dx)$$

step 2 $\varphi(x) = \sum_{j=1}^n a_j \mathbb{1}_{A_j}(x)$ $a_j \geq 0$

step 3 $\varphi \geq 0$ b'dd Borel o.e.s

$\exists \varphi_n(x)$ step function st $\varphi_n(x) \uparrow \varphi(x)$

$$\therefore \int_{\Omega} \varphi(X(\omega)) P(d\omega) = \lim_n \int_{\Omega} \varphi_n(X(\omega)) P(d\omega)$$

$$= \lim_n \int_E \varphi_n(x) X_{\#} P(dx) = \int_E \varphi(x) X_{\#} P(dx)$$

step 4 φ b'dd Borel $\varphi = \varphi_+ - \varphi_-$ //

Gaussian measures

基本 $(2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|x|^2}{2t}\right) dx = 1$

$$(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ik \cdot x} e^{-\frac{|x|^2}{2}} dx = e^{-\frac{|k|^2}{2}}$$

$e^{-\frac{|x|^2}{2}}$ is Fourier tr. 2" 不变函数.

$$(a, \lambda) \in \mathbb{R} \times \mathbb{R}_+ \quad \mathbb{R}_+ = [0, \infty)$$

$$\textcircled{1} N_{a, \lambda}(B) = (2\pi\lambda)^{-\frac{1}{2}} \int_B e^{-\frac{(x-a)^2}{2\lambda}} dx$$

$$\textcircled{2} N_{a, 0}(B) := \begin{cases} 1 & a \in B \\ 0 & a \notin B \end{cases}$$

$$\text{Prop 1.3} \quad \int x N_{a,\lambda}(dx) = a$$

$$\int (x-a)^2 N_{a,\lambda}(dx) = \lambda$$

$$\int e^{ihx} N_{a,\lambda}(dx) = e^{iah - \frac{1}{2}\lambda h^2}$$

$$\therefore (2\pi\lambda)^{-\frac{1}{2}} \int (y+a) e^{-\frac{|y|^2}{2\lambda}} dy = a$$

$$(2\pi\lambda)^{-\frac{1}{2}} \int y^2 e^{-\frac{|y|^2}{2\lambda}} dy$$

$$(2\pi)^{-\frac{1}{2}} \int |y|^2 e^{-\frac{|y|^2}{2}} dy = 1$$

$$= \frac{1}{2} \sqrt{2\pi}$$

$$= (2\pi\lambda)^{-\frac{1}{2}} \int \frac{|y|^2}{\lambda} e^{-\frac{1}{2} \left| \frac{y}{\sqrt{\lambda}} \right|^2} \frac{dy}{\sqrt{\lambda}} \cdot \sqrt{\lambda} \lambda = \lambda.$$

$$(2\pi\lambda)^{-\frac{1}{2}} \int e^{ihx} e^{-\frac{(x-a)^2}{2\lambda}} dx$$

$$= (2\pi\lambda)^{-\frac{1}{2}} \int e^{ih(y+a)} e^{-\frac{|y|^2}{2\lambda}} dy$$

$$= e^{iha} (2\pi\lambda)^{-\frac{1}{2}} \int e^{ihy} e^{-\frac{|y|^2}{2\lambda}} dy$$

$$= e^{iha} (2\pi\lambda)^{-\frac{1}{2}} \int e^{i\sqrt{\lambda}h \frac{y}{\sqrt{\lambda}}} e^{-\frac{1}{2} \left| \frac{y}{\sqrt{\lambda}} \right|^2} \frac{dy}{\sqrt{\lambda}} \cdot \sqrt{\lambda}$$

$$= e^{iha} e^{-\frac{\lambda}{2} h^2}$$

\mathcal{H} : ^{real separable} Hilbert space $(\cdot, \cdot)_{\mathcal{H}}$

$(\mathcal{H}, B(\mathcal{H}))$ measurable space

$L(\mathcal{H})$ b'dd linear op $\mathcal{H} \rightarrow \mathcal{H}$ 全体

$$L_1^+(\mathcal{H}) \subset L^+(\mathcal{H}) \subset L(\mathcal{H})$$

\uparrow
 sym b'dd op \uparrow $(Tx, x) \geq 0$
 $(Tx, y) = (x, Ty) \uparrow$

$$\text{Tr} Q = \sum_{j=1}^{\infty} (Q e_j, e_j) < \infty \quad \forall \text{CONS } (e_j)$$

$L_1^+ \ni Q$ trace class $\Leftrightarrow (e_j)$ CONS s.t.

$$Q e_j = \lambda_j e_j \quad (\lambda_j \geq 0)$$

~~...~~
 0 が d 重の λ_j になる

$$(\Omega_i, \mathcal{F}_i, P_i) \quad i=1, \dots, d$$

- $B_1 \times \dots \times B_d$ rectangle 全体を含む最小の σ -field $\mathcal{F} = \mathcal{F}_1 \times \dots \times \mathcal{F}_d = \mathcal{F}$ と表す.
 - $P(B_1 \times \dots \times B_d) = \prod_{j=1}^d P_j(B_j)$ は $\mathcal{F}_1 \times \dots \times \mathcal{F}_d$ 上の prob には一意的に存在する長 $d \geq 2$, $P_1 \times \dots \times P_d$ と表す
- $(\Omega, \mathcal{F}, P) \quad \Omega = \Omega_1 \times \dots \times \Omega_d$ "P"

$\dim \mathcal{H} = d < \infty$ のとき

$a \in \mathcal{H}, Q \in L^+(\mathcal{H}) = L_1^+(\mathcal{H})$

Q : Hermitian matrix $a + \tau = 1$ である

$(e_j)_{j=1}^d$ base かつ $Q e_j = \lambda_j e_j, \lambda_j > 0.$

Lem 1.4 $\mathcal{H} \cong \mathbb{R}^d$

isomorphic
(位相空間同型)
かつ 基底型空間 \mathbb{R}^d と
同型になる

☺ $\gamma: \mathcal{H} \rightarrow \mathbb{R}_0^d$
 $x \mapsto \begin{pmatrix} \langle x, e_1 \rangle \\ \vdots \\ \langle x, e_d \rangle \end{pmatrix} = \gamma(x)$ bijective, linear

γ is cont $\gamma^{-1}(a) = a_1 e_1 + \dots + a_n e_n \in \text{cont}$

Lem 1.5

$(\mathcal{H}, \mathcal{B}(\mathcal{H})) \cong (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ measurable space である

☺ \mathcal{L} on Lem 2 $(\mathcal{H}, \sigma_{\mathcal{H}}) \cong (\mathbb{R}^d, \sigma_{\mathbb{R}^d})$

i.e., $\gamma: \mathcal{H} \rightarrow \mathbb{R}^d$ bijective linear

$\gamma(\sigma_{\mathcal{H}}) = \sigma_{\mathbb{R}^d}$

$\mathcal{B}(\mathcal{H}) = \sigma(\sigma_{\mathcal{H}})$

$\gamma(\sigma(\sigma_{\mathcal{H}})) \supset \sigma_{\mathbb{R}^d}$

$\gamma(\sigma(\sigma_{\mathcal{H}})) \ni A_n \in \mathcal{L} \text{ と } A_n = \gamma(B_n)$
 $B_n \in \sigma(\sigma_{\mathcal{H}})$

$\bigcup_n A_n = \bigcup_n \gamma(B_n) = \gamma(\bigcup_n B_n) \in \gamma(\sigma(\sigma_{\mathcal{H}}))$

$\gamma(\sigma(\sigma_{\mathcal{H}})) \supset \sigma(\mathbb{R}^d) \in \mathcal{L} \in \mathcal{L}$

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②

存在定理

$$\dim \mathcal{X} = d \quad \text{のとき}$$

$$(\mathcal{X}, \mathcal{B}(\mathcal{X})) \cong (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \quad \text{と見做し可い。}$$

$$a = (a_1, \dots, a_d) \in \mathcal{X} = \mathbb{R}^d$$

$$Q \in L^+(\mathcal{X}) \quad Q \text{ は } \mathbb{R} \text{ の対称正定行列}$$

$$= Q \cong \begin{pmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_d \end{pmatrix} \quad \lambda_j > 0$$

$$N_{a,Q}(dx) = \left((2\pi)^d \det Q \right)^{-\frac{1}{2}} \prod_{j=1}^d \frac{1}{\sqrt{\lambda_j}} e^{-\frac{(x_j - a_j)^2}{2\lambda_j}} \prod_{j=1}^d dx_j$$

Prop 4.6 $a \in \mathcal{X}, Q \in L^+(\mathcal{X}), \mu = N_{a,Q}$

$$\textcircled{1} \int_{\mathcal{X}} x \mu(dx) = a$$

$$\textcircled{2} \int_{\mathcal{X}} (y, x-a)(z, x-a) N_{a,Q}(dx) = (Qy, z)_{\mathcal{X}}$$

$$\textcircled{3} \int_{\mathcal{X}} e^{i(h, x)} N_{a,Q}(dx) = e^{i(a, h) - \frac{1}{2}(Qh, h)}$$

$$\textcircled{\text{註}} \textcircled{1} \int_{\mathbb{R}^d} x_j \mu(dx) = a_j \quad \text{のとき}$$

$$\textcircled{2} \int_{\mathbb{R}^d} \left[\sum_j y_j (x-a)_j \right] \left[\sum_j z_j (x-a)_j \right] N_{a,Q}(dx)$$

$$\textcircled{3} \int_{\mathbb{R}^d} e^{i \sum_j h_j x_j} N_{a,Q}(dx) = e^{i \sum_j a_j h_j - \frac{1}{2} \sum_j \lambda_j h_j^2}$$

$$\textcircled{1} \int x_j e^{-\frac{(x_j - a_j)^2}{2\lambda_j}} \times \left((2\pi)^{\lambda_j} \right)^{-\frac{1}{2}}$$

$$\times \left((2\pi)^{d-1} \lambda_1 \dots \lambda_d \right)^{-\frac{1}{2}} \int \dots \prod_{i \neq j} \frac{\pi}{\lambda_i} e^{-\frac{(x_i - a_i)^2}{2\lambda_i}} dx_i \dots$$

$$= a_j$$

$$\textcircled{2} \sum_j y_j z_j \int (x - a)_i (x - a)_j N_{a, Q}(dx)$$

$$= \sum_j y_j z_j \int (x - a)_j^2 N_{a, Q}(dx)$$

$$= \sum_j \lambda_j y_j z_j = (Qy, z)$$

③ $\int_{\mathbb{R}^d} e^{i \langle h, x \rangle} N_{a, Q}(dx)$

$$= \prod_j \int_{\mathbb{R}} e^{i h_j x_j} e^{-\frac{(x_j - a_j)^2}{2\lambda_j}} \left[(2\pi\lambda_j) \right]^{-1/2}$$

$$= \prod_j e^{i (a_j h_j)} e^{-\frac{1}{2} \lambda_j h_j h_j}$$

$$\textcircled{\text{注}} N_{a, Q}(dx) = \left[(2\pi)^d \det Q \right]^{-1/2} e^{-\frac{1}{2} \left(\vec{Q}(x-a), (x-a) \right)} dx$$

$$\hat{\mu}(u) = \int_{\mathbb{R}^d} e^{i u x} d\mu(x) \quad \mu \text{ is finite measure on } (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$$

μ is characteristic function

Thm ~~1.7~~ $\hat{\mu}_1 = \hat{\mu}_2 \Leftrightarrow \mu_1 = \mu_2$

準備: E cpt Hausdorff space

$C_b(E, \mathbb{K})$ \mathbb{K} -valued cont, bdd
Banach $\|\cdot\|_\infty$

$\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}$

$$\mathcal{C} \subset C_b(E, \mathbb{K})$$

$$\mathcal{C} \text{ is algebra} \Leftrightarrow f, g \in \mathcal{C} \Rightarrow \begin{matrix} f+g \\ \alpha f \end{matrix} \in \mathcal{C} \quad \alpha \in \mathbb{K}$$

$$\mathcal{C} \text{ separates points} \Leftrightarrow \forall x, y \in E \exists f \in \mathcal{C} \text{ st } f(x) \neq f(y)$$

Proposition (Stone-Weierstrass).

\mathcal{C} is algebra and separates points
($\mathbb{K} = \mathbb{C}$ or \mathbb{R} complex conjugate $\bar{z} = \overline{z}$)
 $\Rightarrow \mathcal{C}$ is $C_b(E, \mathbb{K})$ dense

$M_f(E)$ E is a finite measure E metric space

$\mathcal{C} \subset C_b(E, \mathbb{C})$ separating class

$\Leftrightarrow \mu, \nu \in M_f(E) \mu \neq \nu$

$$\underbrace{\exists f}_{\substack{= \lambda > 1 \\ \text{st}}} \int f d\mu \neq \int f d\nu.$$

Cor of SW theorem

Let E cpt metric. $K = \mathbb{R}$ or \mathbb{C} .

$\mathcal{C} \subset C_b(E, K)$ alg. (\mathcal{C} is $f+g \in \mathcal{C}$ is $1 \rightarrow 1$)
separating points

(if $K = \mathbb{C}$, closed under the complex conjugation)

Then \mathcal{C} is a separating class.

Assume \mathcal{C} is not separating class i.e. $\mathcal{C} \ni g \int g d\mu_1 = \int g d\mu_2$

For any $f \in C_b(E, \mathbb{R})$ any $\varepsilon > 0$.

$\exists g \in \mathcal{C}$ with $\|f - g\|_\infty < \varepsilon$

$$\left| \int f d\mu_1 - \int f d\mu_2 \right|$$

$$\leq \left| \int f d\mu_1 - \int g d\mu_1 \right| + \left| \int g d\mu_1 - \int g d\mu_2 \right|$$

$$+ \left| \int g d\mu_2 - \int f d\mu_2 \right| \leq \|f - g\|_\infty (\mu_1(E) + \mu_2(E))$$

$$\leq \varepsilon.$$

$\therefore \mathcal{C}$ is $\mu_1 = \mu_2$ (\mathcal{C} is \mathbb{R}) \mathcal{C} is \mathbb{R} .

☹ of Thm 1.7.

$$f_u(y) = e^{i u \cdot y}$$

$$\mathcal{C} = \text{L.H.} \{ f_u; u \in \mathbb{R}^d \}$$

alg. sep. points
closed under -

$C_b(\mathbb{R}^d, \mathbb{C})$ is closed & \mathcal{C} is dense in it $\because \mathbb{R}^d$ not compact

Let $f \in C_b(\mathbb{R}^d)$, $\epsilon > 0$ fix.

$$\mathcal{C}' = \text{L.H.} \{ f_{2\pi m}; m \in \mathbb{Z}^d \}$$

$$\mathcal{C}_N = \{ g \mid [-N, N]^d; g \in \mathcal{C}' \}$$

alg. sep. closed under -

\mathcal{C}_N is dense in $C([-N, N]^d; \mathbb{C})$. SW OK

$$\sup_{x \in [-N, N]^d} |f(x) - g(x)| < \epsilon \quad \exists g \in \mathcal{C}_N$$

$$\begin{aligned} \left| \int f d\mu_1 - \int f d\mu_2 \right| &\leq \left| \int_{[-N, N]^d} f - g d\mu_1 + \int_{[-N, N]^d} g d\mu_1 \right. \\ &\quad \left. - \int_{[-N, N]^d} f - g d\mu_2 - \int_{[-N, N]^d} g d\mu_2 \right| \\ &\quad + \left| \int_{\mathbb{R}^d \setminus [-N, N]^d} f - g d\mu_1 \right| \\ &\leq \delta (\mu_1(\mathbb{R}^d) + \mu_2(\mathbb{R}^d)) + \|f - g\|_\infty (\mu_1 + \mu_2)(\mathbb{R}^d \setminus [-N, N]^d) \rightarrow 0 \end{aligned}$$

$$\|f - g\|_\infty \leq \|f\|_\infty + \|g\|_\infty \leq 2\|f\|_\infty + 1 \quad \exists \epsilon > 0$$

$$\Rightarrow \|g\|_\infty = \sup_{x \in [-N, N]^d} |g(x)|$$

$N \rightarrow \infty \Rightarrow \mathbb{R}^d = \bigcup_{N \in \mathbb{N}} [-N, N]^d$

$$\leq \sup_{x \in [-N, N]^d} |g(x) - f(x)| + \|f\|_\infty \leq \epsilon + \|f\|_\infty$$

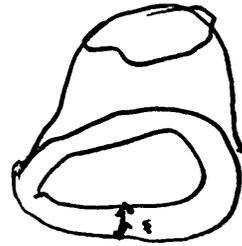
$$\therefore \int f d\mu_1 = \int f d\mu_2 \quad \text{for } \forall f \in C_b(\mathbb{R}^d)$$

Let $A \subset \mathbb{R}^d$ be a closed set.

$$P_\varepsilon(x) = 1 - \varphi(d(x, A)/\varepsilon)$$

$d(x, A)$ is $x \in A$ or metric

$$\varphi(t) = \begin{cases} 1 & t \leq 1 \\ 1 - t & t > 1 \end{cases}$$



$$P_\varepsilon(x) = \begin{cases} 1 & x \in A \\ 0 & d(x, A) > \varepsilon \end{cases}$$

$$\int P_\varepsilon(x) d\mu_1 = \int P_\varepsilon(x) d\mu_2 \rightarrow \mu_1(A) = \mu_2(A) //$$

$$\int_\varepsilon P_\varepsilon(x) \xrightarrow{(\varepsilon \rightarrow 0)} \mathbb{1}_A(x)$$

Example

$$\int_{\mathcal{H}} e^{i\langle h, x \rangle} \mu(dx) = e^{i\langle a, h \rangle - \frac{1}{2}\langle Qx, x \rangle}$$

$$\Rightarrow \mu(dx) = N_{a, Q}(dx) \quad //$$

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over \mathbb{R}
 $\dim \mathcal{X} = \infty^{\vee}$ separable, $\{e_k\}_{k=1}^{\infty}$ CONS.

$P_n(\mathcal{X}) = \text{L.H. } \{e_1, \dots, e_n\} \leftarrow \text{cont} \rightarrow \text{measurable}$
 $P_n : \mathcal{X} \rightarrow P_n(\mathcal{X})$ defined by $P_n x = \sum_{k=1}^n (x, e_k) e_k$

$M(\mathcal{X})$ --- Set of prob measures on \mathcal{X} .

prop 1.8 Let $\mu, \nu \in M(\mathcal{X})$ st.

$$\int_{\mathcal{X}} \varphi(x) \mu(dx) = \int_{\mathcal{X}} \varphi(x) \nu(dx)$$

\forall cont, bdd $\varphi : \mathcal{X} \rightarrow \mathbb{R}$.

Then $\mu = \nu$

⊙ $C \subset \mathcal{X}$ closed.

$\exists \varphi_n$ st (1) cont, bdd
 (2) $\varphi_n(x) \uparrow \mathbb{1}_C(x)$

Ex 1.10

$$\varphi_n(x) = \begin{cases} 1 & x \in C \\ 1 - n d(x, C) & d(x, C) \leq \frac{1}{n} \\ 0 & d(x, C) \geq \frac{1}{n} \end{cases}$$

$$\therefore \int \varphi_n(x) \mu(dx) = \int \varphi_n(x) \nu(dx)$$

↓

$$\int \mathbb{1}_C(x) \mu(dx) = \int \mathbb{1}_C(x) \nu(dx)$$

$\therefore \mu(C) = \nu(C) \quad \forall C \text{ closed } "$

$$\int_{P_n(\mathcal{H})} e^{i(\xi, P_n h)} (P_n \#)_\mu(d\xi) = \int_{P_n(\mathcal{H})} e^{i(\xi, P_n h)} (P_n \#)_\nu(d\xi)$$

$$P_n(\mathcal{H}) \cong \mathbb{R}^n \quad \text{と } \mathbb{R}^n \text{ 上 } \mu = \nu \quad \text{先 } \mathbb{R}^n \text{ 上 } \mu, \nu \text{ が } \mu = \nu$$

$$(P_n \#)_\mu(d\xi) = (P_n \#)_\nu(d\xi) \stackrel{\leftarrow n}{\nu} \therefore \mu = \nu \quad "$$

.....

($\mathcal{H}, B(\mathcal{H}), \mu$) μ prob meas.

① Suppose that $\int_{\mathcal{H}} \|x\| \mu(dx) < \infty$

$$F : \mathcal{H} \rightarrow \mathbb{R}$$

$$\begin{matrix} \uparrow \\ h \end{matrix} \mapsto \int_{\mathcal{H}} (x, h) \mu(dx)$$

$$\|F(h)\| \leq \|h\| \int_{\mathcal{H}} \|x\| \mu(dx) \quad \text{linear functional}$$

By Riesz rep. thm. $\exists m \in \mathcal{H}$ st

$$F(h) = (m, h) \quad m = \int_{\mathcal{H}} x \mu(dx) \in \mathbb{R}^c.$$

② Suppose that $\int_{\mathcal{H}} \|x\|^2 \mu(dx) < \infty$

$$G : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$$

$$\begin{matrix} \uparrow \\ h \times k \end{matrix} \mapsto \int_{\mathcal{H}} (h, x-m)(k, x-m) \mu(dx)$$

$$|G(h, k)| \leq \|h\| \cdot \|k\| \int \|x - m\|^2 \mu(dx)$$

Riesz rep thm.

$$G(h, k) = (x_h, k)$$

$h \mapsto x_h$ linear

$$\|x_h\| \leq \int \|x - m\|^2 \mu(dx) \cdot \|h\|$$

$\therefore h \mapsto x_h$ bounded op.

$$x_h = Qh \quad \text{where } Q = \int (x - m)(x - m)^T \mu(dx)$$

$$G(h, k) = (Qh, k)$$

$$(Qh, k) = \int (h, x - m)(k, x - m) \mu(dx)$$

$Q \in L(\mathcal{H})$
 covariance $\sigma_B \mu$
 m mean $\sigma_B m$
 \mathcal{H}

Prop 1.11 $\mu \in M(\mathcal{H})$, mean m , cov. Q .

Then $Q \in L_1^+(\mathcal{H})$ sym, positive, trace class.

$$\odot (Qh, k) = \int (h, x - m)(k, x - m) \mu(dx)$$

$$(Qk, h) = \int (k, x - m)(h, x - m) \mu(dx)$$

$$\therefore (Qh, k) = (Qk, h) = (h, Qk)$$

$$\exists \{e_k\} (Qe_k, e_k) \geq 0.$$

$$\exists \{e_k\} \sum_k (Qe_k, e_k) = \sum_k \int |(e_k, x - m)|^2 \mu(dx) = \int \|x - m\|^2 \mu(dx) < \infty$$

Prop 1.9 $\mu, \nu \in M(\mathcal{X})$. $(P_u \#)_\mu = (P_u \#)_\nu \quad \forall u$
 Then $\mu = \nu$.

i.e., $\mu(P_u^{-1}(A)) = \nu(P_u^{-1}(A)) \quad \forall A \in \mathcal{B}(\mathcal{X})$

☺ $\varphi: \mathcal{X} \rightarrow \mathbb{R}$ bdd cont.

$$\int_{P_u(\mathcal{X})} \varphi(z) (P_u \#)_\mu(dz) = \int_{\mathcal{X}} \varphi(P_u x) \mu(dx)$$

change of variable

$$= \int_{\mathcal{X}} \varphi(P_u x) \mu(dx) = \int_{\mathcal{X}} \varphi(P_u x) \nu(dx)$$

$$= \int_{\mathcal{X}} \varphi(x) \mu(dx) = \int_{\mathcal{X}} \varphi(x) \nu(dx) \quad //$$

by Prop 1.8 による.

Prop 1.10. $\hat{\mu}(u) = \int_{\mathcal{X}} e^{i(x, u)} \mu(dx) \quad \forall u$

$\mu, \nu \in M(\mathcal{X})$. $\hat{\mu}(u) = \hat{\nu}(u) \quad \forall u \in \mathcal{X}$

$\Rightarrow \mu = \nu$

$$\begin{aligned} \text{☺} \quad \hat{\mu}(P_u u) &= \int_{\mathcal{X}} e^{i(x, P_u u)} \mu(dx) = \int_{\mathcal{X}} e^{i(P_u x, P_u u)} \mu(dx) \\ &= \int_{P_u(\mathcal{X})} e^{i(\underbrace{P_u x}_z, P_u u)} (P_u \#)_\mu(dz) \end{aligned}$$

Def 1.12

Let $a \in \mathcal{H}$, $Q \in L_1^+(\mathcal{H})$.

A Gaussian meas $\mu = N_{a,Q}$ on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$

- \Leftrightarrow
- (1) mean a
 - (2) covariance Q

(3)
$$\widehat{N_{a,Q}}(h) = \exp\left(i(a, h) - \frac{1}{2}(Qh, h)\right)$$

Gaussian measure μ is said to be nondegenerate if $\text{Ker } Q = \{0\}$.

$$\mathcal{H} \cong_{\gamma} \ell^2$$

$$\prod_{k=1}^{\infty} \mathbb{R} = \mathbb{R}^{\infty} \quad \text{I.I.} \quad \mu = \prod_{k=1}^{\infty} N_{a_k, \lambda_k}$$

\exists measure $\varepsilon \frac{1}{\sqrt{2\pi}} \lambda$ (I.I.)

iii) $\forall y \in \mathcal{H}$

$$y = \underbrace{\left(y - \frac{T y}{T x_0} x_0 \right)}_{\mathcal{N}} + \underbrace{\frac{T y}{T x_0} x_0}_{\alpha x_0}$$

$$\therefore T y = (x_T, y) \quad //$$

(- 實. 性) $(x_T', y) = (x_T, y) \quad \forall y \in \mathcal{H} //$

$$(\|x_T\| = \|T\|)$$

$$\bullet \|T\| = \sup_{\|x\| \leq 1} |Tx| = \sup_{\|x\| \leq 1} |(x_T, x)| \leq \|x_T\|$$

$$\bullet \|T\| = \| \| \Rightarrow T \left(\frac{x_T}{\|x_T\|} \right) = \|x_T\|$$

$$\text{以上より } \|T\| = \|x_T\| //$$

Cor $G : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ ^{anti-linear} ^{linear} st
 $|G(h, k)| \leq C \|h\| \cdot \|k\|$

$\Rightarrow \exists Q \in L(\mathcal{H}) \text{ st } G(h, k) = (Qh, k)$

$$\odot G(h, k) = (x_h, k) \quad \text{7312}$$

$$\|x_h\| \leq C \|h\| \quad \therefore h \mapsto x_h \text{ bdd}$$

$$\therefore x_h = Qh \quad //$$

Riesz's Representation thm.

\mathcal{H} : Hilbert space over \mathbb{C} (or \mathbb{R})

$T: \mathcal{H} \rightarrow \mathbb{C}$ (or \mathbb{R}) cont i.e. $T \in \mathcal{H}^*$

$$\Rightarrow \begin{aligned} (1) & \exists! x_T \in \mathcal{H} \text{ s.t. } Ty = (x_T, y) \\ (2) & \|x_T\| = \|T\| \end{aligned}$$

$$\text{Thm 1: } \mathcal{H} \cong \mathcal{H}^*$$

☺ (Thm 2) $\mathcal{N} = \ker T$

$$(1) \mathcal{N} = \mathcal{H} \text{ a.k.a. } x_T = 0$$

$$(2) \mathcal{N} \neq \mathcal{H} \text{ a.k.a. } \mathcal{N} \text{ is closed}$$

$$\therefore \exists x_0 \in \mathcal{N}^\perp \text{ (}\neq 0\text{)}$$

$$x_T = \overline{Tx_0} \|x_0\|^{-2} x_0 \text{ a.k.a.}$$

$$i) y \in \mathcal{N} \text{ a.k.a.}$$

$$Ty = 0, (x_T, y) = 0$$

$$ii) y = \alpha x_0 \text{ a.k.a.}$$

$$Ty = \alpha Tx_0$$

$$(x_T, y) = \alpha (\overline{Tx_0} \|x_0\|^{-2} x_0, x_0) = \alpha \overline{Tx_0}$$

④ 5/20(8)

Gaussian measures ξ 構成 \mathcal{M}

$a \in \mathcal{X}$, $Q \in L_1^+(\mathcal{X})$, $\mu = N_{a, Q}$ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$
 $\exists (e_k)_{k=1}^\infty \quad \langle e_k, e_k \rangle = \lambda_k e_k$

$$x_k = \langle x, e_k \rangle \in \mathbb{R} \quad \forall k \in \mathbb{N}$$

$$\gamma : \mathcal{X} \rightarrow \ell^2 = \{ (a_k) \mid a_k \in \mathbb{R}, \sum a_k^2 < \infty \}$$

$\downarrow \qquad \qquad \downarrow$
 $x \mapsto (x_k), \text{ where } x_k = \langle x, e_k \rangle$

Lemma 1.13 $\gamma : \mathcal{X} \rightarrow \ell^2$ unitary op.

⊙ $\gamma(x) = (x_k) \therefore \|\gamma(x)\|_{\ell^2}^2 = \sum |x_k|^2 = \|x\|^2$

\therefore isometry, 1:1.

$\forall (a_k) \in \ell^2 \exists x = \sum a_k e_k \in \mathcal{X}$

$\gamma(x) = (a_k) \in \ell^2$

\therefore onto. //

$\mu = \prod_{k=1}^\infty N_{a_k, \lambda_k} \quad \text{if} \quad \prod_{k=1}^\infty \lambda_k > 0 \quad \text{is a meas.}$

$$I_{n,A} = \{ x = (x_k) \in \mathbb{R}^\infty \mid (x_1 \dots x_n) \in A \}$$

$n \geq 1, A \in \mathcal{B}(\mathbb{R}^n)$ cylindrical subsets

$$\mathcal{C} : I_{n,A} \text{ 全体.}$$

Prop 1.14 \mathcal{C} is alg i.e., $\forall I_{n,A} I_{m,B} \in \mathcal{C}$

$$\rightarrow I_{n,A} \cup I_{m,B} \in \mathcal{C}, I_{n,A}^c \in \mathcal{C}$$

$$\textcircled{=} I_{n,A} = I_{n+k, A \times X_{n+1} \times \dots \times X_{n+k}}$$

$$= I_{n,A} \cup I_{m,B}$$

$$= I_{n+m, A \times X_{n+1} \times \dots \times X_{n+m}} \text{ (i)}$$

$$\cup I_{n+m, B \times X_{n+1} \times \dots \times X_{n+m}} \text{ (ii)}$$

$$= I_{n+m} \text{ (i) } \cup \text{ (ii)}$$

$$I_{n,B}^c = I_{n,A^c}$$

$$\mu : \mathcal{C} \rightarrow [0, \infty) \quad \varepsilon > 0 \quad \exists \delta > 0$$

$$\mu(I_{n,A}) = \mu_1 \times \dots \times \mu_n(A)$$

$(\mathbb{R}^\infty, \mathcal{C}, \mu)$ 有限加法族 ν
有限加法族の測度

Hopf の主張で μ は測度 ν の主張である。

$$\mathcal{C} \ni A_n \text{ なら } \bigcup_n A_n \in \mathcal{C} \Rightarrow \mu\left(\bigcup_n A_n\right) = \sum \mu(A_n)$$

σ -additivity である。

* $E_j \downarrow$ かつ $\mu(E_j) \geq \varepsilon \quad \forall j \rightarrow \bigcap_{j=1}^{\infty} E_j \neq \emptyset$
 247777:

$$E_j(\alpha) = \{x \in \mathbb{R}_1^\infty \mid (\alpha, x) \in E_j\} \quad \alpha \in \mathbb{R}$$

$$\circ \mathbb{R}_n^\infty = \prod_{k=n+1}^{\infty} \mathbb{R}$$

$$F_j^{(1)} = \left\{ \alpha \in \mathbb{R} \mid \mu^{(1)}(E_j(\alpha)) \geq \frac{\varepsilon}{2} \right\}$$

$$\circ \mu^{(m)} = \prod_{k=n+1}^{\infty} \mu_k$$

← Fubini's theorem

$$\mu(E_j) = \int \mu^{(1)}(E_j(\alpha)) \mu_1(d\alpha)$$

$$= \int_{F_j^{(1)}} \mu^{(1)} + \int_{F_j^{(1)c}} \mu^{(1)}$$

$$\leq \mu_1(F_j^{(1)}) + \frac{\varepsilon}{2}$$

$$\therefore \mu_1(F_j^{(1)}) \geq \frac{\varepsilon}{2}$$

$$\mu_1(F_j^{(1)}) \geq \frac{\varepsilon}{2}$$

μ_1 is meas $F_j^{(1)} \downarrow$ P: "A3"

$$\liminf_j \mu_1(F_j^{(1)}) = \mu_1(\bigcap_j F_j^{(1)}) \geq \frac{\varepsilon}{2}$$

$$\bigcap_j F_j^{(1)} \neq \emptyset \quad \exists \bar{\alpha}_1 \in \mathbb{R} \text{ s.t. } \mu^{(1)}(E_j(\bar{\alpha}_1)) \geq \frac{\varepsilon}{2}$$

$$\therefore E_j(\bar{\alpha}_1) \neq \emptyset \quad \forall_j$$

$$\text{同定義} \quad E_j(\bar{\alpha}_1, \alpha_2) = \{x_2 \in \mathbb{R}_2^\infty \mid (\bar{\alpha}_1, \alpha_2, x) \in E_j\}$$

$$F_j^{(2)} = \{\alpha_2 \in \mathbb{R} \mid \mu^{(2)}(E_j(\alpha_1)) \geq \frac{\varepsilon}{4}\}$$

$$\mu^{(1)}(E_j(\bar{\alpha}_1)) = \int \mu^{(2)}(E_j(\bar{\alpha}_1, \alpha_2)) \mu_2(d\alpha_2)$$

$$= \int_{F_j^{(2)}} + \int_{F_j^{(2)c}}$$

$$\leq \mu_2(F_j^{(2)}) + \frac{\varepsilon}{4}$$

$$\therefore \mu_2(F_j^{(2)}) \geq \frac{\varepsilon}{4}$$

$$\therefore \mu^{(2)}(E_j(\bar{\alpha}_1, \bar{\alpha}_2)) \geq \frac{\varepsilon}{4} \quad \forall_j$$

$$\text{すなわち} \quad E_j(\bar{\alpha}_1, \bar{\alpha}_2) \neq \emptyset$$

$$\exists (\bar{\alpha}_k) \in \mathbb{R}^{\infty} \text{ s.t. } \forall_j$$

$$E_j(\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n) = \left\{ x \in \mathbb{R}_m^{\infty} \mid (\bar{\alpha}_1, \dots, \bar{\alpha}_n, x) \in E_j \right\} \neq \emptyset$$

$$\Rightarrow \forall_j (\bar{\alpha}_k) \in E_j \quad \forall_j$$

$$\therefore (\bar{\alpha}_k) \in \bigcap_j E_j \neq \emptyset \quad //$$

$$\textcircled{1} \quad \mathbb{R}^{\infty} \text{ is } d(x, y) = \sum_n \frac{1}{2^n} \frac{\max_{1 \leq k \leq n} |x_k - y_k|}{1 + \max_{1 \leq k \leq n} |x_k - y_k|}$$

\mathbb{R}^{∞} is Polish space $\Rightarrow \exists$

$$(\mathbb{R}^{\infty}, d) \quad \mathcal{B}(\mathbb{R}^{\infty}) = \mathcal{G}(C)$$

Prop 1.15 $\exists \mu$ prob meas. on $(\mathbb{R}^{\infty}, \mathcal{B}(\mathbb{R}^{\infty}))$

$$\text{st } \mu(A) = \mu_n(A) \quad A = I_n A.$$

Kolmogorov extension thm.

Ω Polish space, $\prod_{\mathbb{R}}^n \Omega = \Omega^n$, $\mathcal{B}(\Omega^n)$,

$\pi_{nm} : \Omega^m \rightarrow \Omega^n$ ($m \geq n$) projection
 $(\omega_1 \dots \omega_m) \mapsto (\omega_1 \dots \omega_n)$

$(\mu_n)_{n \geq 1}$ μ_n meas on Ω^n

$$A = \{ \pi_{n\infty}^{-1}(E) \in \Omega^\infty \mid E \in \mathcal{B}(\Omega^n), n \geq 1 \}$$

consistency $\forall n \neq m$

$$\mu_n(E) = \mu_m(\pi_{nm}^{-1}(E))$$

$$(\Omega^\infty, \mathcal{G}(A), \exists \mu) \text{ s.t. } \mu(\pi_{n\infty}^{-1}(E)) = \mu_n(E)$$

例 $\Omega = \mathbb{R}$, μ prob meas. $\mu_n = \mu \times \dots \times \mu$ on $\mathcal{B}(\mathbb{R}^n)$

$$I_{n,A} = \{ (x_k) \in \mathbb{R}^\infty \mid (x_1 \dots x_n) \in A \}$$

$n \geq 1, A \in \mathcal{B}(\mathbb{R}^n)$

$$C = \bigcup_{\substack{n \geq 1 \\ A \in \mathcal{B}(\mathbb{R}^n)}} I_{n,A}$$

$$\mu_n(A_1 \times \dots \times A_n) = \mu_n(A_1 \times \dots \times A_n \times \mathbb{R} \times \dots \times \mathbb{R})$$

$$\text{is the case } \therefore \mu_n(A_1 \times \dots \times A_n) = \mu_n(\pi_{nn}^{-1}(A_1 \times \dots \times A_n))$$

$$(\mathbb{R}^\infty, \mathcal{G}(C), \exists \mu) \text{ s.t. } \mu(A) = \mu_n(A) \quad A \in \mathcal{B}(\mathbb{R}^n)$$

$$\mathbb{R}^\infty \supset \ell^2 = \left\{ (x_k) \in \mathbb{R}^\infty \mid \sum_{k=1}^{\infty} |x_k|^2 < \infty \right\}$$

Lemma 1.16 $\ell^2 \in \mathcal{B}(\mathbb{R}^\infty)$ (Borel meas.)

☺ $P_n : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ projection onto $P(x)$
 $= \{x_1, x_2, \dots, x_n, 0, 0, \dots\}$

$$f_n(x) = \|P_n x\|_{\ell^2}^2 = \sum_{k=1}^n |x_k|^2$$

$\therefore f_n$ is cont. $\therefore x^{(k)} \rightarrow x^{(\infty)} \quad k \rightarrow \infty$

$\frac{1}{n} \neq 0$ - 不要 $\frac{1}{n}$ 乘 f_n $\rightarrow 0$ $\frac{1}{n}$ $\frac{1}{n}$ $\frac{1}{n}$ $\frac{1}{n}$

$$\|P_n(x^{(k)} - x^{(\infty)})\| \rightarrow 0$$

$\therefore f_n$ ~~meas.~~ meas. $f(x)$

$\therefore \sup_n f_n(x) = \|x\|_{\ell^2}^2$ meas

$\therefore \{x \in \mathbb{R}^\infty \mid f(x) < \infty\}$ meas.

Lemma 1.17 $\mu(\ell^2) = 1$

i.e., $\mu(\{x \in \mathbb{R}^\infty \mid \|x\|_{\ell^2} < \infty\}) = 1$

$$\text{☺} \int_{\mathbb{R}^\infty} f(x) d\mu = \lim_n \int_{\mathbb{R}^\infty} f_n(x) d\mu$$

$$= \lim_n \sum_{k=1}^n \int_{\mathbb{R}^\infty} |x_k|^2 d\mu = \lim_n \sum_{k=1}^n (\lambda_k + a_k^2) < \infty$$

$$\therefore \mu(\{x \mid f(x) < \infty\}) = 1$$

Thm 1.18 $\exists \mu$ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$
 with mean $a \in \mathcal{X}$, covarian $Q \in L^1_+(\mathcal{X})$ and

$$\hat{\mu}(h) = e^{i(a, h) - \frac{1}{2}(Qh, h)}$$

$$\textcircled{\ast} \quad \mathcal{X} \hat{=} \ell^2 \subset \mathbb{R}^\infty$$

$$\mu|_{\ell^2} = \mu \quad \text{とある}$$

$$\textcircled{\ast} \int_{\mathcal{X}} \|x\|^2 \mu(dx) = \text{Tr } Q + \|a\|^2 \quad \text{OK.}$$

$$\text{簡単のため: } \text{Ker } Q = \{0\} \quad \lambda_1 \geq \lambda_2 \geq \dots$$

(mean) \downarrow 有界 $\textcircled{\ast}$ による

$$\begin{aligned} \int_{\mathcal{X}} (x, h) \mu(dx) &= \lim_n \int_{\mathcal{X}} (P_n x, h) \mu(dx) \\ &= \lim_n \sum_{k=1}^n h_k \int_{\mathcal{X}} x_k \mu(dx) = \lim_n h_k a_k = (a, h) \end{aligned}$$

(covariance)

$$\begin{aligned} &\int_{\mathcal{X}} (x-a, y) (x-a, z) d\mu \\ &= \lim_n \int_{\mathcal{X}} (P_n(x-a), y) (P_n(x-a), z) d\mu \\ &= \lim_n \sum_{k=1}^n y_k z_k \int_{\mathcal{X}} \frac{(x-a)_k^2}{h_k} d\mu \Rightarrow \lim_n \sum y_k z_k \lambda_k \\ &= (Qy, z) \end{aligned}$$

Fourier tr:

$$\int_{\mathcal{X}} e^{i(x, h)} \mu(dx) = \lim_n \int_{\mathcal{X}} e^{i(P_n x, h)} \mu(dx)$$

$$= \lim_n \frac{1}{\pi} \int_{\mathbb{R}} e^{i x_n h_n} N_{a_n, \sigma_n} (dx_n)$$

$$= \lim_n \frac{1}{\pi} \int_{\mathbb{R}} e^{i a_n h_n - \frac{1}{2} \cdot \sigma_n^2 h_n^2} \rightarrow e^{i(a, h)} e^{-\frac{1}{2}(\sigma h, h)}$$

⑤ 5/27 (金)

§ 2 Cameron-Martin spaces

Def 2.1 (Ω, \mathcal{F}, P) K Hilbert sp
separable

$X: \Omega \rightarrow K$ Gaussian r.v.

$\Leftrightarrow X_{\#} P$ is G. meas on $(K, \mathcal{B}(K))$

\uparrow
(X or law \Leftrightarrow)

mean

$$(m_x, h) = \int_K (y, h) X_{\#} P(dy) = \int_{\Omega} (X(\omega), h) P(d\omega)$$

cov.

$$\begin{aligned} (Q_x h, k) &= \int_K (y - m_x, h) (y - m_x, k) X_{\#} P(dy) \\ &= \int_{\Omega} (X(\omega) - m_x, h) (X(\omega) - m_x, k) P(d\omega) \end{aligned}$$

$$\hat{\mu}_x(h) = \int e^{i(y, h)} X_{\#} P(dy) = \int e^{i(X(\omega), h)} P(d\omega).$$

Example 1 $(\mathcal{H}, \mathcal{B}(\mathcal{H}), N_{a, \alpha})$

$$X : \mathcal{H} \rightarrow \mathcal{H} \quad \text{is G.v.v.}$$

$$x \mapsto x+b$$

$$\int e^{i(y, h)} X_{\#} P(dy) = \int e^{i(x+b, h)} P(dx) = e^{i(b, h)} \int e^{i(x, h)} P(dx)$$

$$\therefore X_{\#} P(dy) = N_{a+b, \alpha}$$

Example 2 $(\mathcal{H}, \mathcal{B}(\mathcal{H}), N_{a, \alpha}) \quad T \in L(\mathcal{H}, \mathcal{K})$

is G.v.v.

$$\therefore \int e^{i(y, h)} T_{\#} P(dy) = \int e^{i(Ty, h)} P(dy)$$

$$= e^{i(a, T^*h)} - \frac{1}{2} (QT^*h, T^*h)$$

$$T_{\#} P = N_{(Ta, TQT^*)}$$

Example 3 $(\mathcal{H}, \mathcal{B}(\mathcal{H}), N_a) \quad N_a = N_{0, Q}$

$$T : \mathcal{H} \rightarrow \mathbb{R}^n \quad \text{is odd nt}$$

$$x \mapsto (x, z_1), \dots, (x, z_n) \quad z_j \text{ fix}$$

is G.v.v.

Example 2 54).

$$T_{\#} P = N_{TQT^*}$$

$$TQT^* = \left((z_i, Qz_j) \right)_{1 \leq i, j \leq n}$$

Def 2.2 $X: (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

$X = (X_1 \dots X_n)$ $X_1 \dots X_n$ indep

$$\Leftrightarrow X_{\#}P = \prod_{j=1}^n (X_j)_{\#}P$$

Lemma 2.3 $X_1 \dots X_n$ indep \Leftrightarrow

$$\int \varphi_1(X_1(\omega)) \dots \varphi_n(X_n(\omega)) P(d\omega) = \prod \int \varphi_j(X_j(\omega)) P(d\omega)$$

$$\textcircled{\text{1}} (\Rightarrow) \varphi_1(\xi_1) \dots \varphi_n(\xi_n) = \varphi(\xi_1 \dots \xi_n) \text{ etc}$$

$$\textcircled{\text{2}} = \int \varphi(X(\omega)) P(d\omega) = \int \varphi(\xi) X_{\#}P(d\xi)$$

$$= \int \varphi(\xi) \prod_j (X_j)_{\#}P(d\xi) = \prod \int \varphi_j(\xi_j) (X_j)_{\#}P(d\xi) \\ = \prod \int \varphi_j(X_j(\omega)) P(d\omega)$$

$$\textcircled{\text{3}} (\Leftarrow) \varphi_i(\xi_i) = \mathbb{1}_{I_i} \text{ etc}$$

$$X_{\#}P(I_1 \times \dots \times I_n) = \prod (X_j)_{\#}P(I_j)$$

Example. $(\mathcal{H}, B(\mathcal{H}), N_Q)$ (e_k)
CONS

$$Qe_k = \lambda_k e_k, \quad Q \in L'_t(\mathcal{H})$$

$$X_j(x) = (x, e_j) \quad X_j : \mathcal{H} \rightarrow \mathbb{R} \quad \text{G.r.v.}$$

\Rightarrow Ψ is not $\frac{1}{2}$ $\int \Psi^2$ $\int \Psi^2$ $X = (X_1 \dots X_n)$

\cdot $(X_j)_{\#} \mu = N_{(Qe_j, e_j)} = N_{\lambda_j}$

\cdot $X_{\#} \mu = N_{(Qe_i, e_j)} = N_{(\lambda_1, \dots, \lambda_n)}$

$$= \prod (2\pi\lambda_j)^{-\frac{1}{2}} e^{-\frac{1}{\lambda_j} \frac{|x_j|^2}{2}} dx_j$$

$$= \prod N_{\lambda_j}$$

\cdot (注) $N_{Q, Q}(dx) = \left[(2\pi)^d \det Q \right]^{-1/2}$
 $\text{ker } Q \neq \{0\}$ $\times \exp\left[-\frac{1}{2} (Q^{-1}x, x)\right] dx.$

Q degenerato 155

Lemma 2.4 ($\mathcal{H}, B(\mathcal{H}), N_Q$)

$f_1 \dots f_n \in \mathcal{H}$

$$X_j(x) = (x, f_j) \quad x_j: \mathcal{H} \rightarrow \mathbb{R}$$

G r.v.

$$X = (X_1 \dots X_n)$$

$$X \text{ is independent} \iff (Q f_i, f_j) = 0 \quad i \neq j$$

☺ (\Leftarrow)

$$(X_{\#})_{\mu} = N((Q f_i, f_j)) = N(\lambda_1, \dots, \lambda_n)$$

$$= \prod N_{\lambda_j}$$

$$(X_{j\#})_{\mu} = N_{\lambda_j} \quad \text{77の2" 完了}$$

(\Rightarrow)

$$(X_{\#} P)(I_1 \times \dots \times I_n) = \prod_j (X_{j\#} P)(I_j)$$

~~$$\sqrt{(2\pi)^d / \det Q} e^{-\frac{1}{2} (Q^{-1} x, x)} = \prod_j \frac{1}{\sqrt{2\pi \lambda_j}} e^{-\frac{1}{2} \frac{|x_j|^2}{\lambda_j}} \frac{dx_j}{\lambda_j}$$~~

$(Q^{-1} x, x)$ の 正則分解成分は λ_j の逆.

$$\underbrace{N(Q f_i, f_j)}_{\equiv Q} = \prod N(Q f_i, f_j) \quad //$$

Prop 2.5 $(\mathcal{X}, B(\mathcal{X}), N_Q)$ $z_j \in Q^{\frac{1}{2}}(\mathcal{X})$

$$X_j(x) = (x, Q^{-\frac{1}{2}} z_j) \quad j=1, \dots, n.$$

$$X = (X_1 \dots X_n)$$

$$\textcircled{1} X \# N_Q = N_{\tilde{Q}} \quad \textcircled{2} X_j \text{ 独立} \Leftrightarrow \sum_{i \neq j} (z_i, z_j) = 0$$

$$\tilde{Q} = ((z_i, z_j))_{1 \leq i, j \leq n}$$

☺ 早速 check する

~~$$X \# N_Q = N((Q Q^{-\frac{1}{2}} z_i, Q^{-\frac{1}{2}} z_j)) = N_{\tilde{Q}}$$~~

$$X_1 \dots X_n \text{ 独立} \Leftrightarrow (Q Q^{-\frac{1}{2}} z_i, Q^{-\frac{1}{2}} z_j) = 0 \quad i \neq j //$$

$X_j(x) \in \forall z_j$ により det して... z の 2 次元に

$$X_j(x) := (Q^{-\frac{1}{2}} x, z_j) \quad \forall z_j \in \mathbb{R}^2 \text{ def する.}$$

$$T = T^* \quad x \in Q^{\frac{1}{2}}(\mathcal{X})$$

Prop 2.6 $(\mathcal{X}, B(\mathcal{X}), N_Q)$ $\text{Ker } Q = \{0\}$

$$\mu(Q^{\frac{1}{2}}(\mathcal{X})) = 0$$

$$\textcircled{1} Q^{\frac{1}{2}}(\mathcal{X}) = \bigcup_m \left\{ y \mid \sum_{j=1}^{\infty} \frac{1}{\lambda_j} y_j^2 \leq n^2 \right\}$$

$$= \bigcup_{n=1}^{\infty} U_n$$

$$U_{n+k} = \left\{ y \mid \sum_{j=1}^{2k} \frac{1}{\lambda_j} y_j^2 \leq n^2 \right\} \quad U_{n+k} \downarrow U_n$$

$$\mu(U_n) = \lim_k \mu(U_{nk})$$

$$\mu(U_{nk}) = \int \mathbb{1}_{\left\{y \mid \sum \frac{1}{\lambda_j} y_j^2 \leq n^2\right\}} N(\lambda_1, \dots, \lambda_{2k})$$

$$= \int \mathbb{1}_{\left\{\dots\right\}} e^{-\frac{1}{2} \sum \frac{1}{\lambda_j} |y_j|^2} dy \quad (2\pi)^{-\frac{n}{2}} (\lambda_1 \dots \lambda_{2k})^{-\frac{1}{2}}$$

$$= \int \mathbb{1}_{\left\{\dots\right\}} e^{-\frac{1}{2} \sum |z_j|^2} dz \quad (2\pi)^{-\frac{n}{2}}$$

$$= \int_0^{n^2} e^{-\frac{1}{2} r^2} r^{2k-1} dr \quad S_{2k-1} (2\pi)^{-k}$$

$$S_{2k-1} = \frac{2k \pi^k}{\Gamma(k+1)} = \frac{2k \pi^k}{k!} = \frac{2 \pi^k}{(k-1)!}$$

$$\begin{aligned} \therefore \frac{1}{(2\pi)^k} \frac{2\pi^k}{(k-1)!} \int_0^{n^2} e^{-\frac{1}{2} r^2} r^{2k-1} dr &= \frac{2\pi^k}{(k-1)!} \frac{1}{2k} n^{2k} \frac{1}{2^{2k} \pi^k} \\ &= \frac{1}{2^k k!} n^{2k} = \frac{1}{k!} \left(\frac{n^2}{2}\right)^k \rightarrow 0 \end{aligned} \quad //$$

$Q^{\frac{1}{2}}(\partial)$ is Cameron-Martin space (e.g.)

⑥ 6/3 (全)

Prop 2.7 (X_n) Gaussian v.v. on (Ω, \mathcal{F}, P)

$$X_n: \Omega \rightarrow \mathcal{H}$$

mean $a_n \in \mathcal{H}$

covariance $Q_n \in L_+(\mathcal{H})$

$$X_n \rightarrow X \text{ in } L^2(\Omega).$$

Then X is also Gaussian v.v. s.t. \exists

$$X \# P = N_{a, Q} \text{ where } (a, h) = \lim_n (a_n, h)$$

$$(Qh, h) = \lim_n (Q_n h, h)$$

$$\textcircled{:} \lim_n (a_n, h) = \lim_n \int_{\Omega} (X_n(\omega), h) P(d\omega)$$

$$= \int_{\Omega} (X(\omega), h) P(d\omega) = (a, h)$$

同様にして

$$\begin{aligned} \lim_n (Q_n h, k) &= \lim_n \int_{\Omega} (X_n(\omega) - a_n, h) \\ &\quad (X_n(\omega) - a_n, k) P(d\omega) \\ &= (Qh, k) \end{aligned}$$

$$\int_{\mathcal{H}} e^{i(y, k)} X \# P(dy) = \int_{\mathcal{H}} e^{i(X(\omega), k)} P(d\omega)$$

$$= \lim_n \int_{\mathcal{H}} e^{i(X_n(\omega), k)} P(d\omega) = e^{i(a_n, k) - \frac{1}{2}(Q_n k, k)}$$

$$= e^{i(a, h) - \frac{1}{2}(Qh, h)}$$

//

$$\textcircled{a} X_z = (\alpha, Q^{-1/2} z) \quad z \in Q^{1/2} \mathcal{H} \quad T = \cdot, T_z$$

$$\int X_{z_1} X_{z_2} \mu(dw) = (Q Q^{-1/2} z_1, Q^{-1/2} z_2) = (z_1, z_2) \quad \text{ok}$$

$$\textcircled{b} X_z = (Q^{-1/2} \alpha, z) \quad \text{etc} \quad \mu(Q^{1/2} \mathcal{H}) = 0.$$

$$Z = Z'' \quad X_0 : Q^{1/2}(\mathcal{H}) \rightarrow L^2(\mathcal{H}, \mu) \quad \mu = N_Q$$

$$\cap \mathcal{H} \quad \cup$$

$$z \mapsto X_z$$

isometry $\hookrightarrow Q^{1/2}(\mathcal{H}) \subset \mathcal{H}$
dense.

Prop 2.8 $T : \mathcal{H} \rightarrow \mathcal{K}$

$D(T) \subset \mathcal{H}$ dense $\hookrightarrow \|Tf\| \leq M \|f\|$
 $\Rightarrow \exists \tilde{T} \text{ s.t. } \tilde{T} \supset T \quad D(\tilde{T}) = \mathcal{H} \quad \hookrightarrow \|\tilde{T}f\| \leq M \|f\|$

$$\textcircled{!} \tilde{T}f = \lim_n T f_n \quad (f_n \rightarrow f) \quad \text{etc}$$

isometry
 $\alpha \in \mathcal{H}$
 isometry
 $\in \mathcal{H}$
 $\#T \in \mathcal{H}$

$$\exists \tilde{X}_0 : \mathcal{H} \rightarrow L^2(\mathcal{H}) \quad \text{s.t.} \quad \tilde{X}_0 \supset X$$

$$\hookrightarrow \int \tilde{X}_{z_1} \tilde{X}_{z_2} \mu(dw)$$

$$\tilde{X}_0 \text{ is white noise mapping} = (z_1, z_2).$$

Thm 2.9 $\tilde{X} = (\tilde{X}_{z_1}, \dots, \tilde{X}_{z_n})$ $z_j \in \partial \mathcal{D}$

$\Leftrightarrow \tilde{X}_{\#} P = N(z; z_0)$

\tilde{X} is analytic $\Leftrightarrow z_i$ is $\bar{z}_i = 1 = \text{analytic}$

$\odot z_i^N = \sum_{j=1}^N (z_i, e_j) e_j \in \mathcal{D}^{1/2}(\partial \mathcal{D})$

$\tilde{X}_N = (\tilde{X}_{z_1^N}, \dots, \tilde{X}_{z_n^N}) = (X_{z_1^N}, \dots, X_{z_n^N})$

is Gaussian i.e.

$\tilde{X}_N_{\#} P = N(z_i^N, z_j^N)$ etc

$(z_i^N, z_j^N)_{ij} \rightarrow (z_i, z_j)_{ij} \quad (N \rightarrow \infty)$

prop 2.7 $\#$) \tilde{X} is Gaussian z^n

$\tilde{X}_{\#} P = N(z; z_0)$ \parallel

$\#$) $\int_{\mathbb{R}^n} e^{i(y, z)} \tilde{X}_{\#} P(dy) = e^{-\frac{1}{2}(Qz, z)}$, where $Q = (z_i, z_j)_{1 \leq i, j \leq n}$

analyticity \Leftrightarrow is Lem 2.4 & 10.11 \parallel

§ 3 Cameron-Martin Formula

$N_{a,Q}$ & N_Q relations $\Rightarrow \mu \ll \nu \Leftrightarrow \nu(A)=0 \Rightarrow \mu(A)=0$.

$$\mu \ll \nu \Leftrightarrow \nu(A)=0 \Rightarrow \mu(A)=0$$

$$\mu \ll \nu \Rightarrow \exists f \in L^1(\Omega) \text{ s.t.}$$

$$\mu(A) = \int_A f(x) d\nu(x)$$

Radon-Nikodym thm.

$$f = \frac{d\mu}{d\nu}$$

$\exists \beta \in \mathbb{R}^n$

$$N_Q(dn) = \frac{(2\pi)^{-\frac{n}{2}}}{\det Q} \prod_{j=1}^n \exp\left(-\frac{1}{2} \frac{1}{\lambda_j} |x_j|^2\right) dx_j$$

$$N_{Q,a}(dn) = (2\pi \det Q)^{-\frac{n}{2}} \prod_{j=1}^n \exp\left(-\frac{1}{2} \frac{1}{\lambda_j} (x_j - a_j)^2\right)$$

$$\frac{dN_{Q,a}}{dN_Q} = \frac{e^{-\frac{1}{2} (Q^{-1}(x-a), x-a)}}}{e^{-\frac{1}{2} (Q^{-1}x, x)}} = e^{-\frac{1}{2} \|Q^{-1/2}a\|^2 + (Q^{-1/2}a, Q^{-1/2}x)}$$

$N_{Q,a}$ & N_Q are equivalent

目標:

$$(i) \quad a \in Q^{1/2}(\mathcal{A}) \Rightarrow N_a \mathcal{A} \cong N_a$$

$$(ii) \quad a \notin Q^{1/2}(\mathcal{A}) \Rightarrow N_a \mathcal{A} \subset N_a \text{ is singular}$$

(Ω, \mathcal{F}) μ, ν two prob meas

$\mu \ll \nu$ is $\frac{1}{2}(\mu + \nu)$ is abs cont.

$$\mu \ll \frac{1}{2}(\mu + \nu) \quad \nu \ll \frac{1}{2}(\mu + \nu)$$

$$\frac{1}{2}(\mu + \nu) = \xi$$

$$\frac{d\mu}{d\xi} \in L^1(d\xi)$$

$$\frac{d\nu}{d\xi} \in L^1(d\xi) \quad \mathbb{R}-N$$

$$\int \sqrt{\frac{d\mu}{d\xi} \frac{d\nu}{d\xi}} d\xi = H(\mu, \nu)$$

= 4E Hellinger integral 4E

$$H(\mu, \nu) \leq \left(\int \frac{d\mu}{d\xi} d\xi \right)^{1/2} \left(\int \frac{d\nu}{d\xi} d\xi \right)^{1/2} = \mu^{1/2} \nu^{1/2} = 1$$

Prop 3.1 let $H(\mu, \nu) = 0$. Then $\mu \perp \nu$

☺ $f = \frac{d\mu}{d\xi}$ $g = \frac{d\nu}{d\xi}$

$$H(\mu, \nu) = \int_{\Omega} \sqrt{fg} \, d\xi = 0$$

Hence $fg = 0$ a.e. ξ

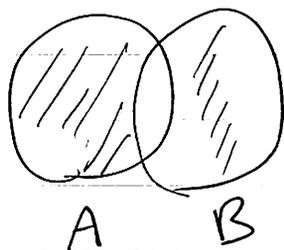
$$A = \{ \omega \mid f = 0 \}$$

$$B = \{ \omega \mid g = 0 \}$$

$$C = \{ \omega \mid fg = 0 \} \leftarrow \text{true}$$

$$\mu(A) = \int_A f \, d\xi = 0, \quad \nu(B) = \int_B g \, d\xi = 0$$

$$C = A \cup B \quad \text{so} \quad \mu(C) = \nu(C) = 1$$



$\mu \perp \nu$ ^{disjoint} $B \setminus A$
 $\nu \perp \mu$ $A \setminus B$ is disjoint

$$\mu(B \setminus A) = 1 = \nu(A \setminus B)$$

$$(B \setminus A) \cap (A \setminus B) = \emptyset$$

$$\sqrt{\frac{d\mu}{d\nu} \frac{d\nu}{d\mu}} d\nu$$

$$\frac{d\mu}{d\nu} d\nu$$

(7) 6/10 (全)

Lemma $\mu \ll \nu$ iff equivalent $\Leftrightarrow \exists$

$$= a \Leftrightarrow H(\mu, \nu) = \int_{\Omega} \sqrt{\frac{d\nu}{d\mu}} \frac{d\mu}{d\nu} d\nu$$

☺ $\frac{d\mu}{d\nu} \frac{d\mu}{d\nu} = \frac{d\mu}{d\nu} \frac{d\nu}{d\mu} \frac{d\nu}{d\mu} \frac{d\mu}{d\nu}$

$$= \sqrt{\frac{d\mu}{d\nu} \frac{d\nu}{d\mu}} d\nu = \sqrt{\frac{d\mu}{d\nu} \frac{d\nu}{d\mu}} d\nu = \sqrt{\frac{d\nu}{d\mu}} d\mu = d\mu$$

$$H(\mu, \nu) = \int \sqrt{\frac{d\mu}{d\nu} \frac{d\nu}{d\mu}} d\nu = \int \sqrt{\frac{d\nu}{d\mu}} d\mu = \sqrt{\frac{d\nu}{d\mu}} d\mu = d\mu$$

Example: $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ $\mu = N_{2, \lambda}$, $\nu = N_{a, \lambda}$
 $\Leftrightarrow \mu \ll \nu$ $\mu \ll \nu$ iff equivalent

$$\frac{d\nu}{d\mu} = \exp\left(-\frac{a^2}{2\lambda} + \frac{ax}{\lambda}\right)$$

$$H(\mu, \nu) = \int_{\mathbb{R}} \sqrt{\frac{d\nu}{d\mu}} d\mu = \int e^{-\frac{a^2}{4\lambda} + \frac{ax}{2\lambda}} N_{2, \lambda}(d\mu)$$

$$= e^{-\frac{a^2}{8\lambda}}$$

Example 2 $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ $\mu = N(\Lambda, \Sigma)$
 $\nu = N(a, \Sigma)$

$$\therefore \frac{d\nu}{d\mu} = \pi \exp\left(-\frac{a^2}{2\lambda_j} + \frac{a_j x_j}{\lambda_j}\right)$$

$$\Rightarrow H(\mu, \nu) = \pi e^{-\frac{1}{8} \sum_{j=1}^n \frac{a_j^2}{\lambda_j}}$$

Lemma 3.2 $\mu_1, \nu_1, \mu_2, \nu_2$ prob. meas. on (Ω, \mathcal{F})

$$H(\mu_1 \times \mu_2, \nu_1 \times \nu_2) = H(\mu_1, \nu_1) \cdot H(\mu_2, \nu_2)$$

$$\odot \quad \xi_1 = \frac{1}{2}(\mu_1 + \nu_1) \quad \xi_2 = \frac{1}{2}(\mu_2 + \nu_2)$$

$$\begin{array}{cc} \text{alt} & \mu_1 \ll \xi_1 & \mu_2 \ll \xi_2 \\ & \nu_1 \ll \xi_1 & \nu_2 \ll \xi_2 \end{array}$$

= alt

$$\mu_1 \times \mu_2 \ll \xi_1 \times \xi_2$$

$$\therefore \xi_1 \times \xi_2(C) = 0 \iff \xi_1(C) = 0 \text{ and } \xi_2(C) = 0$$

$$\int d\xi_1 \int d\xi_2 \mathbb{1}_C(x_1, x_2) = 0$$

$$\dots \int d\xi_2 \mathbb{1}_C(x_1, x_2) = 0 \quad \forall x_1 \in \Omega \setminus N \quad \xi_1(N) = 0$$

$$\dots \int d\mu_2 \mathbb{1}_C(x_1, x_2) = 0 \quad \xi_1(N) = 0 \text{ and } \mu_1(N) = 0$$

$$\uparrow$$

$$x_1 \in \Omega \setminus N \text{ where } \mu_1(N) = 0$$

$$\therefore \int d\mu_1 \int d\mu_2 \mathbb{1}_C(x_1, x_2) = 0$$

$$f_1 = \frac{d\mu_1}{d\xi_1} \quad g_1 = \frac{d\nu_1}{d\xi_1} \quad f_2 = \frac{d\mu_2}{d\xi_2} \quad g_2 = \frac{d\nu_2}{d\xi_2}$$

$$\frac{d(\mu_1 \times \mu_2)}{d(\xi_1 \times \xi_2)} = f_1 f_2$$

$$\frac{d(\nu_1 \times \nu_2)}{d(\xi_1 \times \xi_2)} = g_1 g_2$$

$$(\mu_1 \times \mu_2)(A \times B) = \mu_1(A) \mu_2(B) = \int_A f_1 d\xi_1 \int_B f_2 d\xi_2$$

$$= \int_{A \times B} f_1 \otimes f_2 d\xi_1 \times \xi_2$$

rectangle $C = \xi_1 \times \xi_2$ $(\mu_1 \times \mu_2)(C) = \int_C f_1 \otimes f_2 d\xi_1 \times \xi_2$

一意の測度の存在を
 $\mu_1 \times \mu_2$ と書く

$$\int_C f_1 \otimes f_2 d\xi_1 \times \xi_2 = f(C)$$

$$\therefore \mu_1 \times \mu_2(C) = f(C)$$

一意測度

$$H(\mu_1 \times \mu_2, \nu_1 \times \nu_2) = \int \sqrt{f_1 f_2 g_1 g_2} d\xi_1 \times \xi_2$$

$$= \int \sqrt{f_1 g_1} d\xi_1 \int \sqrt{f_2 g_2} d\xi_2 = H(\mu_1, \nu_1) \cdot H(\mu_2, \nu_2)$$

$$\mu = \prod_{k=1}^{\infty} \mu_k, \nu = \prod_{k=1}^{\infty} \nu_k \Rightarrow H(\mu, \nu) = \int_{\Omega} \sqrt{\frac{d\mu}{d\zeta} \frac{d\nu}{d\zeta}} d\zeta$$

$$= \prod_{k=1}^{\infty} \int_{\Omega} \sqrt{\frac{d\mu_k}{d\zeta_k} \frac{d\nu_k}{d\zeta_k}} d\zeta_k$$

Thm (Kakutani)

$(\mu_k), (\nu_k)$ prob m. on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\mu_k \stackrel{\sim}{=} \nu_k$ let $\mu = \prod_{k=1}^{\infty} \mu_k, \nu = \prod_{k=1}^{\infty} \nu_k$

かつ $H(\mu, \nu) > 0 \Rightarrow \mu \stackrel{\sim}{=} \nu$ and

$$\frac{d\nu}{d\mu} = \lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{d\nu_k}{d\mu_k}$$

$$H(\mu, \nu) = 0 \Rightarrow \mu \perp \nu \quad \text{in } L^1(\mathbb{R}^{\infty}, \mu)$$



$$\frac{d\nu_k}{d\mu_k} = f_k(x_k) \quad f_n(x_1, \dots, x_n) = \prod_{k=1}^n f_k(x_k)$$

$$\mu^{(n)} = \prod_{k=1}^n \mu_k, \quad \nu^{(n)} = \prod_{k=1}^n \nu_k$$

\mathbb{R}^{∞} 上の
measure = $\prod_{k=1}^{\infty} \mu_k$
1276C

$$H(\mu^{(n)}, \nu^{(n)}) = \int_{\mathbb{R}^{\infty}} \prod_{k=1}^n \sqrt{f_k(x_k)} \mu^{(n)}(dx)$$

$$= \int_{\mathbb{R}^{\infty}} \sqrt{f_n(x_1, \dots, x_n)} \mu^{(n)}(dx)$$

実は $\sqrt{f_n}$ は $L^2(\mathbb{R}^{\infty}, \mu)$ 上の Cauchy seq. $\rightarrow \oplus$

$$\exists g \text{ s.t. } \|\sqrt{f_n} - g\| \rightarrow 0 \Leftrightarrow \int |\sqrt{f_n} - g|^2 \rightarrow 0$$

$$\int |f_n - 2\sqrt{f_n} \cdot g + g^2| \rightarrow 0$$

$$\int |f_n - g^2| = \int |f_n - 2\sqrt{f_n}g + g^2 + 2\sqrt{f_n}g - g^2 - g^2| \rightarrow 0$$

↓ 10

$$g^2 = f \text{ とおくと } f = \frac{dV}{d\mu} \text{ となる}$$

$$\therefore \varphi = \varphi(x_1 \dots x_k) \quad k < n$$

$$\int_{\mathbb{R}^{\infty}} \varphi(x) V^{(n)}(dx) = \int \varphi(x) f_n^{(n)} \mu^{(n)}(dx)$$

$$= \int_{\mathbb{R}^{\infty}} \varphi(x) V(dx) = \int_{\mathbb{R}^{\infty}} \varphi(x) \underline{f_n(x)} \mu(dx)$$

φ 6dd

lim $\epsilon \ll \delta$

$$\int_{\mathbb{R}^{\infty}} \varphi(x) V(dx) = \int_{\mathbb{R}^{\infty}} \varphi(x) f(x) \mu(dx)$$

$$V(C) = \int_C f(x) \mu(dx) \quad \forall C: \text{rectangle}$$

$\therefore V \ll \mu$ 同程度に $V \gg \mu$ //

(*) の (:) $\int_{\mathbb{R}^{\infty}} |\sqrt{f_{n+p}} - \sqrt{f_n}|^2 = \int \prod_{k=1}^n P_k \left| \frac{\prod_{k=n+1}^{n+p} P_k}{\prod_{k=n+1}^n P_k} - 1 \right|^2 \mu(dx)$

$$= \int \underbrace{\prod_{k=1}^n P_k(x_1 \dots x_n)}_{\int V(dx) = 1} \int \left| \frac{\prod_{k=n+1}^{n+p} P_k}{\prod_{k=n+1}^n P_k} - 1 \right|^2 \mu(dx)$$

$$= \int \frac{\prod_{k=n+1}^{n+p} P_k}{\prod_{k=n+1}^n P_k} - 2 \frac{\prod_{k=n+1}^{n+p} P_k}{\prod_{k=n+1}^n P_k} + 1 \mu(dx)$$

$$= 2 \left(1 - \frac{\prod_{k=n+1}^{n+p} P_k}{\prod_{k=n+1}^n P_k} \int_{\mathbb{R}} \sqrt{P_k(x_k)} \mu(dx_k) \right) = 2 \left(1 - \prod_{k=n+1}^{n+p} H(\mu_k, \nu_k) \right)$$

$$H(\mu, \nu) = \prod_{k=1}^{\infty} H(\mu_k, \nu_k) \geq 0$$

$$-\log H(\mu, \nu) = -\sum \log H(\mu_k, \nu_k) < +\infty$$

$$-\sum_{k=n+1}^{n+p} \log H(\mu_k, \nu_k) < \varepsilon \quad \text{+ 分式可小}$$

$$\sum \log H(\mu_k, \nu_k) > -\varepsilon$$

$$\prod H(\mu_k, \nu_k) > e^{-\varepsilon}$$

$$-\prod H(\mu_k, \nu_k) < -e^{-\varepsilon}$$

$$\therefore \int |\sqrt{f_{n+p}} - \sqrt{f_n}|^2 \leq 2(1 - e^{-\varepsilon}) \quad \checkmark$$

2016/6/17 (A)

先週の続き

$$H(\mu_1 \times \mu_2, \nu_1 \times \nu_2) = \int \sqrt{\frac{d\mu_1 \times \mu_2}{d\xi_1 \times \xi_2} \frac{d\nu_1 \times \nu_2}{d\xi_1 \times \xi_2}}$$

$$\frac{1}{2}(\mu_1 \times \mu_2 + \nu_1 \times \nu_2) = \int \chi_{12}$$

$$\int \sqrt{\frac{d\mu_1 \times \mu_2}{d\xi} \frac{d\nu_1 \times \nu_2}{d\xi}} d\xi$$

$$\frac{d\mu_1 \times \mu_2}{d\xi} = \frac{d\mu_1 \times \mu_2}{d\xi_1 \times \xi_2} \frac{d\xi_1 \times \xi_2}{d\xi} \quad \chi_{12} \dots$$

$$\xi_1 \times \xi_2 \ll \xi \quad \leftarrow \quad \text{示す } \odot$$

$$\xi_1 = \frac{1}{2}(\mu_1 + \nu_1) \quad \xi_2 = \frac{1}{2}(\mu_2 + \nu_2)$$

$$\odot \quad \int(A) = \frac{1}{2}(\mu_1 \times \mu_2(A) + \nu_1 \times \nu_2(A)) = 0$$

χ 仮定す。 $A \in \mathcal{F} \times \mathcal{F}$

特 $A = B \times C \in \mathcal{F} \times \mathcal{F}$ とす

$$(\xi_1 \times \xi_2)(B \times C) = \xi_1(B) \cdot \xi_2(C)$$

$$= \frac{1}{2}(\mu_1(B) + \nu_1(B)) \frac{1}{2}(\mu_2(C) + \nu_2(C)) = 0$$

$$\Rightarrow \mu_1(B) + \nu_1(B) = 0 \text{ とす } \mu_1(B) = \nu_1(B) = 0$$

$$\int(B \times C) = \frac{1}{2}(\mu_1(B)\nu_2(C) + \nu_1(B)\mu_2(C)) = 0$$

$$\therefore \xi_1 \times \xi_2(B \times C) = 0 \Rightarrow \int(B \times C) = 0$$

$$\frac{d\nu}{d\mu} = \lim_n e^{-\frac{1}{2} \|Q_n^{-1/2} a\|^2 + (Q_n^{-1/2} a, Q_n^{-1/2} x)}$$

$$\rightarrow e^{-\frac{1}{2} \|Q^{-1/2} a\|^2 + (Q^{-1/2} a, Q^{-1/2} x)}$$

in L^1

Thm (Feldman-Hajek thm)

$$Q, R \in L_1^+(\mathcal{X}) \text{ mit } [Q, R] = 0$$

$$\mu = N_Q, \nu = N_R$$

$$\mu \perp \nu \Leftrightarrow \sum_{k=1}^{\infty} \left(\frac{\lambda_k - \gamma_k}{\lambda_k + \gamma_k} \right)^2 < \infty$$

$$\nexists! \mu \not\perp \nu \Leftrightarrow \mu \perp \nu$$

$$\odot \quad H(\mu, \nu) = \prod_k H(\mu_k, \nu_k)$$

$$\mu_k = (2\pi \lambda_k)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{|x|^2}{\lambda_k}\right) dx$$

$$\nu_k = (2\pi \gamma_k)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{|x|^2}{\gamma_k}\right) dx$$

$$\therefore \mu_k \cong \nu_k$$

$$\therefore H(\mu_k, \nu_k) = \int \sqrt{\frac{d\nu_k}{d\mu}} d\mu_k = \int \sqrt{\frac{\gamma_k^{-1/2} e^{-\frac{1}{2} \frac{|x|^2}{\gamma_k}}}{\lambda_k^{-1/2} e^{-\frac{1}{2} \frac{|x|^2}{\lambda_k}}} d\mu_k}$$

$$= \int \sqrt{\frac{\lambda_k}{\gamma_k} e^{-\frac{1}{2} \frac{|x|^2}{\gamma_k} + \frac{1}{2} \frac{|x|^2}{\lambda_k}}} \cdot (2\pi \lambda_k)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{|x|^2}{\lambda_k}\right) dx$$

$$= \left[\frac{4 \gamma_k \lambda_k}{(\gamma_k + \lambda_k)^2} \right]^{\frac{1}{4}} = \left[\frac{4 \xi_k}{(1 + \xi_k)^2} \right]^{\frac{1}{4}}$$

$(\mathcal{H}, B(\mathcal{H})) \quad \mu = N_Q, \quad \nu = N_{a, Q}$ Gaussian m.
 $a \in \mathcal{H}, \quad Q \in L_1^+(\mathcal{H})$

Thm. (i) $a \notin Q^{1/2}(\mathcal{H}) \Rightarrow \mu \perp \nu$

(ii) $a \in Q^{1/2}(\mathcal{H}) \Rightarrow \mu \cong \nu$

(iii) (ii) $a \in \mathcal{H}$

$$\frac{d\mu}{d\nu} = \exp \left\{ -\frac{1}{2} \| Q^{-1/2} a \|^2 + W_{Q^{-1/2} a}(x) \right\}$$

$$W_{Q^{-1/2} a}(x) = (Q^{1/2} a, x)$$

$$\textcircled{ii} \quad (i) \quad H(\mu, \nu) = \prod_k H(\mu_k, \nu_k) = \prod_k e^{-\frac{a_k^2}{8\lambda_k}}$$

$$\mu_k = N_{\lambda_k}, \quad \nu_k = N_{a_k, \lambda_k} = e^{-\frac{1}{8} \frac{(Q^{-1/2} a, a_k)^2}{\lambda_k}} = e^{-\frac{1}{8} \| Q^{-1/2} a \|^2} = 0.$$

\textcircled{iii} $N_{a, Q}$ の def 1-711?

$$\mathbb{R}^\infty \ni \mathbb{1} = \prod_{k=1}^{\infty} N_{\lambda_k} \quad \text{1-712 def 1-712}$$

$$\mathbb{R}^\infty \supset \ell^2 \quad \text{1-712} \quad \prod_{k=1}^{\infty} N_{\lambda_k} \uparrow \ell^2$$

(ii) $H(\mu, \nu) < \infty \quad \therefore \mu \cong \nu$

(iii) 2-512 Kakutani の 定理 1-711

$$H(n_r)^4 = \prod \frac{4\xi_k}{(1+\xi_k)^2} = \prod \frac{(1+\xi_k)^2 - (1-\xi_k)^2}{(1+\xi_k)^2}$$

$$= \prod \left(1 - \frac{(1-\xi_k)^2}{(1+\xi_k)^2} \right) > 0$$

$$\log \prod \left(1 - \frac{(1-\xi_k)^2}{(1+\xi_k)^2} \right) = \sum \log \left(1 - \frac{(1-\xi_k)^2}{(1+\xi_k)^2} \right) > -\infty$$

$$\prod (1-a_n) > 0 \Leftrightarrow \sum a_n < \infty$$

$$\sum \log(1-a_n) > -\infty \Leftrightarrow -\sum \log(1-a_n) < \infty$$

$$\Leftrightarrow \sum \frac{(1-\xi_n)^2}{(1+\xi_n)^2} < \infty$$

Remarks: $R = a \mathbb{Q} \ a \in \mathbb{Z}$

$$r_k = a \lambda_k \quad \therefore \sum \frac{\lambda_k^2 (1-a)^2}{\lambda_k^2 (1+a)^2} = \infty$$

$\mathbb{E} \Leftrightarrow a \neq 1$

$$\therefore N_{a\mathbb{Q}} \perp N_{\mathbb{Q}} \quad (a \neq 1)$$

$$R = a\mathbb{Q}$$

$$\mathbb{Q} \ a \in \mathbb{Z}$$

(9)

2016/6/24 (金)

先を問の答に

$$\int \sqrt{\frac{d\mu_1 \times \mu_2}{dn} \cdot \frac{dv_1 \times v_2}{dn}} d\eta = H(\mu_1 \times \mu_2, v_1 \times v_2)$$

$$\exists \frac{dn}{d\xi_1 \times \xi_2} \quad \therefore d\eta = \frac{dn}{d\xi_1 \times \xi_2} \cdot d\xi_1 \times \xi_2 \quad \text{絶対連続性}$$

d7

$$\begin{aligned} H(\mu_1 \times \mu_2, v_1 \times v_2) &= \int \sqrt{\frac{d\cdot}{dn} \cdot \frac{d\cdot}{dn}} \frac{dn}{d\xi_1 \times \xi_2} d\xi_1 \times \xi_2 \\ &= \int \sqrt{\frac{d\cdot}{dn} \frac{dn}{d\xi_1 \times \xi_2} \cdot \frac{d\cdot}{dn} \frac{dn}{d\xi_1 \times \xi_2}} d\xi_1 \times \xi_2 \end{aligned}$$

where $\frac{d\cdot}{dn} \frac{dn}{d\xi_1 \times \xi_2} \quad \mu_1 \times \mu_2 \ll \xi_1 \times \xi_2$

$$\Downarrow$$

$$\exists \frac{d\mu_1 \times \mu_2}{d\xi_1 \times \xi_2}$$

 ~~$\mu_1 \times \mu_2$~~

$$\text{— 仮定 } \mu_1 \ll \mu_2 \ll \mu_3$$

$$\frac{d\mu_1}{d\mu_2} \frac{d\mu_2}{d\mu_3} = \frac{d\mu_1}{d\mu_3} \quad \therefore \mu_1(A) = \int_A \frac{d\mu_1}{d\mu_2} d\mu_2$$

$$= \int \frac{d\mu_1}{d\mu_2} \mathbb{1}_A d\mu_2$$

$$= \int \frac{d\mu_1}{d\mu_2} \mathbb{1}_A \frac{d\mu_2}{d\mu_3} d\mu_3$$

$$H(\mu_1 \times \mu_2, \nu_1 \times \nu_2) = H(\mu_1, \nu_1) \cdot H(\mu_2, \nu_2)$$

$$\text{一般: } H(\mu_1 \times \dots \times \mu_n, \nu_1 \times \dots \times \nu_n) = \prod_{k=1}^n H(\mu_k, \nu_k) \quad \text{if OK}$$

$$\mu = \prod_{n=1}^{\infty} \mu_n, \quad \nu = \prod_{n=1}^{\infty} \nu_n \quad \text{or } \bar{z}$$

$$H(\mu, \nu) = \prod_{k=1}^{\infty} H(\mu_k, \nu_k) \quad (\mu_n \sim \nu_n)$$

$$\text{☺ } \psi_n(x) = \psi_n(x_1, x_2, \dots) = \prod_{k=1}^n \sqrt{\frac{d\nu_k}{d\mu_k}}(x_k) \quad \underline{n \leq m}$$

$$\|\psi_n(x) - \psi_m(x)\|^2 = \int_{\mathbb{R}^{\infty}} \left(\frac{n}{\pi} - \frac{m}{\pi} \right)^2 \mu(dx)$$

$$= \int_{\mathbb{R}^{\infty}} \frac{n}{\pi} \left(1 - \frac{m}{n} \right)^2 = \int \left(1 - 2\frac{m}{n} + \left(\frac{m}{n}\right)^2 \right)$$

$$= 2 \left(1 - \frac{m}{n} \int \sqrt{\frac{d\nu_k}{d\mu_k}} \mu_k(dx_k) \right)$$

$$= 2 \left(1 - \prod_{k=1}^m H(\mu_k, \nu_k) \right)$$

$$\textcircled{1} \quad \prod_{k=1}^{\infty} H(\mu_k, \nu_k) > 0 \quad \text{or } \omega \neq \emptyset$$

$$\Leftrightarrow \sum_{k=1}^{\infty} \log H(\mu_k, \nu_k) > -\infty$$

$$\sum_{k=1}^{\infty} -\log H(\mu_k, \nu_k) < +\infty$$

$\therefore \{ \sum_{k=1}^n -\log H(\mu_k, \nu_k) \}$ is bounded

$\therefore -\log H(\mu_k, \nu_k)$ is Cauchy sum

$$\therefore \lim_{n \rightarrow \infty} \prod_{k=1}^n H(\mu_k, \nu_k) = 1$$

i.e., $\{\psi_n\}$ is Cauchy sequence

$$\therefore \exists \psi \in L^2(\mathbb{R}^d) \text{ s.t. } \|\psi_n - \psi\|_{L^2} \rightarrow 0$$

$$\frac{d\nu}{d\mu} = (\psi(x))^2 \quad \text{E.T.C.} \dots$$

$$\left[\int_{\mathbb{R}^d} |\psi_m^2 - \psi_n^2|^2 \right]^{\frac{1}{2}} \leq \int_{\mathbb{R}^d} \underbrace{|\psi_m + \psi_n|^2}_{\leq 4} |\psi_m - \psi_n|^2$$

$$\leq 4 \int_{\mathbb{R}^d} |\psi_m - \psi_n|^2$$

$$= 8 \left(1 - \prod_{k=n+1}^m H(\mu_k, \nu_k) \right)$$

ψ_m^2 is $L^1(\mathbb{R}^d)$ Cauchy

$$B \in \mathcal{B}(\mathbb{R}^n) \quad \chi_n(x) = \mathbb{1}_B(p_n x)$$

$$\text{where } p_n x = (x_1 \dots x_n 0 \dots)$$

$$\begin{aligned} \int_{\mathbb{R}^{2n}} \chi_n(x) \nu(dx) &= \int \chi_n(x_1 \dots x_n 0 \dots 0) \nu_1(dx_1) \dots \nu_n(dx_n) \\ &= \int_{\mathbb{R}^{2n}} \chi_n(x_1 \dots x_n 0 \dots 0) \frac{d\nu_1}{d\mu_1} \dots \frac{d\nu_n}{d\mu_n} \prod_{i=1}^n d\mu_i \\ &= \int_{\mathbb{R}^{2n}} \chi_n(x_1 \dots x_n 0 \dots 0) \psi_n(x)^2 \prod_{i=1}^n d\mu_i \mu(dx) \end{aligned}$$

$$\lim_n \chi_n(x) = \mathbb{1}_B(x) \quad \text{a.s.}$$

$$\nu(B) = \lim_n \int_{\mathbb{R}^{2n}} \chi_n(x) \psi_n(x)^2 \mu(dx) \quad \mathbb{1}_B(x)$$

Lebesgue 収束定理が使えないから

$$\underbrace{\int \mathbb{1}_B |\varphi_n(x)|^2}_{\text{Lebesgue}} - \int \chi_n |\varphi(x)|^2 + \underbrace{\int \chi_n |\varphi(x)|^2 - \int_n \chi_n |\varphi_n(x)|^2}_{L^1 \text{ 収束}} = 0$$

$$\rightarrow \int \mathbb{1}_B |\varphi(x)|^2 d\mu \quad \therefore \nu \ll \mu \quad \text{a.s.}$$

$$\frac{d\nu}{d\mu} = |\varphi(x)|^2$$

$$H(\mu \nu) = \int \sqrt{\frac{d\mu}{d\nu}} d\mu = \int \psi(x) \mu(dx)$$

$$= \lim_n \int_{\mathbb{R}^n} \psi_n(x) \mu(dx)$$

$$= \lim_n \prod_{k=1}^n \int \sqrt{\frac{d\nu_k}{d\mu_k}} \mu_k(dx_k)$$

$$= \lim_n \prod_{k=1}^n H(\mu_k \nu_k) \quad //$$

② $\prod_{k=1}^n H(\mu_k \nu_k) = 0$ $\alpha \in \mathbb{R}^n / \sqrt{2}$ ~~$\prod_{k=1}^n H(\mu_k \nu_k) < \epsilon$~~

$\downarrow B_n = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \psi_n(x_1, \dots, x_n, 0, \dots, 0) = \prod_{k=1}^n \frac{d\nu_k}{d\mu_k}(x_k) > 1 \}$

$$\begin{aligned} \left(\prod_{k=1}^n \mu_k \right) (B_n) &= \int_{B_n} \prod_{k=1}^n \mu_k(dx_k) < \int_{B_n} \psi_n(x_1, \dots, x_n, 0, \dots, 0) \prod_{k=1}^n \mu_k(dx_k) \\ &= \int_{B_n} \prod_{k=1}^n \sqrt{\frac{d\nu_k}{d\mu_k}} \prod_{k=1}^n \mu_k(dx_k) = \prod_{k=1}^n H(\mu_k \nu_k) < \epsilon \end{aligned}$$

\downarrow 单调

(n+1) \times \dots \times \dots \times \dots

$$\prod_{k=1}^n \mu_k(\mathbb{R}^n \setminus B_n) = \int_{\mathbb{R}^n \setminus B_n} \prod_{k=1}^n \mu_k(dx_k) \leq \int_{\mathbb{R}^n \setminus B_n} \psi_n(x_1, \dots) \prod_{k=1}^n \mu_k(dx_k)$$

$$\mathbb{R}^n \setminus B_n \subseteq \{x \mid \prod_{k=1}^n \frac{d\nu_k}{d\mu_k} \leq 1\}$$

$$\prod_{k=1}^n \frac{d\nu_k}{d\mu_k} \cdot \prod_{k=1}^n \frac{d\mu_k}{d\nu_k} = 1 \quad \& \cdot \quad \prod_{k=1}^n \frac{d\mu_k}{d\nu_k} > 1$$

$$\therefore \left(\prod_{k=1}^n \mu_k \right) (\mathbb{R}^n \setminus B_n) < \epsilon \quad \text{证毕}$$

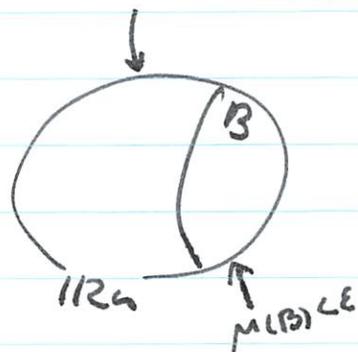
$$\tilde{B}_n = B_n \times \prod_{k=n+1}^{\infty} \mathbb{R}$$

と可測性

$$v(\mathbb{R}^{\infty} \setminus B) < \varepsilon$$

$$\mu(\tilde{B}_n) < \varepsilon$$

$$v(\mathbb{R}^{\infty} \setminus \tilde{B}_n) < \varepsilon$$



$$B = \bigcap_{n=1}^{\infty} \tilde{B}_n$$

$\mu(B) < \varepsilon, \quad v(\mathbb{R}^{\infty} \setminus B) < \varepsilon \Rightarrow \mu \perp v$ (これは下を見よ)

$\therefore \mu(B) = 0, \quad v(\mathbb{R}^{\infty} \setminus B) = 0 \quad \mu \perp v.$

$$H(\mu, \nu) = \int_B \sqrt{\frac{d\mu}{ds} \frac{d\nu}{ds}} + \int_{\mathbb{R}^{\infty} \setminus B} \sqrt{\frac{d\mu}{ds} \frac{d\nu}{ds}}$$

$$\leq \left(\int_B \frac{d\mu}{d\eta} d\eta \right)^{1/2} \left(\int_B \frac{d\nu}{d\eta} d\eta \right)^{1/2} + \left(\int_{\mathbb{R}^{\infty} \setminus B} \frac{d\nu}{d\eta} d\eta \right)^{1/2} \left(\int_{\mathbb{R}^{\infty} \setminus B} \frac{d\mu}{d\eta} d\eta \right)^{1/2}$$

$$= [\mu(B) \nu(B)]^{1/2} + [\mu(\mathbb{R}^{\infty} \setminus B) \nu(\mathbb{R}^{\infty} \setminus B)]^{1/2} = 0$$

$$\therefore H(\mu, \nu) = \prod_n H(\mu_n, \nu_n) = 0 \quad \square$$

(*)

$$\exists B_n \text{ st } \mu(B_n) < \frac{\varepsilon}{2^n} \quad v(\mathbb{R}^{\infty} \setminus B_n) < \frac{\varepsilon}{2^n}$$

$$\bigcap_n B_n = C \quad \mu(C) < \frac{\varepsilon}{2^n} \quad \forall n \quad \therefore \mu(C) = 0$$

$$v(\mathbb{R}^{\infty} \setminus C) \leq \sum \frac{\varepsilon}{2^n} = \varepsilon$$

$$\therefore \mu(C) = 0 \quad v(\mathbb{R}^{\infty} \setminus C) < \varepsilon$$

$$C = C_n \text{ と } C_n^c$$

$$\mu(C_n^c) = 0$$

$$v(\mathbb{R}^{\infty} \setminus C_n^c) < \frac{\varepsilon}{n}$$

$$\bigcup_n C_n^c = D \quad \left. \begin{array}{l} \mu(D) = 0 \\ v(\mathbb{R}^{\infty} \setminus D) = 0 \end{array} \right\}$$

§4 Borel measures on \mathcal{H} .

$$(\mathcal{H}, \mathcal{B}(\mathcal{H})) \quad \hat{\mu}(x) = \int_{\mathcal{H}} e^{i\langle x, y \rangle} \mu(dy)$$

- Lemma 4.1
- ① $\hat{\mu}(0) = \mu(\mathcal{H})$
 - ② $\hat{\mu}(\cdot)$ is cont on \mathcal{H}
 - ③ $\hat{\mu}$ is positive definite

$$\sum_{l, k} \hat{\mu}(x_l - x_k) z_l \bar{z}_k \geq 0$$

$\forall n, \forall z_l \in \mathbb{C}$

∴ ① ② ok ③

$$\sum \hat{\mu}(x_l - x_k) z_l \bar{z}_k = \sum \int_{\mathcal{H}} e^{i\langle x_l, y \rangle - i\langle x_k, y \rangle} z_l \bar{z}_k \mu(dy)$$

$$= \sum \int \begin{pmatrix} e^{i\langle x_l, y \rangle} \\ z_l \end{pmatrix} \begin{pmatrix} e^{i\langle x_k, y \rangle} \\ z_k \end{pmatrix}$$

$$= \begin{pmatrix} \sum e^{i\langle x_l, y \rangle} & \sum e^{i\langle x_k, y \rangle} \\ z_l & z_k \end{pmatrix} \geq 0$$

Natural question ①②③ $\Rightarrow \hat{\mu}$?

有限次元 $\Rightarrow z_l \bar{z}_k \geq 0 \Rightarrow$ Bochner's \mathbb{R}^n

例 11 $\exp\left(-\frac{1}{2}\|x\|_2^2\right)$ ① ② ③ \mathcal{E} かつ \mathcal{E}

例 12 $\int_{\mathcal{X}} e^{i\langle x, y \rangle} \mu(dy) \neq e^{-\frac{1}{2}\|x\|_2^2}$ $\mathcal{E} \rightarrow \mathcal{E}$

Lemma 4.2 Let $\phi: \mathcal{X} \rightarrow \mathbb{C}$ positive definite

Then

(1) $|\phi(x)| \leq |\phi(0)|$ $\overline{\phi(x)} = \phi(-x)$

(2) $|\phi(x) - \phi(y)| \leq 2\sqrt{\phi(0)}\sqrt{\phi(0) - \phi(x-y)}$

(3) $|\phi(0) - \phi(x)| \leq 2\sqrt{\phi(0)(\phi(0) - \operatorname{Re}(\phi(x)))}$

③ $A = \begin{pmatrix} \phi(0) & \phi(x) \\ \phi(-x) & \phi(0) \end{pmatrix}$

Lemma 4.3 \mathcal{X} は \mathbb{R}^d 値

① $\int_{\mathcal{X}} \|x\|^2 \mu(dx) < \infty$

② $\exists Q \in L_+^1(\mathcal{X})$ s.t. $(Qx, y) = \int (x, z)(y, z) \mu(dz)$

If ② holds. $\operatorname{Tr} Q = \int \|x\|^2 \mu(dx)$

let
① holds

$\therefore \{e_n\}$ CONS

$$\int \|x\|^2 \mu(dm) = \sum |(x, e_n)|^2 \mu(dm)$$

$$\sum (x, e_n)(x, e_n) = \sum (Q e_n, e_n) < \infty$$

let ② hold.

$$\left| \int_{\mathcal{X}} (x, z)(y, z) \mu(dz) \right| \leq \|x\| \cdot \|y\| \int \|z\|^2 \mu(dz) < \infty$$

$$= (Qx, y) \quad \text{Riesz rep. \#7}$$

$$(Qx, y) = \int (x, z)(y, z) \mu(dz) \quad \text{a.s.}$$

$Q \geq 0$, symmetric (a.s.)

$$\text{Tr } Q = \int \|x\|^2 \mu(dm) < \infty.$$

$\therefore Q$ is trace class.

①

⑩

⑦/①

Lemma 4.7] ϕ positive definite functional on \mathcal{D} .

$$(1) |\phi(x)| \leq \phi(0) \quad \overline{\phi(x)} = \phi(-x)$$

$$(2) |\phi(x) - \phi(y)| \leq 2\sqrt{\phi(0)} \sqrt{\phi(0) - \phi(x-y)}$$

$$(3) |\phi(0) - \phi(x)| \leq \sqrt{2\phi(0)(\phi(0) - \operatorname{Re}(\phi(x)))}$$

Upper bound
 Continuity
 $\phi(0) = \phi(0)$
 $\phi(x) = \phi(x)$
 $\phi(y) = \phi(y)$
 (3)

$$\odot \quad A = \begin{pmatrix} \phi(0) & \phi(x) \\ \phi(-x) & \phi(0) \end{pmatrix}$$

$$B = \begin{pmatrix} \phi(0) & \phi(x) & \phi(y) \\ \phi(-x) & \phi(0) & \phi(y-x) \\ \phi(-y) & \phi(x-y) & \phi(0) \end{pmatrix}$$

$$\begin{aligned} (z, Az) &= (\bar{z}_1, \bar{z}_2) \cdot \begin{pmatrix} z_1 \phi(0) + z_2 \phi(x) \\ z_1 \phi(-x) + z_2 \phi(0) \end{pmatrix} \\ &= [\bar{z}_1 z_1 \phi(0) + \bar{z}_1 z_2 \phi(x)] + [\bar{z}_1 z_2 \phi(-x) + \bar{z}_2 z_2 \phi(0)] \\ &\geq 0 \end{aligned}$$

$$\forall z \in \mathcal{D} \quad A \text{ is s.g.} \quad \therefore \overline{\phi(x)} = \phi(-x) \quad \therefore (1)$$

$$\exists z \quad \det A = \phi(0)^2 - |\phi(x)|^2 \geq 0$$

$$\begin{aligned} \exists z \quad \det B &= \phi(0)^3 - \phi(0) |\phi(x-y)|^2 \\ &\quad - \phi(0) |\phi(x) - \phi(y)|^2 + \\ &\quad 2 \operatorname{Re} [\phi(y) \overline{\phi(x)} (\phi(x-y) - \phi(0))] \geq 0 \end{aligned}$$

$$\begin{aligned} \therefore 0 \leq \det B &\leq 2\phi(0)^2 |\phi(0) - \phi(x-y)| \\ &\quad - \phi(0) |\phi(x) - \phi(y)|^2 \\ &\quad + 2 \operatorname{Re} [\phi(y) \overline{\phi(x)} (\phi(x-y) - \phi(0))] \end{aligned}$$

$$\therefore \quad a^3 - a|b|^2 \leq 2a^2|a-b| \quad \text{if } |b| < a$$

$$\frac{a^2 + |b|^2}{(a - |b|)^2} \leq \frac{2a}{|a - b|}$$

$$0 \leq 4\phi(0)^2 |\phi(0) - \phi(x-y)| - \phi(0) |\phi(x) - \phi(y)|^2 \quad \therefore (2)$$

②

$$|\phi(0) - \phi(x)|^2 = \phi(0)^2 - 2 \operatorname{Re}(\phi(0)\phi(x)) + |\phi(x)|^2$$

real
complex
↓
↓

$$\leq 2\phi(0)^2 - 2 \operatorname{Re}(\phi(0)\phi(x))$$

$$\therefore |\phi(0) - \phi(x)| \leq \sqrt{2\phi(0) (\phi(0) - \operatorname{Re}\phi(x))} \quad "$$

Lemma 4.3

Let μ be a finite Borel meas. on \mathcal{X} .

Then

$$\int_{\mathcal{X}} \|x\|^2 \mu(dx) < \infty \Leftrightarrow \exists Q \in L_+^1(\mathcal{X}) \text{ s.t.}$$

$$(Qx, y) = \int (xz)(yz) \mu(dz)$$

⊙ $(Qe_n, e_n) = \int \|ze_n\|^2 \mu(dz)$

$$\cdot \sum (Qe_n, e_n) = \sum \int (\cancel{x}e_n)(\cancel{y}e_n) \mu(dz)$$

$$\infty > \operatorname{Tr} Q$$

monotone conv. thm $\int \|z\|^2 \mu(dz) < \infty$

$$\Leftrightarrow \int_{\mathcal{X}} |(xz)(yz)| \mu(dz) \leq \|x\| \cdot \|y\| \int_{\mathcal{X}} \|z\|^2 \mu(dz)$$

Riesz rep thm $\exists Q \in L(\mathcal{X})$

$$\int (xz)(yz) \mu(dz) = (Qx, y)$$

$$\sum_n (Qe_n, e_n) = \sum \int (e_n z)(e_n z) \mu(dz)$$

$$= \int \sum_n |e_n z|^2 \mu(dz) \uparrow \int \|z\|^2 \mu(dz)$$

$$\therefore Q \in L_+^1(\mathcal{X})$$

3

Kolmogorov Extension thm.

$$(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu_n) \quad \Lambda \subset \mathbb{N}$$

* $\Lambda = \{i_1, \dots, i_n\} \quad \text{or } \mathbb{Z} \quad \mathbb{R}^n \cong \mathbb{R}^m \quad \text{or } \mathbb{Z}$
 consistency condition

$$\mu_{i_1, \dots, i_n, i_{n+1}, \dots, i_{n+m}}(A_1 \times \dots \times A_n \times \mathbb{R} \times \dots \times \mathbb{R})$$

$$= \mu_{i_1, \dots, i_n}(A_1 \times \dots \times A_n)$$

$$\mathcal{A} = \{ \pi_\Lambda^{-1}(E) \subset \mathbb{R}^\infty \mid \Lambda \subset \mathbb{N}, \# \Lambda < \infty, E \in \mathcal{B}(\mathbb{R}^n) \}$$

$$\text{or } \pi_\Lambda: \mathbb{R}^\infty \rightarrow \mathbb{R}^n \quad \text{projection}$$

$$\text{or } \exists (\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mu) \quad \text{s.t.}$$

$$\mu(\pi_\Lambda^{-1}(E)) = \mu_\Lambda(E)$$

$$\forall \mathbb{Z} \quad \mu(\pi_{i_1, \dots, i_n}^{-1}(E)) = \mu_{i_1, \dots, i_n}(E)$$

Bochner's thm - $\varphi: \mathbb{R}^n \rightarrow \mathbb{C}$

- ① cont
- ② $\varphi(0) = 1$
- ③ positive definite

$\Leftrightarrow \exists \mu$ prob meas on \mathbb{R}^n
 s.t.

$$\varphi(x) = \int_{\mathbb{R}^n} e^{-i(x \cdot y)} d\mu(y)$$

$\mathbb{R}^n \rightarrow$ locally cpt abelian group G

$$\varphi(g) = \int_{\hat{G}} \xi(g) d\mu(\xi) \quad | \quad \hat{\mathbb{Z}}^n = [-\pi, \pi]^n \text{ etc.}$$

4

Thm (Mincos-Sazanov thm.)

\mathcal{H} sep. real Hilbert space

$\phi: \mathcal{H} \rightarrow \mathbb{C}$ positive def functional
(1) \sim (3) \Leftrightarrow (2)

(1) $\exists \mu$ finite Borel meas

$$\text{s.t. } \phi(x) = \int_{\mathcal{H}} e^{i(x,z)} \mu(dz)$$

(2) For every $\varepsilon > 0 \exists Q_\varepsilon \in L^1_+(\mathcal{H})$ s.t.

$$(Q_\varepsilon x, x) < 1 \Rightarrow \operatorname{Re}(\phi(\omega) - \phi(x)) < \varepsilon$$

(3) $\exists Q \in L^1_+(\mathcal{H})$ s.t. ϕ is cont w.r.t $\|\cdot\|_Q$

$$\text{where } \|x\|_Q = \sqrt{(Qx, x)}$$

\therefore (1) \Leftrightarrow (2)

$$\operatorname{Re}(\phi(\omega) - \phi(x)) = \operatorname{Re} \int_{\mathcal{H}} (1 - e^{i(x,z)}) \mu(dz) = \int_{\mathcal{H}} (1 - \cos(xz)) \mu(dz)$$

$$\leq \frac{1}{2} \int_{\mathcal{H}} |xz|^2 \mu(dz) = \frac{1}{2} \int_{|z| \leq \delta} |xz|^2 \mu(dz) + \frac{1}{2} \int_{|z| > \delta} |xz|^2 \mu(dz)$$

$$= \frac{1}{2} \int_{|z| \leq \delta} |xz|^2 \mu(dz) + \frac{1}{2} \int_{|z| > \delta} \|x\|^2 \|z\|^2 \mu(dz)$$

$$= \int_{|z| \leq \delta} |xz|^2 \mu(dz) + \int_{|z| > \delta} \|x\|^2 \|z\|^2 \mu(dz)$$

$$\leq \frac{1}{2} \int_{|z| \leq \delta} |xz|^2 \mu(dz) + 2 \int_{|z| > \delta} \mu(dz)$$

Minlos - Sazonov thm.

(1) \rightarrow (2)

$$\operatorname{Re}(\phi(0) - \phi(x)) = \int 1 - \cos(x \cdot z) \mu(dz)$$

$$= \int_{|z| \leq \gamma} \dots + \int_{|z| > \gamma} \dots$$

$$\leq \frac{1}{2} \int_{|z| \leq \gamma} |(x, z)|^2 \mu(dz) + 2 \mu(|z| > \gamma)$$

① $\mu_1(A) = \mu_1(A \cap \{|z| \leq \gamma\})$ by the \mathbb{R}^n Lemma
 apply $\exists \epsilon$

$$\exists B_\gamma \in L_+^2(\mathcal{H}) \text{ s.t.}$$

$$(B_\gamma z_1, z_2) = \int_{|z| \leq \gamma} (z, z_1)(z, z_2) \mu(dz)$$

$$Q_\epsilon = \frac{1}{\epsilon} B_\gamma \text{ s.t. } \epsilon > 0.$$

② $\gamma \gg 1$ s.t. $\mu(|z| > \gamma) < \frac{\epsilon}{4}$ s.t. \exists .

$$\therefore \operatorname{Re}(\phi(0) - \phi(x)) \leq \frac{\epsilon}{2} (Q_\epsilon x, x) + \frac{\epsilon}{2}$$

(2) \rightarrow (1)

$$\left(\overset{\exists}{Q_\varepsilon} x, x \right) < 1 \Leftrightarrow \operatorname{Re}(\phi(0) - \phi(x)) < \varepsilon$$

$$\therefore \left| \left(Q_\varepsilon x, x \right) \right| \leq \|Q_\varepsilon\| \cdot \|x\|^2 < 1 \quad \text{と } \varepsilon > 0 \text{ 任意}$$

$$\|x\| \leq \frac{1}{\sqrt{\|Q_\varepsilon\|}} \quad \text{かつ } \operatorname{Re}(\phi(0) - \phi(x)) < \varepsilon$$

$\therefore \phi(x) \leftarrow x$ 連続.

(e_n) $\subset \mathcal{H}$ CONS

$$a) f_{i_1 \dots i_n}(\omega_1 \dots \omega_n) = \phi\left(\sum_{j=1}^n \omega_j e_{ij}\right) \text{ は}$$

positive def on \mathbb{R}^n

• Bochner's thm \mathbb{R}^1

$$f_{i_1 \dots i_n} = \hat{\mu}_{i_1 \dots i_n} = \int e^{i(\tilde{\omega} \cdot \tilde{z})} \mu_{i_1 \dots i_n}(d\tilde{z})$$

$\{ \mu_{i_1 \dots i_n} \}$ は consistency 条件を満たす.

$\therefore \exists \gamma$ on $(\mathbb{R}^{\infty}, \mathcal{B}(\mathbb{R}^{\infty})) \rightarrow \tau.$

$$\mu_{i_1 \dots i_n}(E) = \gamma\left(\pi_{i_1 \dots i_n}^{-1}(E)\right) \quad \forall E \in \mathcal{B}(\mathbb{R}^n)$$

$:= \tau$

$\pi_{i_1 \dots i_n} : \mathbb{R}^{\infty} \rightarrow \mathbb{R}^n$ projection

$$(w_1 \dots) \mapsto (w_{i_1} \dots w_{i_n})$$

$$X_j = \pi_j : \mathbb{R}^{\omega} \rightarrow \mathbb{R} \quad \text{and} \quad \sum_{k=1}^{\infty} X_k(\omega)^2 < \infty$$

$$X(\omega) = \sum_{j=1}^{\infty} X_j(\omega) e_j \in \mathcal{H} \quad \text{and} \quad \text{etc}$$

$$X : \mathbb{R}^{\omega} \rightarrow \mathcal{H} \quad \therefore \mu = X_{\#} \delta$$

$$\text{令 } \nu = \sum_{j=1}^n \omega_j e_j \quad \text{and} \quad \omega = (\omega_j) = ((x_j)) \in \mathbb{R}^{\omega}$$

次の計算をする:

$$\begin{aligned} \hat{\mu} \left(\sum_{j=1}^n \omega_j e_j \right) &= \int_{\mathbb{R}^{\omega}} e^{i \left(\sum_{j=1}^n \omega_j e_j, X(\omega) \right)} \delta(d\omega) \\ &= \int_{\mathbb{R}^{\omega}} e^{i \sum_{j=1}^n \omega_j X_j(\omega)} \delta(d\omega) \\ &= \int_{\mathbb{R}^n} e^{i \sum_{j=1}^n \omega_j z_j} \mathcal{M}_{1 \dots n}(dz) = f_{1 \dots n}(\omega_1, \dots, \omega_n) \\ &= \phi \left(\sum_{j=1}^n \omega_j e_j \right) \end{aligned}$$

$\therefore \mu = \nu$

$$\hat{\mu}(x) = \phi(x) \quad //$$

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\therefore of $\sum X_{k2}^2 < \infty$ γ -a.e.

$$\phi(0) - R_\epsilon(\phi)(x) \leq \epsilon + 2\phi(0) (Q_\epsilon x, x) \quad \text{Ⓢ}$$

$$|\phi(x)| \leq \phi(0)$$

(i) $(Q_\epsilon x, x) \geq 1$ \Rightarrow $\epsilon \geq 1$ Ⓢ \Rightarrow self

(ii) $(Q_\epsilon x, x) < 1$ \Rightarrow $\epsilon < 1$ Ⓢ \Rightarrow self

$$\frac{\phi(0) - R_\epsilon(\phi)(x)}{\phi(0)} = (Q_\epsilon x, x)$$

$$\therefore \phi(0) - \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} \sum_{j=1}^n X_{k2j}^2\right) \gamma(dw)$$

$$= \phi(0) - \int_{\mathbb{R}^n} \gamma(dw) \int_{\mathbb{R}^n} e^{i \sum_{j=1}^n y_j X_{k2j}} P_n(dy) \quad \text{Gauss.}$$

$$= \phi(0) - \int_{\mathbb{R}^n} P_n(dy) \int_{\mathbb{R}^n} e^{i \sum_{j=1}^n y_j X_{k2j}} \gamma(dw)$$

$$= \phi(0) - \int_{\mathbb{R}^n} P_n(dy) \phi\left(\sum_{j=1}^n y_j e_{k2j}\right)$$

$$= \int_{\mathbb{R}^n} \left[\phi(0) - R_\epsilon(\phi)(\dots) \right] P_n(dy)$$

$$\leq \epsilon + 2\phi(0) \int (Q_\epsilon(\dots), (\dots)) dP_n$$

⑨

$$= \varepsilon + 2\phi(0) \sum_{j=1}^n (Q_\varepsilon e_{k+j}, e_{k+j}) \int_{\mathbb{R}^n} y_j y_e \mathbb{P}_n(dy)$$

$$= \varepsilon + 2\phi(0) \sum_{j=1}^n (Q_\varepsilon e_{k+j} e_{k+j})$$

Hence

$$\therefore \phi(0) - \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} \sum_{j=1}^n X_{k+j}^2\right) \gamma(dw)$$

$$\leq \varepsilon + \phi(0) \sum_{j=k+1}^{\infty} (Q_\varepsilon e_j e_j)$$

$$\therefore \lim_{n \rightarrow \infty} \int \exp\left(-\frac{1}{2} \sum_{j=1}^n X_{k+j}^2\right) \gamma(dw) = \phi(0)$$

(or $\gamma(\mathbb{R}^n) \neq 0$)

$$\Rightarrow \int \left| \frac{1}{\phi(0)} - \exp\left(-\frac{1}{2} \sum_{j=1}^n X_{k+j}^2\right) \right| \gamma(dw) \Rightarrow 0$$

constant function 1 is $L^1(\mathbb{R}^n, \gamma)$ 42 葉 33

$$\therefore -\frac{1}{2} \sum_{j=1}^n X_{k+j}^2 \rightarrow 0$$

$$\Rightarrow \sum_{j=1}^n X_j^2 = a_n \quad \text{where } a_n \rightarrow 0$$

{ a_n } is ~~Cauchy~~ Cauchy series

$$\therefore \sum_{j=1}^{\infty} X_j^2 < \infty \quad \text{a.e. } \gamma$$

(12) 7/15

§ 5 Measures on Group

Def 5.1 G : group, \mathcal{B} ; σ -field

(G, \mathcal{B}) meas group \Leftrightarrow (1) $x \mapsto x^{-1}$
(2) $(x, y) \mapsto xy$
both measurable

Lemma 5.2

- ① ~~$E \in \mathcal{B} \Rightarrow E^{-1} = \{x^{-1} \mid x \in E\} \in \mathcal{B}$~~
- ② $E \in \mathcal{B} \Rightarrow E_y, yE \in \mathcal{B}$.

① $\varphi: x \mapsto x^{-1}$ meas $\forall x \in G$

$$\varphi^{-1}(E) \in \mathcal{B} \quad \forall E \in \mathcal{B}$$

~~$E^{-1} \in \mathcal{B}$~~ $E^{-1} \subset \varphi^{-1}(E)$ $\exists \exists \exists \exists \exists$

$$\forall x \in \varphi^{-1}(E) \text{ s.t. } \varphi(x) \in E \quad \therefore x = (x^{-1})^{-1} \in E^{-1}$$

$$E^{-1} \supset \varphi^{-1}(E)$$

$$\therefore E^{-1} = \varphi^{-1}(E) \quad \therefore E^{-1} \in \mathcal{B}$$

② $\Phi(x, y) = xy \in G$

$$\Phi^{-1}(E)(y) \in \mathcal{B} \quad \text{section at } y$$

$$\therefore x \in \Phi^{-1}(E)(y) \Leftrightarrow (x, y) \in \Phi^{-1}(E) \Leftrightarrow xy \in E$$

$$\Leftrightarrow x \in E y^{-1} \quad \therefore \Phi^{-1}(E)(y) = E y^{-1} \quad \therefore \Phi^{-1}(E)(y^{-1}) = E_y \in \mathcal{B}$$

$yE \in \mathcal{B} \neq$ 同い.

③ \mathcal{B} は 左, 右 不変, 逆で不変.

Def 5.3 (G, \mathcal{B}, μ) ① μ が右不変
 $\Leftrightarrow \mu(Ey) = \mu(E) \quad \forall E \in \mathcal{B}, \forall y \in G.$

② μ が左不変 $\Leftrightarrow \mu(yE) = \mu(E) \quad \forall E \in \mathcal{B}, \forall y \in G$

③ μ が右導不変 $\Leftrightarrow \mu(\cdot y) \cong \mu(\cdot)$

④ 左 $\Leftrightarrow \mu(y \cdot) \cong \mu(\cdot)$

Ex $\mu(E) = \begin{cases} = \infty & (\#E = \infty) \\ = \#E & \end{cases} \quad \checkmark \sigma\text{-finite} \\ \text{counting meas.}$

これは右不変測度

Ex λ : Lebesgue meas
 $\lambda(E+a) = \lambda(E)$

これは右不変測度 記号 $R_y \mu(\cdot) = \mu(\cdot y)$
 $L_y \mu(\cdot) = \mu(y \cdot)$

Thm 5.5 (G, \mathcal{B}) σ -finite meas. Group.

\exists 右導不変測度 μ かつ一意性 μ 左導不変測度 ν にもなる

① $\checkmark \checkmark \mu(E) = \mu(E^{-1}) \quad \forall E \in \mathcal{B}$

u
v
a
Lemma

$\mu \sim \nu \Rightarrow \checkmark \checkmark \mu \sim \checkmark \checkmark \nu$ i) $\mu(E) = 0 \Leftrightarrow \nu(E) = 0$
 $\checkmark \checkmark \mu(Ey) = \mu((Ey)^{-1}) = \mu(y^{-1}E^{-1}) \quad \checkmark \checkmark \nu(E) = 0 \Leftrightarrow \nu(E^{-1}) = 0 \Rightarrow \mu(E^{-1}) = 0 = \checkmark \checkmark \mu(E) = 0$

~~$R_y \mu$~~ $R_y \checkmark \mu = (L_{y^{-1}} \checkmark \mu)$

μ が左導不変 $\checkmark \mu$ は右導不変

$\mu \sim L_y \mu \Rightarrow R_y \checkmark \mu \sim \checkmark \mu$

$[\checkmark \mu] := [\checkmark \mu]$ well-def

~~scribbles~~

↓ σ -finite & 仮定終了

Lemma 5.4 μ, ν 右準不変測度 とある
 $= \sigma$ & $\tilde{\mu} \sim \nu$

☺ $\Psi : (x, y) \mapsto xy^{-1}$ measurable

$$x \in \Psi^{-1}(E) \Rightarrow \Psi^{-1}(E)(y) = Ey$$

$$\therefore x \in \Psi^{-1}(E)(y) \Leftrightarrow (x, y) \in \Psi^{-1}(E) \Leftrightarrow xy^{-1} \in E \\ \Leftrightarrow x \in Ey$$

○ 同不変性より $(x) \Psi^{-1}(E) = E^{-1}x$ とある

$$\text{以上より} \quad \begin{aligned} \Psi^{-1}(E)(y) &= Ey \\ (x) \Psi^{-1}(E) &= E^{-1}x \end{aligned}$$

$$I = \int_G \nu \times \mu (\Psi^{-1}(E))$$

$$I = \int_G \nu(\Psi^{-1}(E)(y)) d\mu(y) = \int_G \nu(Ey) d\mu(y)$$

$$= \int_G \mu((x) \Psi^{-1}(E)) d\nu(x) = \int_G \mu(E^{-1}x) d\nu(x)$$

$$\nu(\circ y) \sim \nu(\circ)$$

$$I = 0 \Leftrightarrow \nu(Ey) = 0 \Leftrightarrow \nu(E) = 0$$

$$I = 0 \Leftrightarrow \mu(E^{-1}x) = 0 \Leftrightarrow \mu(E^{-1}) = 0 \Leftrightarrow \tilde{\mu}(E) = 0$$

$$\therefore \nu \sim \tilde{\mu}$$

変数変換: の公式'

$$\int_{E_y} f(y) d\mu(y) = \int_E f(\varphi(x)) dR_{y,\mu}(x)$$

☺ $y = \varphi(x)$ とおく

$$\int_E f(\varphi(x)) dR_{y,\mu}(x)$$

$$= \int_E f(\varphi(x)) \nu(dx)$$

$$= \int_{\varphi(G)} f(z) \varphi'_\nu(dz)$$

$$\varphi'_\nu(A) = \nu(\varphi^{-1}(A)) = R_{y,\mu}(\varphi^{-1}(A))$$

$$= R_{y,\mu}(A \varphi^{-1}) = \mu(A) =$$

$$\int_{E_y} f(z) \mu(dz)$$

Thm 5.7 (G, \mathcal{B}) 右不変 μ が有限 ($\mu(G) < \infty$) ならば
 μ は左不変でもある. i.e. 両側不変

☺ $\mu = \check{\mu}$ ^{左不変性} 証明. 十分

$$\bar{\Psi}: (x, y) \mapsto xy^{-1}$$

$$\bar{\Psi}^{-1}(E)(y) = E y \quad (x) \bar{\Psi}^{-1}(E) = E x^{-1}$$

$$\begin{aligned} \therefore (\mu * \mu)(\bar{\Psi}^{-1}(E)) &= \int \mu(E y) \mu(dy) = \int \mu(E x^{-1}) \mu(dx) \\ &= \int \mu(E) \mu(dy) = \int \mu(E^{-1}) \mu(dx) \end{aligned}$$

$$\therefore \mu(E) \mu(G) = \mu(E^{-1}) \mu(G) \quad \mu(G) < \infty \text{ より}$$

$$\mu(E) = \mu(E^{-1})$$

例 1 $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{L})$

$(\mathbb{R}, +)$ 加法群 $\mathcal{L}(A+x) = \mathcal{L}(A)$

例 2 $(\mathbb{R} \setminus \{0\}, \mathcal{B}(\mathbb{R} \setminus \{0\}), \mu)$

$$\mu(A) = \int_A \frac{dx}{|x|}$$

例 1 例 2 $\int_{[a,b]} \frac{dx}{|x|} = \int_a^b \frac{dx}{|x|}$

$$\int_{\alpha[a,b]} \frac{dx}{|x|} = \int_{\alpha b}^{\alpha a} \frac{dx}{|x|} \quad \begin{matrix} dx=y \\ dx = \frac{dy}{\alpha} \end{matrix} \quad (\alpha < 0)$$

$$= \int_b^a \frac{|d\alpha| dy}{|\alpha| |y|} = \int_a^b \frac{dy}{|y|} = \mu([a,b])$$

$$\therefore \bigcup_{j=1}^{\infty} [a_j, b_j] = A \Rightarrow \mu(A) = R_{\alpha} \mu(A)$$

$$\left\{ \bigcup_{j=1}^{\infty} [a_j, b_j] \right\} \text{ 上 } \mu = R_{\alpha} \mu$$

$\mathcal{B}(\mathbb{R} \setminus \{0\})$ 上の測度は μ の α 倍 (Hopf の測度変換定理).

例 3 $(\mathbb{R}^4, \mathcal{B}(\mathbb{R}^4), \mathcal{L})$

$\mathbb{R}^4 \ni \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$ とし 群構造を定める

$$d\mu = \frac{d\alpha d\beta d\gamma d\delta}{|\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}|^2} \quad G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mu \left(\begin{matrix} AG \\ G/A \end{matrix} \right) \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \in G$$

Jacobian is $\frac{\partial(\alpha' \beta' \gamma' \delta')}{\partial(\alpha \beta \gamma \delta)} = \begin{vmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{vmatrix}$

$$\begin{aligned} \alpha' &= \alpha a + \beta c \\ \beta' &= \alpha b + \beta d \\ \gamma' &= \gamma a + \delta c \\ \delta' &= \gamma b + \delta d \end{aligned}$$

$$= \begin{vmatrix} a & c \\ b & d \end{vmatrix}^2 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^2$$

$$\frac{\partial(\alpha' \beta' \gamma' \delta')}{\partial(\alpha \beta \gamma \delta)} d\alpha d\beta d\gamma d\delta = d\alpha' d\beta' d\gamma' d\delta'$$

i.e. ~~$\frac{d\alpha d\beta d\gamma d\delta}{|\alpha \beta \gamma \delta|^2} = \frac{d\alpha' d\beta' d\gamma' d\delta'}{|\alpha' \beta' \gamma' \delta'|^2}$~~

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}^2 \frac{d\alpha d\beta d\gamma d\delta}{|\alpha \beta \gamma \delta|^2} = \frac{d\alpha' d\beta' d\gamma' d\delta'}{|\alpha' \beta' \gamma' \delta'|^2}$$

$$= \frac{d\alpha d\beta d\gamma d\delta}{|\alpha \beta \gamma \delta|^2} = \frac{d\alpha' d\beta' d\gamma' d\delta'}{\underbrace{|\alpha b|^2}_{1} |\alpha \beta \gamma \delta|^2} = \frac{d\alpha' d\beta' d\gamma' d\delta'}{|\alpha' \beta' \gamma' \delta'|^2}$$

実は 左不変測り度 = 右不変測り度

G は Abel 2-次元群, 有界 2-次元

131 | G 的 subgroup $G \supset H$ ($\mathbb{R}^2, B(\mathbb{R}^2), \mu$)

$$H = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}; \alpha \neq 0 \right\} \quad \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & \alpha\beta' + \beta \\ 0 & 1 \end{pmatrix}$$

$$\text{I} \quad \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ 0 & 1 \end{pmatrix} \quad \text{I} \text{ 对}$$

$$= \begin{pmatrix} \alpha a & \alpha b + \beta \\ 0 & 1 \end{pmatrix}$$

$$\frac{\partial(\alpha' \beta')}{\partial(\alpha \beta)} = \begin{vmatrix} a & 0 \\ b & 1 \end{vmatrix} = a$$

$$\alpha' = \alpha a$$

$$\therefore a = \frac{\alpha'}{\alpha}$$

$$\frac{\partial(\alpha' \beta')}{\partial(\alpha \beta)} d\alpha d\beta = d\alpha' d\beta'$$

$$\frac{\alpha'}{\alpha} d\alpha d\beta = d\alpha' d\beta' \quad \therefore \frac{d\alpha d\beta}{\alpha} = \frac{d\alpha' d\beta'}{\alpha'}$$

$$\therefore \frac{d\alpha d\beta}{\alpha} \text{ 右不变} \Rightarrow \text{右不变}$$

$$\text{II} \quad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a\alpha & a\beta + b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ 0 & 1 \end{pmatrix}$$

$$\frac{\partial(\alpha' \beta')}{\partial(a \beta)} = \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} = a^2 = \left(\frac{\alpha'}{\alpha}\right)^2$$

$$\therefore \frac{\partial(\alpha' \beta')}{\partial(a \beta)} d\alpha d\beta = d\alpha' d\beta'$$

$$\frac{d\alpha d\beta}{\alpha^2} = \frac{d\alpha' d\beta'}{\alpha'^2}$$

$$\therefore \frac{d\alpha d\beta}{\alpha^2} \text{ 左不变} \Rightarrow \text{左不变}$$

右不变 \neq 左不变

この定理により $(X, \mathcal{B}), (Y, \mathcal{B}') = (X, \mathcal{B})$

$f := i_x : X \rightarrow X$ identity と 思 之 ば

$$G(f) = \{ (x, y) \mid y = x \} = \{ (x, y) \mid xy^{-1} = e \} \in \mathcal{B} \times \mathcal{B}'$$

X の濃度
 \geq cont 濃度

Thm 6.3 局所コンパクトかつ可算コンパクト位相群 G において Baire 集合族 \mathcal{B} に対し (G, \mathcal{B}) は可測群.

① 略.

② 局所コンパクト.
 可算コンパクト.
 コンパクト

近傍とは cont. 集合がとれる
 可算開被覆 あるいは
 有限開被覆をとれる
 開被覆は有限開被覆とれる

③ Baire 集合族 : (X, σ) Hausdorff sp.

$f : X \rightarrow \mathbb{R}^1$ cont
 包含する最小の σ -field

$$\mathcal{B} = \sigma \left(\bigcup_{f \in C(X)} f^{-1}(A), A \in \mathcal{B}(\mathbb{R}^1) \right) \text{ の } \sigma$$

④ 距離空間 かつ

$$\text{Baire} = \text{Borel}$$

X Hausdorff space, \mathcal{B} Borel- σ field

$\mu(x) < \infty$ meas on (X, \mathcal{B})

$C(X) = \{ f: X \rightarrow \mathbb{C} \mid \text{cont + bdd} \}$

$I: C(X) \rightarrow \mathbb{C}$ st. $I(f) = \int_X f(x) d\mu(x)$

- i) $I(\cdot)$ linear
- ii) $f \geq 0 \Rightarrow I(f) \geq 0$
- iii) $\sup_{f \leq 1} I(f) < \infty$

\exists (i), (ii), (iii) $\Rightarrow \mu$?

X local cpt $C_0(X) = \{ f \in C(X) \mid \text{supp } f \text{ compact} \}$

Thm $I: C_0(X) \rightarrow \mathbb{C}$ st. i), ii), iii) $\Leftrightarrow \exists \mu$

bounded $\Leftrightarrow \exists \mu \in \mathcal{B}$ meas μ st. $I(f) = \int f d\mu \quad \forall f \in C_0(X)$

① の outline $\exists \mu$ の \Leftrightarrow ②: $O \subset X$ open set $\Rightarrow \nu(O)$

$\nu(O) := \sup \{ I(f) \mid 0 \leq f \leq 1, \text{supp } f \subset O, f \in C_0(X) \}$

$\forall \text{ disjoint } \nu(\bigsqcup_{n=1}^{\infty} O_n) = \sum_{n=1}^{\infty} \nu(O_n)$ \Leftarrow ②

$\exists \mu$ $\mu^+(E) = \inf_{E \subset O} \nu(O)$ \Leftarrow ② $E_n \subset O_n \cup E_n \subset \bigcup O_n$ \Leftarrow ②

$\mu^+(\bigsqcup E_n) \leq \nu(\bigsqcup O_n) \leq \sum \nu(O_n)$

$\therefore \mu^+(\bigsqcup E_n) \leq \sum \mu^+(E_n) \quad \therefore \mu^+$ is Caratheodory σ -field μ^+

一般論 ①) μ^+ -meas σ -field $\tilde{\mathcal{B}}$ is

σ -field \Rightarrow ②

i.e. $\mu^+(A) \geq \mu^+(E \cap A) + \mu^+(E^c \cap A)$
 $\forall A \in \tilde{\mathcal{B}}$

$\left(\begin{array}{l} \nu(\emptyset) = 0 \\ \nu(A_1) \leq \nu(A_2) \quad (A_1 \subset A_2) \\ \nu(\cup A) \leq \sum \nu(A) \end{array} \right)$

$\mu^+|_{\tilde{\mathcal{B}}}$ is measure μ^+ .

$\Rightarrow \tilde{\mathcal{B}} \subset \mathcal{B}$ \Leftarrow ②. $\mu^+|_{\mathcal{B}} = \mu$ \Leftarrow ② (X, \mathcal{B}) is μ measure \in def ③ / $f \geq 0$ \Leftarrow ③

$A_n = \{ x \mid f(x) > n \}$ \Leftarrow ③ $A_n \downarrow$

$\sum_n \mu(A_n \setminus A_{n+1}) = \int_X f(x) d\mu(x) \leq \sum_n \mu(A_n \setminus A_{n+1}) + \epsilon$
 $\epsilon I(\sum_n f_n) \leq \dots \leq \epsilon I(\sum_n f_n) + 2\epsilon \mu(X)$

上 σ -finite measure の正則性 $T \rightarrow T_*$

i.e., $\mu(O) = \sup_{K \subset O} \mu(K) \quad K \subset \text{cpt}$

$\mu(E) = \inf_{F \subset O} \mu(O)$

正則性を課せば μ の存在は一意的

(Cor 2. $\mu(X) < \infty$ は仮定)

X 可算cpt + locally cpt.

X 上の正則性測度 $\mu(K) < \infty \quad \forall K \subset \text{cpt}$

\leftrightarrow i) ii) と同等 $C_0(X)^*$ 双対性空間

可算cpt + locally cpt

$\Rightarrow X = \bigcup_{n=1}^{\infty} O_n$

O_n open $\bar{O}_n \subset \text{cpt}$
 $O_n \uparrow \quad \text{cpt}$

Cor 1. X 可算cpt + locally cpt 距離空間 X 上

有限 Borel measure \leftrightarrow i) ii) iii) と同等 $C_0(X)^*$

は 1:1 に対応する

⊙ 有限 Borel measure は正則性 (X 可算cpt 距離空間 X 上) である。

↓ 位相群 G の場合 \rightarrow 同様に

Thm. G locally cpt 位相群

$C_0(G)$ 上には \exists 測度 $\exists I: C_0(G) \rightarrow \mathbb{R}$

(i) $I(\cdot)$ \mathbb{R} -linear

(ii) $f \geq 0 \Rightarrow I(f) \geq 0$

(iii) $I(f) = I(f_y) \quad \forall y \in G \quad \because f_y(x) = f(xy)$

~~*/ locally cpt + 可算cpt / Bohr 集合族 = Borel 集合族~~

Cor locally cpt + 可算cpt 位相群 G , B Borel

(G, B) 上は正則性かつ右不変測度 μ が

定義倍を除く μ - 一意に存在する (測度群 μ_y は μ の右不変測度)

⊙ $I(f) = \int_G f(x) d\mu(x)$ は既知である。

\Rightarrow $R_y \mu$ は正則性がある

i.e. $R_y \mu(O) = \mu(O_y) = \sup_{K \subset O_y} \mu(K)$

$= \sup_{K_y \subset O} \mu(K)$

$= \sup_{K \subset O} \mu(K_y) = \sup_{K \subset O} R_y \mu(K)$

$\int_G f(x) dR_y \mu(x) = \int_G f_{y^{-1}}(x) d\mu(x) = I(f_{y^{-1}}) = I(f)$

一意性より $R_y \mu = \mu$ almost everywhere

$R_y \mu(E) = \inf_{F \subset O} R_y \mu(F)$ と同様

Thm G 局cpt + 可算cpt B Baire ~~集合族~~ 集合族
 (G, B) 上 ϵ 右不变测度が一意的 (定数 \times 倍)

\therefore at ∞ に存在する.

Lemma G locally cpt 位相群

$C_0^+(G) = \{f \in C_0(G) \mid f(x) \geq 0\}$ $J: C_0^+(G) \rightarrow \mathbb{R}$ は
 ≥ 0 かつ π $i) J(af) = aJ(f)$

$ii) J(f+g) = J(f) + J(g)$

$iii) J(f) \geq 0$

$iv) J(f) = J(f_y) \quad \forall y \in G$

\therefore $f \in C_0(G)$ に対し $I(f) = J(f^+) - J(f^-)$ であり
 $I(f)$ は (i) ~ (iii) を満たす.

\therefore 略.

$\mathcal{U}: \mathbb{1} \in G$ の近接近傍 $(u = u')$ 全体とする

$\mathcal{U} \ni U, k$ cpt

$X \subset K$ が \mathcal{U} 分離的 i.e. $x, y \in X \Rightarrow xy^{-1} \notin U$
 $x \neq y$

つまり $\#X < \infty$

$\max_x \#X = s(\mathcal{U}, K)$ と表す.

$\mathcal{U} \ni U_0$ fix

$$J_{U_0}(f) = \frac{1}{s(U_0, U_0)} \sup_X \sum_{x \in X} f(x)$$

$\therefore X$ は U 分離的集合を重く.

$$\Phi: \mathcal{U} \ni U \mapsto (J_U(f)) \in \prod_{f \in C_0^+(X)} [0, C_f]$$

$$H(U) = \{ \Phi(V) \mid V \in \mathcal{U}, V \subset U \}$$

$\bigcup_{U \in \mathcal{U}_0} \overline{H(U)} \ni J(\cdot) \leftarrow = \text{by } i) \sim iv) \text{ を満たす.}$

Cor 2 に 2))
 $C_0(G)^* \ni I \Leftrightarrow (G, \mathcal{B})$ 正則 μ かつ $\mu(K) < \infty$
 1:1

事実 Thm. $I \in C_0(X)^*$ かつ (i) ~ (iii)

⇔ (i) linear / \mathbb{R} (ii) positive

(iii) $\sup_{f \in \mathcal{L}} I(f) < \infty$

\exists Borel meas μ s.t. (1) $\mu(X) < \infty$
 (2) $I(f) = \int_X f(x) d\mu(x)$

☹ 言正則は μ 中 \mathbb{Z}

$\nu(X) = \sup \{ I(f) : 0 \leq f \leq 1, \text{supp } f \subset X \} < \infty$

なる \mathbb{Z}

~~$\mu^*(X) < \infty$~~ $\mu^*(X) < \infty \iff \mu(X) < \infty$.

Cor 1. X 可算cpt + locally cpt metric sp. \downarrow (4)

\mathcal{B}_∞ 有界 Borel meas $\Leftrightarrow \mathcal{F}$
 1:1

$\mathcal{F} := \{ I \in C_0(X)^* \mid (i) \sim (iii) \text{ 成立} \}$

☹ (4) \uparrow 有界 Borel は 正則

$\mathcal{B}_\infty \ni \mu \mapsto I(f) = \int f d\mu$
 \leftarrow

Cor 2 X 可算cpt + locally cpt.

$\mathcal{B} \ni \mu \Leftrightarrow I \in C_0(X)^*$

Thm 6.4 G 局所 cpt 位相群

$$C_0(G) = \{ f: G \rightarrow \mathbb{R} \mid \text{cont. supp } f \text{ cpt} \}$$

$\exists C_0(G) \ni f \mapsto I(f) \in \mathbb{R}$ s.t.

(1) $I(\cdot)$ は線形型 / \mathbb{R}

(2) $I(\cdot)$ は正値 i.e., $f(x) \geq 0 \forall x \in G \Rightarrow I(f) \geq 0$

(3) 右不変 $I(f) = I(f_y) := \int_G f_y(x) dx = \int_G f(xy) dx$.

Lemma 6.5, G 局所 cpt 位相群

$$C_0^+(G) = \{ f \in C_0(G) \mid f \geq 0 \} \quad J: C_0^+(G) \rightarrow \mathbb{R}$$

(i) $J(af) = aJ(f) \quad a \geq 0$

(ii) $J(f+g) = J(f) + J(g)$

(iii) $J(f) \geq 0$

(iv) $J(f) = J(f_y)$

右不変性 Thm 6.4a \mathbb{Z} \mathbb{Z} (4.3).

☺ $f \in C_0(G) \Rightarrow \exists f_+ - f_-$ とす

$$f_+ = \max(f, 0) \quad f_- = -\min(f, 0) \quad f_{\pm} \geq 0$$

$$I(f) := J(f_+) - J(f_-) \quad \text{とすればいい}$$

☺ $(f_y)^+ = (\max(f(\cdot y), 0))^+$

$$(f^+)_y = f^+(xy)$$



$$\textcircled{a} \quad (f_y)^+ = (f^+)_y \\ (f_y)^- = (f^-)_y$$

$$\begin{aligned} \text{p's} \quad I(f_y) &= J((f_y)^+) - J((f_y)^-) \\ &= J((f^+)_y) - J((f^-)_y) = I(f) \end{aligned}$$

\therefore (4) が従う。

$$\textcircled{a} \quad (af)^+ = a f^+ \quad (af)^- = a f^- \quad a > 0 \\ (af)^+ = |a| f^+ \quad (af)^- = |a| f^- \quad a < 0$$

$$I(af) = a I(f) \quad a \neq 0.$$

$$\textcircled{a} \quad I(f+g) = I(f) + I(g)$$

$$\therefore (f+g)^+ \leq f^+ + g^+$$

$$h = f^+ + g^+ - (f+g)^+ \geq 0 \quad h \in C_0^+(G)$$

$$f+g = (f+g)^+ - (f+g)^-$$

$$= f^+ + g^+ - f^- - g^-$$

$$\therefore h = (f+g) + (f^- + g^-) - (f+g - (f+g)^-)$$

$$= f^- + g^- + (f+g)^-$$



$$\therefore I(f+g) = J((f+g)^+) - J((f+g)^-) + \overset{\uparrow}{J(h)} - \overset{\uparrow}{J(h)}$$

(1) (2) (3)

$$\stackrel{EVD}{=} J((f+g)^+ + h) - J((f+g)^- + h)$$

$$= J(f^+ + g^+) - J(f^- + g^-)$$

$$= J(f^+) - J(f^-) + J(g^+) - J(g^-) = I(f) + I(g)$$

~~3) 12~~ I is a \mathbb{R} -valued ± 2 \square $f \geq 0$ and $f^- \equiv 0$ \square ,

$$I(f) = J(f^+) \geq 0 \quad \square$$