Point-wise exponential decay of bound states of the Nelson model with Kato-class potentials

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Abstract Point-wise exponential decay of bound states of the so-called Nelson model with Kato-class potentials is shown by constructing a martingale derived from the semigroup generated by the Nelson Hamiltonian.

1 Introduction

To show the existence of the ground state of a model in quantum field theory has been a crucial issue. In particular the so-called infrared-regular-condition is a critical condition for a scalar model to have the ground state. In this article we are concerned with the so-called Nelson model [17, 16] describing an interaction between non-relativistic nucleons and spinless scalar mesons. The time evolution of the non-relativistic matters studied in this article is given by a Schrödinger operator. Then the model can be regarded as Schrödinger operator coupled to a quantum field.

The existence of the ground state of this kind of model has been shown under some general conditions so far. Next interesting issue concerning the ground state is to make properties of the ground state clear, which includes to estimate the number of bosons in the ground state and the decay properties on both field variable ϕ and matter variable x. In this article we treat Kato-class potentials V which were introduced and studied by Aizenman and Simon [1] and the definition of Kato-class potentials is based on a condition considered by Kato in [11]. The Nelson Hamiltonian H with Kato-class potential is defined via functional integrations. The main purpose of this article is to show *point-wise* exponential decay of bound states Φ of the Nelson Hamiltonian:

$$\|\Phi(x)\|_{\mathscr{F}} \leq Ce^{-c|x|}, \quad a.e. x \in \mathbb{R}^d. \tag{1}$$

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The strategy is an extension of Carmona, Masters and Simon [4]. Let $H\Phi = E\Phi$. Then $\Phi = e^{+tE}e^{-tH}\Phi$. By the functional integration of e^{-tH} we can express Φ as

$$\Phi(x) = e^{tE}\mathbb{E}^x[\xi\Phi(B_t)]$$

for each $x \in \mathbb{R}^d$, where

$$\xi = e^{-\int_0^t V(B_s)ds}\mathrm{I}_0^* e^{-\phi(K)}\mathrm{I}_t \tag{2}$$

is the integral kernel and $\mathbb{E}$ denotes the expectation with respect to Brownian motion $(B_t)_{t\geq 0}$. See (18) below. We estimate $\mathbb{E}^x[\xi\Phi(B_t)]$ to get the bound (1).

Statement (1) is stronger rather than localization: $\int_{\mathbb{R}^d} dx\|e^{C|x|}\Phi(x)\|_{\mathscr{F}}^2 < \infty$, which can be shown by IMS localization [5, Theorem 3.2] in e.g.[2, 6, 8]. The point-wise exponential decay is also shown for the so-called semi-relativistic Pauli-Fierz model in quantum electrodynamics by the author oneself [9]. Here the integral kernel of the semi-group generated by the semi-relativistic Pauli-Fierz Hamiltonian is of the form

$$\xi_{\mathrm{PF}} = e^{-\int_0^t V(X_s)ds}\mathrm{J}_0^* e^{-iA(K_{PF})}\mathrm{J}_t,$$

where $(X_t)_{t\geq 0}$ is a Lévy process and J_t the family of isometries and $e^{-iA(K_{PF})}$ a unitary operator. Hence $\mathrm{J}_0^* e^{-iA(K_{PF})}\mathrm{J}_t$ is contractive, i.e., $\|\mathrm{J}_0^* e^{-iA(K_{PF})}\mathrm{J}_t\| \leq 1$. However the integral kernel of the semi-group generated by the Nelson Hamiltonian is of the form (2), where I_t is also a family of isometries but $e^{-\phi(K)}$ is unbounded. To see the bound $\|\mathrm{I}_0^* e^{-\phi(K)}\mathrm{I}_t\|$ one can apply the hypercontractivity [19] of $\mathrm{I}_0^*\mathrm{I}_t$ for massive cases, but it is non-trivial to have a bound for massless cases. In this article we derive the operator bound

$$\|\mathrm{I}_0^* e^{-\phi(K)}\mathrm{I}_t\| \leq e^{tE(\hat{\varphi})}$$

for massless cases in Corollary 4.6 but under infrared-regular condition, and consequently show point-wise exponential decay (1).

2 Basic facts on Fock space

2.1 Boson Fock space

Let $\mathscr{H}$ be a separable Hilbert space over $\mathbb{C}$. Define $\mathscr{F}^{(n)} = \otimes_s^n \mathscr{H}$, where $\otimes_s^n \mathscr{H}$ denotes the n-fold symmetric tensor product with $\otimes_s^0 \mathscr{H} = \mathbb{C}$. The space

$$\mathscr{F} = \oplus_{n=0}^\infty \mathscr{F}^{(n)}(\mathscr{H})$$

is called boson Fock space over $\mathscr{H}$. The Fock space $\mathscr{F}$ can be identified with the space of ℓ_2-sequences $(\Psi^{(n)})_{n\in\mathbb{N}}$ such that $\Psi^{(n)} \in \mathscr{F}^{(n)}$ and

$$\|\Psi\|_{\mathscr{F}}^2 = \sum_{n=0}^{\infty} \|\Psi^{(n)}\|_{\mathscr{F}^{(n)}}^2 < \infty. \tag{3}$$

Set $(\Psi,\Phi)_{\mathscr{F}} = \sum_{n=0}^{\infty}(\Psi^{(n)},\Phi^{(n)})_{\mathscr{F}^{(n)}}$. The vector $\Omega = (1,0,0,\dots)$ is called the Fock vacuum. The creation operator denoted by $a^*(f)$ and the annihilation operator by $a(f)$ are defined by

$$(a^*(f)\Psi)^{(n)} = \sqrt{n}S_n(f\otimes\Psi^{(n-1)}), \quad n\geq 1,$$
$$(a^*(f)\Psi)^{(0)} = 0$$

with domain $D(a^*(f)) = \left\{(\Psi^{(n)})_{n\geq 0}\in\mathscr{F} \,\middle|\, \sum_{n=1}^{\infty} n\|S_n(f\otimes\Psi^{(n-1)})\|_{\mathscr{F}^{(n)}}^2 < \infty\right\}$ and $a(f) = (a^*(\bar{f}))^*$. Furthermore, since both operators are closable and we denote their closed extensions by the same symbols. The space

$$\mathscr{F}_{\rm fin} = \{(\Psi^{(n)})_{n\geq 0}\in\mathscr{F} \,|\, \Psi^{(m)} = 0 \text{ for all } m\geq M \text{ with some } M\}$$

is called finite particle subspace. Operators a, a^* leave $\mathscr{F}_{\rm fin}$ invariant and satisfy the canonical commutation relations on $\mathscr{F}_{\rm fin}$:

$$[a(f),a^*(g)] = (\bar{f},g), \quad [a(f),a(g)] = 0, \quad [a^*(f),a^*(g)] = 0.$$

Given a contraction operator T on $\mathscr{H}$, the second quantization of T is defined by

$$\Gamma(T) = \oplus_{n=0}^{\infty}(\otimes^n T).$$

Here $\otimes^0 T = 1$. For a self-adjoint operator h on $\mathscr{H}$, $\{\Gamma(e^{ith}) : t\in\mathbb{R}\}$ is a strongly continuous one-parameter unitary group on $\mathscr{F}$. Then by Stone's theorem there exists a unique self-adjoint operator $\mathrm{d}\Gamma(h)$ on $\mathscr{F}$ such that $\Gamma(e^{ith}) = e^{it\mathrm{d}\Gamma(h)}$. Let $N = \mathrm{d}\Gamma(1)$. To obtain the commutation relations between $a^\sharp(f)$ and $\mathrm{d}\Gamma(h)$, suppose that $f\in D(h)$. Then

$$[\mathrm{d}\Gamma(h),a^*(f)] = a^*(hf), \quad [\mathrm{d}\Gamma(h),a(f)] = -a(\overline{h\bar{f}}), \tag{4}$$

for $\Psi\in D(\mathrm{d}\Gamma(h)^{3/2})\cap\mathscr{F}_{\rm fin}$. The Segal field $\Phi(f)$ on the boson Fock space $\mathscr{F}(\mathscr{H})$ is defined by

$$\Phi(f) = \frac{1}{\sqrt{2}}(a^*(f)+a(\bar{f})), \quad f\in\mathscr{H}.$$

Here $\bar{f}$ denotes the complex conjugate of f. Field operator $\Phi(f)$, $f\in\mathscr{H}$, is a self-adjoint operator, but $a^*(f)$ and $a(f)$ are not. Nevertheless we can define $e^{a^*(f)}$ and $e^{a(f)}$ by a geometric series. Let $f\in\mathscr{H}$ and we define the exponential of creation operators F_f by

$$F_f = \sum_{n=0}^{\infty}\frac{1}{n!}a^*(f)^n$$

and $D(F_f) = \left\{ \Phi \in \cap_{n=1}^\infty D(a^*(f)^n) \middle| \sum_{n=0}^\infty \frac{1}{n!} \|a^*(f)^n \Phi\| < \infty \right\}$. Let $\Phi \in \mathscr{F}^{(m)}$. Thus we have

$$\|F_f \Phi\| \leq \|\Phi\| + \sum_{n=1}^\infty \frac{\sqrt{m+n-1} \cdots \sqrt{m}}{n!} \|f\|^n \|\Phi\| < \infty.$$

Then $\mathscr{F}_{\mathrm{fin}} \subset D(F_f)$ follows. We also define the exponential of annihilation operators by

$$G_f = \sum_{n=0}^\infty \frac{1}{n!} a(f)^n$$

with $D(G_f) = \left\{ \Phi \in \cap_{n=1}^\infty D(a(f)^n) \middle| \sum_{n=0}^\infty \frac{1}{n!} \|a(f)^n \Phi\| < \infty \right\}$. We simply write $F_f = e^{a^*(f)}$ and $G_f = e^{a(f)}$. Then we can see that $(e^{a^*(f)})^* \supset e^{a(\bar{f})}$ and this implies that $e^{a^*(f)}$ is closable. The closure of $e^{a^*(f)}$ is denoted by the same symbol. Similarly the closure of $e^{a(f)}$ is denoted by the same symbol. We can represent $e^{\Phi(f)}$ in terms of both $e^{a^*(f)}$ and $e^{a(f)}$. Let $\mathscr{D}_{\mathrm{b}} = \mathrm{L.H.}\{C(g), \Phi | g \in \mathscr{H}, \Phi \in \mathscr{F}_{\mathrm{fin}}\}$.

Proposition 2.1 (Baker-Campbell-Hausdorff formula) *Let $f \in \mathscr{H}$ and $\alpha \in \mathbb{C}$. Then it holds on $\mathscr{D}_{\mathrm{b}}$ that*

$$e^{\alpha \Phi(f)} = e^{\alpha a^*(f)/\sqrt{2}} e^{\alpha a(\bar{f})/\sqrt{2}} e^{\frac{1}{2}\alpha^2 \|f\|^2}. \tag{5}$$

PROOF. We shall show (5) on $C(g)$. The proof of (5) on $\mathscr{F}_{\mathrm{fin}}$ is similar. We have

$$e^{\alpha a^*(f)} e^{\alpha a(\bar{f})} C(g) = e^{\alpha (f,g)} C(\alpha f + g). \tag{6}$$

Let $\psi(f) = a^*(f) + a(\bar{f})$. Then $\psi(f)$ is self-adjoint and it holds that

$$e^{\alpha \psi(f)} = \sum_{n=0}^\infty \frac{\alpha^n \psi(f)^n}{n!} \tag{7}$$

on $\mathscr{F}_{\mathrm{fin}}$. Let $C_m(g) = \sum_{n=0}^m \frac{a^*(g)^n}{n!} \Omega$. By using the expansion (7) we can compute as

$$e^{\alpha \psi(f)} C_m(g) = \sum_{n=0}^m \frac{(a^*(g) + \alpha (f,g))^n}{n!} e^{\alpha \psi(f)} \Omega.$$

Together with $e^{\alpha \psi(f)} \Omega = e^{\frac{1}{2}\alpha^2 \|f\|^2} e^{\alpha a^*(f)} \Omega$ we see that

$$e^{\alpha \psi(f)} C_m(g) = \sum_{n=0}^m \frac{(a^*(g) + \alpha (f,g))^n}{n!} e^{\frac{1}{2}\alpha^2 \|f\|^2} e^{\alpha a^*(f)} \Omega.$$

Hence we have

$$e^{\psi(f)} C(g) = e^{\alpha (f,g)} e^{\frac{1}{2}\alpha^2 \|f\|^2} C(f+g). \tag{8}$$

By (6) and (8) the proposition follows. $\square$

2.2 Bounds

In this section we show several bounds concerning the exponential of annihilation operators and the creation operators. We learned all these bounds from papers by Guneysu, Matte and Møller [7] and Matte and Møller [15]. We consider the case where $\mathscr{H} = L^2(\mathbb{R}^d)$. In this case, for $n \in \mathbb{N}$ the space $\mathscr{F}^{(n)}$ can be identified with the set of symmetric functions on $L^2(\mathbb{R}^{dn})$. The creation and annihilation operators act as

$$(a(f)\Psi)^{(n)}(k_1,\ldots,k_n) = \sqrt{n+1}\int_{\mathbb{R}^d} f(k)\Psi^{(n+1)}(k,k_1,\ldots,k_n)dk, \quad n \geq 0,$$
$$(a^*(f)\Psi)^{(n)}(k_1,\ldots,k_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f(k_j)\Psi^{(n-1)}(k_1,\ldots,\hat{k}_j,\ldots,k_n),, \quad n \geq 1,$$

with $(a^*(f)\Psi)^{(0)} = 0$. Let $H_{\rm f} = d\Gamma(\omega)$ with $\omega = \omega(k) = \sqrt{|k|^2+\nu^2}$. We have

$$(H_{\rm f}\Psi)^{(n)}(k_1,\ldots,k_n) = \left(\sum_{j=1}^{n}\omega(k_j)\right)\Psi^{(n)}(k_1,\ldots,k_n).$$

We can see the lemma below:

Lemma 2.2 *Let $h : \mathbb{R}^d \to \mathbb{C}$ be measurable, and $g_j \in D(h)$ for $j = 1,....,m$. Then for every $\Psi \in D(\mathrm{d}\Gamma(|h|^2)^{m/2})$ we have $\Psi \in D(\prod_{j=1}^m a(hg_j))$, and it follows that*

$$\left\|\prod_{j=1}^{m} a(hg_j)\Psi\right\| \leq \left(\prod_{j=1}^{m}\|g_j\|\right)\|\mathrm{d}\Gamma(|h|^2)^{m/2}\Psi\|. \tag{9}$$

In particular

$$\left\|\prod_{j=1}^{n} a(g_j)\Phi\right\| \leq \left(\prod_{j=1}^{n}\|g_j/\sqrt{\omega}\|\right)\|H_{\rm f}^{n/2}\Phi\|, \quad \Phi \in D(H_{\rm f}^{n/2}).$$

PROOF. Let $\Psi \in D(\mathrm{d}\Gamma(|h|^2)^m)$. First note that

$$(\Psi^{(n)}, (\mathrm{d}\Gamma(|h|^2)^m\Psi)^{(n)}) = \int_{\mathbb{R}^{nd}} |\Psi^{(n)}(k_1,\cdots,k_n)|^2\left(\sum_{i=1}^{n}|h(k_i)|^2\right)^m dk_1\cdots dk_n.$$

By the symmetry we can replace $\left(\sum_{i=1}^n |h(k_i)|^2\right)^m$ with $C(n,m)\prod_{j=1}^m |h(k_j)|^2$. Here $C(n,m) = n(n-1)\cdots(n-m+1)$. Then we have

$$(\Psi^{(n)}, (\mathrm{d}\Gamma(|h|^2)^m\Psi)^{(n)}) = C(n,m)\int_{\mathbb{R}^{nd}}|\Psi^{(n)}(k_1,\cdots,k_n)|^2\prod_{j=1}^{m}|h(k_j)|^2 dk_1\cdots dk_n.$$

On the other hand by the definition of annihilation operators we have

$$\begin{aligned}
&\left\| (\prod_{j=1}^m a(hg_j)\Psi)^{(n-m)} \right\|^2 \\
&\le C(n,m) \left(\prod_{j=1}^m \|g_j\|^2 \right) \int_{\mathbb{R}^{nd}} \left(\prod_{j=1}^m |h(k_j)|^2 \right) |\Psi^{(n)}(k_1,\cdots,k_n)|^2 dk_1\cdots dk_n \\
&= \left(\prod_{j=1}^m \|g_j\|^2 \right) (\Psi^{(n)}, (\mathrm{d}\Gamma(|h|^2)^m\Psi)^{(n)})
\end{aligned}$$

and summation over n gives (9). By the closedness of both operators $\mathrm{d}\Gamma(|h|^2)^{m/2}$ and $\prod_{j=1}^m a(hg_j)$ we can extend to $\Psi \in D(\mathrm{d}\Gamma(|h|^2)^{m/2})$. $\square$

Next we estimate $\left\| \prod_{j=1}^n a^*(f_j)\Phi \right\|$.

Lemma 2.3 *Let $f_i, g_j \in D(1/\sqrt{\omega})$ for $i,j = 1,...,n$ and $\Phi \in D(H_{\mathrm{f}}^{n/2})$. Then*

$$\left| \left(\prod_{j=1}^n a^*(g_j)\Phi, \prod_{j=1}^n a^*(f_j)\Phi \right) \right| \le n!2^n \left(\prod_{l=1}^n \|f_l\|_\omega \|g_l\|_\omega \right) \sum_{m=0}^n \frac{1}{m!} \|H_{\mathrm{f}}^{m/2}\Phi\|^2,$$

where $\|f\|_\omega = \|f\| + \|f/\sqrt{\omega}\|$.

PROOF. Let $\Phi \in \mathscr{F}_{\mathrm{fin}}$ and $f_i, g_j \in \mathscr{H}$ for $i,j = 1,...,m$. Then

$$\begin{aligned}
&\prod_{j=1}^n a(\bar{g}_j) \prod_{j=1}^n a^*(f_j)\Phi \\
&= \sum_{m=0}^n \sum_{C_m \ni A} \sum_{C_{n-m} \ni B} \sum_{\substack{\sigma: A^c \to B \\ \text{bijection}}} \left(\prod_{l \in A^c} (g_l, f_{\sigma(l)}) \right) \left(\prod_{p \in B^c} a^*(f_p) \right) \left(\prod_{q \in A} a(\bar{g}_q) \right) \Phi. \quad (10)
\end{aligned}$$

Here $C_k = \{A \subset \{1,...,n\} | \#A = k\}$, $C_0 = \emptyset$, and $\sum_{\substack{\sigma: A^c \to B \\ \text{bijection}}}$ is understood to take summation over all bijections from A^c to B. In particular

$$\begin{aligned}
&\left[\prod_{j=1}^n a(\bar{g}_j), \prod_{j=1}^n a^*(f_j) \right] \\
&= \sum_{m=0}^{n-1} \sum_{C_m \ni A} \sum_{C_{n-m} \ni B} \sum_{\substack{\sigma: A^c \to B \\ \text{bijection}}} \left(\prod_{l \in A^c} (g_l, f_{\sigma(l)}) \right) \left(\prod_{p \in B^c} a^*(f_p) \right) \prod_{q \in A} a(\bar{g}_q).
\end{aligned}$$

By this formula we have

$$\left(\prod_{j=1}^{n} a^*(g_j)\Phi, \prod_{j=1}^{n} a^*(f_j)\Phi\right)$$
$$= \sum_{m=0}^{n} \sum_{C_m \ni A} \sum_{C_{n-m} \ni B} \sum_{\substack{\sigma: A^c \to B \\ \text{bijection}}} \left(\Phi, \prod_{l \in A^c} (g_l, f_{\sigma(l)}) \prod_{p \in B^c} a^*(f_p) \prod_{q \in A} a(\bar{g}_q)\Phi\right). \quad (11)$$

By $\|\prod_{j=1}^{n} a(h_j)\Phi\| \le \prod_{j=1}^{n} \|h_j\| \|H_{\mathrm{f}}^{n/2}\Phi\|$ and $\#B^c = m = \#A$, the right-hand side of (11) can be estimated as

$$\left|\left(\Phi, \prod_{p \in B^c} a^*(f_p) \prod_{q \in A} a(\bar{g}_q)\Phi\right)\right| \le \left(\prod_{p \in B^c} \|f_p/\sqrt{\omega}\|\right) \left(\prod_{q \in A} \|g_q/\sqrt{\omega}\|\right) \|H_{\mathrm{f}}^{m/2}\Phi\|^2.$$

Since $\|f\| \le \|f\|_\omega$ and $\|f/\sqrt{\omega}\| \le \|f\|_\omega$, we have

$$\left|\left(\Phi, \prod_{l \in A^c} (g_l, f_{\sigma(l)}) \prod_{p \in B^c} a^*(f_p) \prod_{q \in A} a(\bar{g}_q)\Phi\right)\right| \le \left(\prod_{l=1}^{n} \|g_l\|_\omega \|f_l\|_\omega\right) \|H_{\mathrm{f}}^{m/2}\Phi\|^2. \quad (12)$$

Hence by (11) and (12)

$$\left|\left(\Phi, \prod_{j=1}^{n} a(\bar{g}_j) \prod_{j=1}^{n} a^*(f_j)\Phi\right)\right| \le n! 2^n \sum_{m=0}^{n} \frac{1}{m!} \left(\prod_{l=1}^{n} \|g_l\|_\omega \|f_l\|_\omega\right) \|H_{\mathrm{f}}^{m/2}\Phi\|^2.$$

Then the lemma follows. □

By Lemma 2.3 we have bounds for products of annihilation operators and creation operators. We summarise them as follows. Suppose that $f_j \in D(1/\sqrt{\omega})$ for $j = 1, ..., n$. By introducing a scaling parameter $0 < s < 1$ we also have

$$\left\|\prod_{j=1}^{n} a(f_j)\Phi\right\| \le s^{-n/2} \left(\prod_{j=1}^{n} \|f_j/\sqrt{\omega}\|\right) \|(sH_{\mathrm{f}})^{n/2}\Phi\|, \quad (13)$$
$$\left\|\prod_{j=1}^{n} a^*(f_j)\Phi\right\| \le \sqrt{n!} 2^{n/2} s^{-n/2} \left(\prod_{l=1}^{n} \|f_l\|_\omega\right) \left(\sum_{m=0}^{n} \frac{1}{m!} \|(sH_{\mathrm{f}})^{m/2}\Phi\|^2\right)^{1/2}. \quad (14)$$

Although exponential operator $e^{a^*(f)}$ is unbounded, it can be seen in the proposition below that $e^{a^*(f)} e^{-\frac{t}{2}H_{\mathrm{f}}}$ is bounded for any $t > 0$.

Proposition 2.4 *Let $t > 0$ and $f \in D(1/\sqrt{\omega})$. Then both $e^{a^*(f)} e^{-\frac{t}{2}H_{\mathrm{f}}}$ and $\overline{e^{-\frac{t}{2}H_{\mathrm{f}}} e^{a(f)}}$ are bounded.*

PROOF. Let $\Psi \in \cap_{n=1}^{\infty} D(H_{\mathrm{f}}^n)$. Suppose that $t < 1$. By (14) for any $s < t$ we have

$$\left\|\sum_{n=0}^{m}\frac{1}{n!}a^*(f)^n e^{-\frac{t}{2}H_{\rm f}}\Phi\right\| \leq \sum_{n=0}^{m}\frac{1}{\sqrt{n!}}2^{n/2}s^{-n/2}\|f\|_\omega^n\left(\sum_{k=0}^{n}\frac{1}{k!}\|(sH_{\rm f})^{k/2}e^{-\frac{t}{2}H_{\rm f}}\Phi\|^2\right)^{1/2}.$$

We can see that sequence $\{\sum_{n=0}^m \frac{1}{n!}a^*(f)^n e^{-\frac{t}{2}H_{\rm f}}\Phi\}_{m=0}^\infty$ is a Cauchy sequence in $\mathscr{F}$. Hence $e^{-\frac{t}{2}H_{\rm f}}\Phi \in D(e^{a^*(f)})$ and as $m\to\infty$ on both sides above we have

$$\|e^{a^*(f)}e^{-\frac{t}{2}H_{\rm f}}\Phi\| \leq A(f,s)\|e^{-\frac{1}{2}(t-s)H_{\rm f}}\Phi\|,$$

where $A(f,s) = \sum_{n=0}^\infty \frac{1}{\sqrt{n!}}2^{n/2}s^{-n/2}\|f\|_\omega^n$. Choosing s such that $s<t$, we can see that $\|e^{-\frac{1}{2}(t-s)H_{\rm f}}\Phi\| \leq \|\Phi\|$ and $e^{a^*(f)}e^{-\frac{t}{2}H_{\rm f}}$ for $t<1$ is bounded. Suppose $1\leq t$. Choosing $s=1$ in the above discussion, we have

$$\|e^{a^*(f)}e^{-\frac{t}{2}H_{\rm f}}\Phi\| \leq A(f,1)\|e^{-\frac{1}{2}(t-1)H_{\rm f}}\Phi\| \leq A(f,1)\|\Phi\|.$$

Thus $e^{a^*(f)}e^{-\frac{t}{2}H_{\rm f}}$ for $t\geq 1$ is bounded. Finally since $\left(e^{-\frac{t}{2}H_{\rm f}}e^{a(f)}\right)^* \supset e^{a^*(\bar f)}e^{-\frac{t}{2}H_{\rm f}}$, the second statement follows. Then the lemma follows. □

Corollary 2.5 *Let $f\in D(1/\sqrt{\omega})$. Then*

$$\begin{aligned}&\|e^{a^*(f)}e^{-\frac{t}{2}H_{\rm f}}\| \leq \sqrt{2}e^{(2/s)\|f\|_\omega^2}\|e^{-\frac{1}{2}(t-s)H_{\rm f}}\|,\ 0<s<t<1,\\ &\|e^{a^*(f)}e^{-\frac{t}{2}H_{\rm f}}\| \leq \sqrt{2}e^{2\|f\|_\omega^2}\|e^{-\frac{1}{2}(t-1)H_{\rm f}}\|, \quad 1\leq t.\end{aligned}$$

In particular we have

$$\begin{aligned}&\|e^{a^*(f)}e^{-tH_{\rm f}}e^{a(\bar f)}\| \leq 2e^{(4/s)\|f\|_\omega^2},\ 0<s<t<1,\\ &\|e^{a^*(f)}e^{-tH_{\rm f}}e^{a(\bar f)}\| \leq 2e^{4\|f\|_\omega^2}, \quad 1\leq t.\end{aligned}$$

PROOF. We can estimate $A(f,s)$ as

$$A(f,s) \leq \left(\sum_{n=0}^\infty \frac{1}{n!}8^n\|f\|_\omega^{2n}s^{-n}2^{-n}\right)^{1/2}\left(\sum_{n=0}^\infty 1^2\cdot 2^{-n}\right)^{1/2} \leq \sqrt{2}e^{(2/s)\|f\|_\omega^2}.$$

Then (1) follows from Proposition 2.4. □

3 Definition of the Nelson model

Let $\mathscr{S}'(\mathbb{R}^d)$ be the tempered distribution on $\mathbb{R}^d$. Let $s\in\mathbb{R}$ and $H^s(\mathbb{R}^d)$ be the inhomogeneous Sobolev space, i.e.,

$$H^s(\mathbb{R}^d) = \{u\in\mathscr{S}'(\mathbb{R}^d)|\hat u\in L^1_{\rm loc}(\mathbb{R}^d), (1+|k|^2)^{s/2}\hat u\in L^2(\mathbb{R}^d)\}.$$

Here $\hat{u}$ describes the Fourier transform on $\mathscr{S}'(\mathbb{R}^d)$. Homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ is defined by

$$\dot{H}^s(\mathbb{R}^d) = \{u \in \mathscr{S}'(\mathbb{R}^d) | \hat{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^d), |k|^s \hat{u} \in L^2(\mathbb{R}^d)\}.$$

The scalar product on $\dot{H}^s(\mathbb{R}^d)$ is defined by $(f,g)_{\dot{H}^s(\mathbb{R}^d)} = (|k|^s \hat{f}, |k|^s \hat{g})_{L^2(\mathbb{R}^d)}$. It is known that $\dot{H}^s(\mathbb{R}^d)$ is a Hilbert space if and only if $s < d/2$. Now we modify $H^s(\mathbb{R}^d)$ to apply quantum field theory. We define $H^s_\nu(\mathbb{R}^d)$ by $H^s(\mathbb{R}^d)$ with $(1+|k|^2)^{1/2}$ replaced by ω. Hence $H^s(\mathbb{R}^d)$ and $H^s_\nu(\mathbb{R}^d)$ are equivalent for $\nu > 0$, and $\dot{H}^s(\mathbb{R}^d) = H^s_\nu(\mathbb{R}^d)$ for $\nu = 0$. We set

$$\mathscr{H}_s(\mathbb{R}^d) = \begin{cases} H^s_\nu(\mathbb{R}^d), & \nu > 0, \\ \dot{H}^s(\mathbb{R}^d), & \nu = 0. \end{cases}$$

We set $\mathscr{H}_M = \mathscr{H}_{-1/2}(\mathbb{R}^d)$ and $\mathscr{H}_E = \mathscr{H}_{-1}(\mathbb{R}^{d+1})$. We define the Fourier transform (in the sense of tempered distribution) of $\mathscr{H}_M$ and $\mathscr{H}_E$ by $\hat{\mathscr{H}}_M$ and $\hat{\mathscr{H}}_E$, respectively. Although $\mathscr{H}_M$, $\hat{\mathscr{H}}_M$, $\mathscr{H}_E$ and $\hat{\mathscr{H}}_E$ depend on the space dimension and $\nu \geq 0$, we do not write the dependence explicitly. We also define real Hilbert spaces below:

(1)$\mathscr{M} = \{f \in \mathscr{H}_M | f \text{ is real-valued}\}$,
(2)$\mathscr{E} = \{f \in \mathscr{H}_E | f \text{ is real-valued}\}$.

Both $\mathscr{M}$ and $\mathscr{E}$ are Hilbert spaces over $\mathbb{R}$, and note that $\mathscr{M}_\mathbb{C} = \mathscr{H}_M$ and $\mathscr{E}_\mathbb{C} = \mathscr{H}_E$.

Hilbert space $L^2(\mathbb{R}^d)$ describes the state space of the non-relativistic matter, and $\mathscr{F}_\mathrm{N} = \mathscr{F}(\hat{\mathscr{H}}_M)$ that of the scalar bose field. The joint state space is described by the tensor product

$$\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathscr{F}_\mathrm{N}.$$

The free particle Hamiltonian is described by the Schrödinger operator

$$H_\mathrm{p} = -\frac{1}{2}\Delta + V$$

acting in $L^2(\mathbb{R}^d)$. We introduce a class $\mathscr{R}$ of potentials. Let V be relatively bounded with respect to $-(1/2)\Delta$ with a relative bound strictly smaller than one, i.e., $D(V) \subset D(-(1/2)\Delta)$ and

$$\|Vf\| \leq a\| -(1/2)\Delta f\| + b\|f\|$$

for $f \in D(V)$ with some $a < 1$ and $b \geq 0$. Then we say $V \in \mathscr{R}$. We introduce Assumption 3.1.

Assumption 3.1 *The following conditions hold:*

(1) $\varphi \in \mathscr{S}'(\mathbb{R}^d)$, $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k)$ *and* $\hat{\varphi}/\omega, \hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$.
(2) $V \in \mathscr{R}$.

The free field Hamiltonian $H_\mathrm{f} = \mathrm{d}\Gamma(\omega)$ on $\mathscr{F}_\mathrm{N}$ accounts for the energy carried by the field configuration. The matter -field interaction Hamiltonian H_I acting on the Hilbert space $\mathscr{H}$ describes then the interaction energy between the bose field and the matter . To give a definition of this operator we identify $\mathscr{H}$ as the space of $\mathscr{F}_{\rm N}$-valued L^2-functions on $\mathbb{R}^d$:

$$\mathscr{H} \cong \int_{\mathbb{R}^d}^{\oplus} \mathscr{F}_{\rm N} dx = \left\{ F : \mathbb{R}^d \to \mathscr{F}_{\rm N} \,\middle|\, \int_{\mathbb{R}^d} \|F(x)\|^2_{\mathscr{F}_{\rm N}} dx < \infty \right\}.$$

For each $x \in \mathbb{R}^d$, $H_{\rm I}(x)$ is defined by

$$H_{\rm I}(x) = \frac{1}{\sqrt{2}} \left(a_M^*(\hat{\varphi} e^{-ikx}) + a_M(\tilde{\hat{\varphi}} e^{ikx}) \right).$$

Here a_M and a_M^* denote the creation operator and the annihilation operator in the boson Fock space $\mathscr{F}(\hat{\mathscr{H}}_M)$, respectively. Since $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k)$, $H_{\rm I}(x)$ is symmetric, and it can be shown by using Nelson's analytic vector theorem that $H_{\rm I}(x)$ is essentially self-adjoint on $\mathscr{F}_{\rm fin}(\hat{\mathscr{H}}_M)$ of $\mathscr{F}_{\rm N}$. We denote the self-adjoint extension of $H_{\rm I}(x)$ by $\overline{H_{\rm I}(x)}$. The interaction $H_{\rm I}$ is then defined by the self-adjoint operator

$$H_{\rm I} = \int_{\mathbb{R}^d}^{\oplus} \overline{H_{\rm I}(x)} dx.$$

Under the conditions of Assumption 3.1 the operator

$$H = H_{\rm p} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} + H_{\rm I} \tag{15}$$

acting in $\mathscr{H}$ is called the Nelson Hamiltonian. Let

$$H_0 = H_{\rm p} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f}, \quad D(H_0) = D(H_{\rm p} \otimes \mathbb{1}) \cap D(\mathbb{1} \otimes H_{\rm f}).$$

Then H_0 is self-adjoint on $D(H_0)$ and bounded below. Suppose Assumption 3.1. Then H is also self-adjoint on $D(H_0)$ and bounded below, furthermore, it is essentially self-adjoint on any core of H_0. This follows from Kato-Rellich theorem [10].

4 Point-wise exponential decay

4.1 Integral kernels

We review a family of Gaussian random variables indexed by a real vector space $\mathscr{M}$. We say that $(\phi(f), f \in \mathscr{M})$ is a family of Gaussian random variables on a probability space $(\mathscr{Q}, \Sigma, \mu)$ indexed by a real inner product space $\mathscr{M}$ whenever

(1)$\phi : \mathscr{M} \ni f \mapsto \phi(f)$ is a map from $\mathscr{M}$ to a Gaussian random variable on $(\mathscr{Q}, \Sigma, \mu)$ with $\mathbb{E}_\mu[\phi(f)] = 0$ and covariance $\mathbb{E}_\mu[\phi(f)\phi(g)] = \frac{1}{2}(f,g)_{\mathscr{M}}$,
(2)$\phi(\alpha f + \beta g) = \alpha\phi(f) + \beta\phi(g)$, $\alpha, \beta \in \mathbb{R}$,
(3)Σ is the completion of the minimal σ-field generated by $\{\phi(f) | f \in \mathscr{M}\}$.

Also let $(\phi_{\rm E}(f), f \in \mathscr{E})$ be a family of Gaussian random variables on a probability space $(\mathscr{Q}_{\rm E}, \Sigma_{\rm E}, \mu_{\rm E})$ indexed by a real inner product space $\mathscr{E}$. Let $\mathscr{O} \subset \mathbb{R}$ and put

$$\mathscr{E}(\mathscr{O}) = \{f \in \mathscr{E} | {\rm supp} f \subset \mathscr{O} \times \mathbb{R}^d\}$$

and the projection $\mathscr{E} \to \mathscr{E}(\mathscr{O})$ is denoted by $e_{\mathscr{O}}$. Let

$$\Sigma_{\mathscr{O}} = \sigma\left(\{\phi_{\rm E}(f) | f \in \mathscr{E}(\mathscr{O})\}\right).$$

Define

$$\mathscr{E}_{\mathscr{O}} = \{\Phi \in L^2(\mathscr{Q}_{\rm E}) | \Phi \text{ is } \Sigma_{\mathscr{O}}\text{-measurable}\}.$$

Let $e_t = \tau_t \tau_t^*$, $t \in \mathbb{R}$. Then $\{e_t\}_{t\in\mathbb{R}}$ is a family of projections from $\mathscr{E}$ to ${\rm Ran}(\tau_t)$. Let Σ_t, $t \in \mathbb{R}$, be the minimal σ-field generated by $\{\phi_{\rm E}(f) | f \in {\rm Ran}(e_t)\}$. Define

$$\mathscr{E}_t = \{\Phi \in L^2(\mathscr{Q}_{\rm E}) | \Phi \text{ is } \Sigma_t\text{-measurable}\}, \quad t \in \mathbb{R}.$$

We will see below that $F \in \mathscr{E}_{[a,b]}$ can be characterized by ${\rm supp} F \subset [a,b] \times \mathbb{R}^d$.

Lemma 4.1 (1)$\mathscr{E}(\{t\}) = {\rm Ran}(e_t)$ *and any* $f \in {\rm Ran}(e_t)$ *can be expressed as* $f = \delta_t \otimes g$ *for some* $g \in \mathscr{M}$*. In particular,* $e_{\{t\}} = e_t$*.*
(2)$\mathscr{E}([a,b]) = \overline{{\rm L.H.}\{f \in \mathscr{E} | f \in {\rm Ran}(e_t), a \le t \le b\}}^{\|\cdot\|_{-1}}$ *holds.*

PROOF. Refer to see [18] □

We will define a family of transformations ${\rm I}_t$ from $L^2(\mathscr{Q})$ to $L^2(\mathscr{Q}_{\rm E})$ through the second quantization of a specific transformation τ_t from $\mathscr{M}$ to $\mathscr{E}$. Define $\tau_t : \mathscr{M} \to \mathscr{E}$ by $\tau_t : f \mapsto \delta_t \otimes f$. Here $\delta_t(x) = \delta(x-t)$ is the delta function with mass at t. Note that $\overline{\delta_t \otimes f} = \delta_t \otimes f$, which implies that τ_t preserves realness. It follows that

$$\tau_s^* \tau_t = e^{-|s-t|\hat{\omega}}, \quad s,t \in \mathbb{R}.$$

In particular, τ_t is isometry between $\mathscr{M}$ and $\mathscr{E}$ for each $t \in \mathbb{R}$. Let ${\rm I}_t = \Gamma(\tau_t) : L^2(\mathscr{Q}) \to L^2(\mathscr{Q}_{\rm E})$, $t \in \mathbb{R}$, be the family of isometries:

$${\rm I}_t \mathbb{1}_{\rm M} = \mathbb{1}_{\rm E}, \quad {\rm I}_t {:}\phi(f_1)\cdots\phi(f_n){:} = {:}\phi_{\rm E}(\delta_t \otimes f_1)\cdots\phi_{\rm E}(\delta_t \otimes f_n){:}.$$

Let $\hat{\omega} = \omega(-i\nabla) = \sqrt{-\Delta + \nu^2}$. The self-adjoint operator

$$\hat{H}_{\rm f} = {\rm d}\Gamma(\hat{\omega})$$

is called the free field Hamiltonian in $L^2(\mathscr{Q})$. It follows that $\hat{H}_{\rm f} = \theta_{\rm W} H_{\rm f} \theta_{\rm W}^{-1}$. From the identity $\tau_s^* \tau_t = e^{-|s-t|\hat{\omega}}$ it follows that

$$ {\rm I}_t^* {\rm I}_s = e^{-|s-t|\hat{H}_{\rm f}}, \quad s,t \in \mathbb{R}. \tag{16}$$

On $L^2(\mathbb{R}^d) \otimes L^2(\mathscr{Q})$ we define the Nelson Hamiltonian by

$$\hat{H} = H_{\rm p} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}_{\rm f} + g\hat{H}_{\rm I},$$

where $\hat{H}_{\rm I}=\int_{\mathbb{R}^d}^{\oplus}\hat{H}_{\rm I}(x)dx$ with $\hat{H}_{\rm I}(x)=\phi(\tilde{\varphi}(\cdot-x))$. Since H and $\hat{H}$ are unitarily equivalent, we denote H for $\hat{H}$ for simplicity in what follows. Let $(B_t)_{t\geq 0}$ be the brownian motion on the probability space $(\mathscr{X},B(\mathscr{X}),\mathscr{W})$.

Proposition 4.2 *Suppose Assumption 3.1. Then for $t\geq 0$ and $F,G\in\mathscr{H}$,*

$$(F,e^{-tH}G)_{\mathscr{H}}=\int_{\mathbb{R}^d}dx\mathbb{E}^x\left[e^{-\int_0^t V(B_s)ds}(F(B_0),{\rm I}_0^*e^{-\phi_{\rm E}(\int_0^t\delta_s\otimes\varphi(\cdot-B_s)ds)}{\rm I}_tG(B_t))_{L^2(\mathscr{Q})}\right]. \tag{17}$$

Here $F,G\in\mathscr{H}$ are regarded as $L^2(\mathscr{Q})$-valued L^2-functions on $\mathbb{R}^d$.

PROOF. See Appendix and see [14]. □

We call

$${\rm I}_{[0,t]}={\rm I}_0^*e^{-\phi_{\rm E}(\int_0^t\delta_s\otimes\varphi(\cdot-B_s)ds)}{\rm I}_t$$

the integral kernel of the semi-group generated by H. Thus

$$(F,e^{-tH}G)_{\mathscr{H}}=\int_{\mathbb{R}^d}dx\mathbb{E}^x\left[e^{-\int_0^t V(B_s)ds}(F(B_0),{\rm I}_{[0,t]}G(B_t))_{L^2(\mathscr{Q})}\right]$$

and we have

$$e^{-tH}G(x)=\mathbb{E}^x\left[e^{-\int_0^t V(B_s)ds}{\rm I}_{[0,t]}G(B_t)\right].$$

Moreover if $HG=EG$ we have

$$G(x)=e^{tE}\mathbb{E}^x\left[e^{-\int_0^t V(B_s)ds}{\rm I}_{[0,t]}G(B_t)\right]. \tag{18}$$

We discuss the boundedness of the norm of ${\rm I}_a^*e^{\phi_{\rm E}(f)}{\rm I}_b$ for not only massive case but also massless case. We see some intertwining properties of ${\rm I}_t$ and τ_t. We can identify $\phi_{\rm E}(f)$ (resp.$\phi(f)$) with $\frac{1}{\sqrt{2}}(a_E^*(\hat{f})+a_E(\tilde{\hat{f}}))$ (rep. $\frac{1}{\sqrt{2}}(a_M^*(\hat{f})+a_M(\tilde{\hat{f}}))$). Under this identification ${\rm I}_t$ can be recognised as a map from $\mathscr{F}(\hat{\mathscr{H}}_M)$ to $\mathscr{F}(\hat{\mathscr{H}}_E)$, i.e.,

$${\rm I}_t\prod_{j=1}^n a_M^*(\hat{f}_j)\Omega=\prod_{j=1}^n a_E^*(\hat{\tau}_tf_j)\Omega$$

and ${\rm I}_t^*$ as that from $\mathscr{F}(\hat{\mathscr{H}}_E)$ to $\mathscr{F}(\hat{\mathscr{H}}_M)$. It follows that on the finite particle subspace,

$${\rm I}_ta_M^*(\hat{f})=a_E^*(\widehat{\tau_tf}){\rm I}_t,\quad {\rm I}_ta_M(\hat{f})=a_E(\widehat{\tau_tf}){\rm I}_t, \tag{19}$$

$${\rm I}_t^*a_E^*(\hat{f})=a_M^*(\widehat{\tau_t^*f}){\rm I}_t^*,\quad {\rm I}_t^*a_E(\widehat{e_tf})=a_M(\widehat{\tau_t^*f}){\rm I}_t^*, \tag{20}$$

where $e_t=\tau_t\tau_t^*$. In particular

$${\rm I}_ta_M^*(\widehat{\tau_t^*f})=a_E^*(\widehat{e_tf}){\rm I}_t,\quad {\rm I}_ta_M(\widehat{\tau_t^*f})=a_E(\widehat{e_tf}){\rm I}_t=a_E(\hat{f}){\rm I}_t, \tag{21}$$

$${\rm I}_t^*a_E^*(\widehat{\tau_tf})=a_M^*(\hat{f}){\rm I}_t^*,\quad {\rm I}_t^*a_E(\widehat{\tau_tf})=a_M(\hat{f}){\rm I}_t^*. \tag{22}$$

Theorem 4.3 *Suppose that* $\hat{f} \in \hat{\mathscr{H}}_E$ *and* $\widehat{\tau_t^* f}/\sqrt{\omega} \in \hat{\mathscr{H}}_M$ *for* $t = a, b$ *with* $a \neq b$. *Then* $\mathrm{I}_a^* e^{\phi_{\mathrm{E}}(f)} \mathrm{I}_b$ *is bounded and*

$$\|\mathrm{I}_a^* e^{\phi_{\mathrm{E}}(f)} \mathrm{I}_b\| \leq 2 \exp\left(\frac{1}{4}\|\hat{f}\|_{\hat{\mathscr{H}}_E} + \left(1 \vee \frac{1}{|a-b|}\right)(\|\widehat{\tau_a^* f}\|_\omega^2 + \|\widehat{\tau_b^* f}\|_\omega^2)\right).$$

Here $x \vee y = \max\{x, y\}$ *and*

$$\|\widehat{\tau_a^* f}\|_\omega = \|\widehat{\tau_a^* f}/\sqrt{\omega}\|_{L^2(\mathbb{R}^d)} + \|\widehat{\tau_a^* f}/\omega\|_{L^2(\mathbb{R}^d)}.$$

PROOF. By Baker-Campbell-Hausdorff formula we have

$$e^{\phi_{\mathrm{E}}(f)} = e^{\frac{1}{\sqrt{2}} a_E^*(\hat{f})} e^{\frac{1}{\sqrt{2}} a_E(\tilde{\hat{f}})} e^{\frac{1}{4}\|\hat{f}\|_{\hat{\mathscr{H}}_E}^2}.$$

The intertwining property yields that

$$\mathrm{I}_a^* e^{\phi_{\mathrm{E}}(f)} \mathrm{I}_b = e^{\frac{1}{\sqrt{2}} a_M^*(\widehat{\tau_a^* f})} e^{-|a-b| H_{\mathrm{f}}} e^{\frac{1}{\sqrt{2}} a_M(\widetilde{\widehat{\tau_b^* f}})} e^{\frac{1}{4}\|\hat{f}\|_{\hat{\mathscr{H}}_E}^2}. \tag{23}$$

Since $e^{\frac{1}{\sqrt{2}} a_M^*(\widehat{\tau_a^* f})} e^{-\frac{|a-b|}{2} H_{\mathrm{f}}}$ and $e^{-\frac{|a-b|}{2} H_{\mathrm{f}}} e^{\frac{1}{\sqrt{2}} a_M(\widetilde{\widehat{\tau_b^* f}})}$ are bounded operators which operator bounds are given by

$$\|e^{\frac{1}{\sqrt{2}} a_M^*(\widehat{\tau_a^* f})} e^{-\frac{|a-b|}{2} H_{\mathrm{f}}}\| \leq \sqrt{2} \exp\left\{\left(1 \vee \frac{1}{|a-b|}\right)\left(\|\widehat{\tau_a^* f}\|_{\hat{\mathscr{H}}_M}^2 + \|\widehat{\tau_a^* f}/\sqrt{\omega}\|_{\hat{\mathscr{H}}_M}^2\right)\right\},$$
$$\|e^{-\frac{|a-b|}{2} H_{\mathrm{f}}} e^{\frac{1}{\sqrt{2}} a_M(\widetilde{\widehat{\tau_b^* f}})}\| \leq \sqrt{2} \exp\left\{\left(1 \vee \frac{1}{|a-b|}\right)\left(\|\widehat{\tau_b^* f}\|_{\hat{\mathscr{H}}_M}^2 + \|\widehat{\tau_b^* f}/\sqrt{\omega}\|_{\hat{\mathscr{H}}_M}^2\right)\right\}.$$

Hence together with them we have

$$\|\mathrm{I}_a^* e^{\phi_{\mathrm{E}}(f)} \mathrm{I}_b\| \leq 2 \exp\left(\frac{1}{4}\|\hat{f}\|_{\hat{\mathscr{H}}_E}^2 + \left(1 \vee \frac{1}{|a-b|}\right)\left(\|\widehat{\tau_a^* f}\|_\omega^2 + \|\widehat{\tau_b^* f}\|_\omega^2\right)\right).$$

Then the proof is complete. □

Corollary 4.4 (Integral kernel) *The integral kernel is given by*

$$\mathrm{I}_{[0,t]} = e^{\frac{1}{\sqrt{2}} a_M^*(U_t)} e^{-tH_{\mathrm{f}}} e^{\frac{1}{\sqrt{2}} a_M(\tilde{U}_t)} e^{\frac{1}{4} W},$$

where $U_t = \int_0^t ds e^{-s\omega(k)} e^{-ikB_s} \hat{\varphi}(k)/\sqrt{\omega(k)}$, $U_t = \int_0^t ds e^{-s\omega(k)} e^{ikB_s} \hat{\varphi}(k)/\sqrt{\omega(k)}$ *and*

$$W = \int_0^t dr \int_0^t ds \int_{\mathbb{R}^d} dk \frac{|\hat{\varphi}(k)|^2}{\omega(k)} e^{-|s-r|\omega(k)} e^{-ik(B_s - B_r)}.$$

PROOF. This follows from (23) and the definition of $\mathrm{I}_{[0,t]}$. □

We consider special cases of Theorem 4.3.

Corollary 4.5 *Let* $T \geq 0$. *Let* $f \in \mathscr{H}_M$, *i.e.,* $\hat{f}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. *We set*

$$\Phi_X = \phi_{\rm E}\left(\int_0^T (\tau_s f)(\cdot - B_s) ds\right).$$

Then (1) and (2) follow.
(1) *Suppose that* $\hat{f}/\omega, \hat{f}/\sqrt{\omega^3} \in L^2(\mathbb{R}^d)$. *Then*

$$\|\mathrm{I}_0^* e^{\Phi_X} \mathrm{I}_T\| \leq 2\exp\left(\frac{T}{2}\|\hat{f}/\omega\|^2 + 2(T\vee 1)(\|\hat{f}/\omega\|^2 + \|\hat{f}/\sqrt{\omega^3}\|^2)\right). \tag{24}$$

(2) *Suppose that* $\hat{f}/\omega \in L^2(\mathbb{R}^d)$. *Then*

$$\|\mathrm{I}_0^* e^{\Phi_X} \mathrm{I}_T\| \leq 2\exp\left(\frac{T}{2}\|\hat{f}/\omega\|^2 + 2T(T\vee 1)(\|\hat{f}/\sqrt{\omega}\|^2 + \|\hat{f}/\omega\|^2)\right). \tag{25}$$

PROOF. We see that

$$\left\|\int_0^T \tau_s f ds\right\|^2_{\mathscr{H}_{\rm E}} = \int_0^T ds \int_0^T dt (e^{-|s-t|\omega}\hat{f}, \hat{f})_{\mathscr{H}_M} \leq 2T\|\hat{f}/\omega\|^2.$$

We can also see that

$$\left\|\tau_T^* \int_0^T \tau_s f ds\right\|^2_{\mathscr{H}_M} \leq T\|\hat{f}/\omega\|^2, \quad \left\|\tau_T^* \int_0^T \tau_s f ds/\sqrt{\omega}\right\|^2_{\mathscr{H}_M} \leq T\|\hat{f}/\sqrt{\omega^3}\|^2,$$
$$\left\|\tau_0^* \int_0^T \tau_s f ds\right\|^2_{\mathscr{H}_M} \leq T\|\hat{f}/\omega\|^2, \quad \left\|\tau_0^* \int_0^T \tau_s f ds/\sqrt{\omega}\right\|^2_{\mathscr{H}_M} \leq T\|\hat{f}/\sqrt{\omega^3}\|^2.$$

Then (1) follows from Theorem 4.3. For (2) we can estimate as

$$\left\|\tau_T^* \int_0^T \tau_s f ds\right\|^2_{\mathscr{H}_M} \leq T^2\|\hat{f}/\sqrt{\omega}\|^2, \quad \left\|\tau_T^* \int_0^T \tau_s f ds/\sqrt{\omega}\right\|^2_{\mathscr{H}_M} \leq T^2\|\hat{f}/\omega\|^2,$$
$$\left\|\tau_0^* \int_0^T \tau_s f ds\right\|^2_{\mathscr{H}_M} \leq T^2\|\hat{f}/\sqrt{\omega}\|^2, \quad \left\|\tau_0^* \int_0^T \tau_s f ds/\sqrt{\omega}\right\|^2_{\mathscr{H}_M} \leq T^2\|\hat{f}/\omega\|^2.$$

Then (2) follows. □

We can plug (24) and (25). Let

$$E(\hat{f}) = \max\left\{\begin{array}{l} \frac{1}{2}\|\hat{f}/\omega\|^2 + 2(\|\hat{f}/\omega\|^2 + \|\hat{f}/\sqrt{\omega^3}\|^2) \\ \frac{1}{2}\|\hat{f}/\omega\|^2 + 2(\|\hat{f}/\sqrt{\omega}\|^2 + \|\hat{f}/\omega\|^2) \end{array}\right\}. \tag{26}$$

Definition 4.1 (Infrared regular condition) *The condition*

$$\int_{\mathbb{R}^d} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk < \infty$$

is called the infrared regular condition.

Corollary 4.6 (Integral kernel under infrared regular condition) *Let Φ_X be in Corollary 4.5. Suppose that $\hat{f}/\sqrt{\omega}, \hat{f}/\omega \in L^2(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2/\omega(k)^3 dk < \infty$. Then it follows that*

$$\|\mathrm{I}_0^* e^{\Phi_X} \mathrm{I}_T\| \leq 2e^{TE(\hat{\varphi})}, \quad T \geq 0. \tag{27}$$

PROOF. We have

$$\|\mathrm{I}_0^* e^{\Phi_X} \mathrm{I}_T\| \leq 2\exp\left(\frac{T}{2}\|\hat{f}/\omega\|^2 + 2T(\|\hat{f}/\omega\|^2 + \|\hat{f}/\sqrt{\omega^3}\|^2)\right)$$

for $T \geq 1$, and

$$\|\mathrm{I}_0^* e^{\Phi_X} \mathrm{I}_T\| \leq 2\exp\left(\frac{T}{2}\|\hat{f}/\omega\|^2 + 2T(\|\hat{f}/\sqrt{\omega}\|^2 + \|\hat{f}/\omega\|^2)\right)$$

for $T \leq 1$. Then (27) is shown. □

4.2 Kato-class potentials

Definition 4.2 (Kato-class potentials [1, 11])(1) *$V : \mathbb{R}^d \to \mathbb{R}$ is called a Kato-class potential whenever*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{B_r(x)} |g(x-y)V(y)|\, dy = 0$$

holds, where $B_r(x)$ is the closed ball of radius r centered at x, and

$$g(x) = \begin{cases} |x|, & d = 1, \\ -\log|x|, & d = 2, \\ |x|^{2-d}, & d \geq 3. \end{cases}$$

We denote this linear space by $\mathscr{K}(\mathbb{R}^d)$.

(2) *V is Kato-decomposable whenever $V = V_+ - V_-$ with $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $V_- \in \mathscr{K}(\mathbb{R}^d)$.*

Lemma 4.7 ([3]) *Let $0 \leq V \in \mathscr{K}(\mathbb{R}^d)$. Then there exist $\beta, \gamma > 0$ such that*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^x[e^{\int_0^t V(B_s)ds}] < \gamma e^{\beta t}. \tag{28}$$

Furthermore, if $V \in L^p(\mathbb{R}^d)$ with $p > d/2$ and $1 \leq p < \infty$, then

$$\beta \leq c(p)^{1/\varepsilon}\Gamma(\varepsilon)^{1/\varepsilon}\|V\|_p^{1/\varepsilon}, \tag{29}$$

where $\varepsilon = 1 - \frac{d}{2p}$ *and*

$$c(p) = \begin{cases} (2\pi)^{-d/2} & p = 1, \\ (2\pi)^{-d/2p} q^{d/(2q)} & p > 1 \end{cases}$$

with $\frac{1}{p} + \frac{1}{q} = 1$*. In particular* $L^p(\mathbb{R}^d) \subset \mathscr{K}(\mathbb{R}^d)$ *for* $p > d/2$ *and* $1 \le p < \infty$*.*

PROOF. There exists $t^* > 0$ such that $\alpha_t = \sup_{x\in\mathbb{R}^d} \mathbb{E}^x[\int_0^t V(B_s)ds] < 1$, for all $t \le t^*$, and $\alpha_t \to 0$ as $t \to 0$. By Khasminskii's lemma we have

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^x[e^{\int_0^t V(B_s)ds}] < \frac{1}{1-\alpha_t} \tag{30}$$

for all $t \le t^*$. We obtain

$$\mathbb{E}^x[e^{\int_0^{2t^*} V(B_s)ds}] = \mathbb{E}^x[e^{\int_0^{t^*} V(B_s)ds}\mathbb{E}^{B_{t^*}}[e^{\int_0^{t^*} V(B_s)ds}]] \le \left(\frac{1}{1-\alpha_{t^*}}\right)^2.$$

Repeating this procedure we see that

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}^x[e^{\int_0^t V(B_s)ds}] \le \left(\frac{1}{1-\alpha_{t^*}}\right)^{[t/t^*]+1} \tag{31}$$

for all $t > 0$, where $[z] = \max\{w \in \mathbb{Z} | w \le z\}$. Setting $\gamma = (\frac{1}{1-\alpha_{t^*}})$ and $\beta = \log\left\{(\frac{1}{1-\alpha_{t^*}})^{1/t^*}\right\}$. This proves (28). Next we prove (29). Suppose $V \in L^p(\mathbb{R}^d)$ with $p > d/2$ and $1 \le p < \infty$. We let $p > 1$. By Schwarz inequality we have

$$\mathbb{E}^x[V(B_t)] \le (2\pi t)^{-d/2} \left(\int_{\mathbb{R}^d} e^{-|x-y|^2 q/(2t)}\right)^{1/q} \|V\|_p = (2\pi t)^{-d/(2p)} q^{d/(2q)} \|V\|_p.$$

In particular we have

$$\|\mathbb{E}^x[V(B_t)]\|_\infty \le c(p) t^{-d/(2p)} \|V\|_p, \quad p > 1.$$

We introduce a Mittag-Leffler function which is defined by $m_b(x) = \sum_{k=0}^\infty \frac{1}{\Gamma(1+kb)} x^k$, where Γ denotes the Gamma function, $x \in \mathbb{R}$ and $b > 0$. It is known that Mittag-Leffler function $m_b(x)$ satisfies that $\lim_{x\to\infty}(m_b(x) - \frac{1}{b}e^{x^{1/b}}) = 0$ and there exists $k_b > 0$ such that $m_b(x) \le k_b e^{x^{1/b}}$ for all $x > 0$. Let $0 \le s_1 \le s_2 \le \cdots \le s_k$. By the Markov property of Brownian motion we have

$$\mathbb{E}^x[V(B_{s_1}) \cdots V(B_{s_k})] \le c(p)^k \|V\|_p^k s_1^{-d/(2p)} (s_2 - s_1)^{-d/(2p)} \cdots (s_k - s_{k-1})^{-d/(2p)}.$$

Then

$$
\begin{aligned}
&\mathbb{E}^x\left[\frac{1}{k!}\left[\int_0^t V(B_s)ds\right]^k\right] \\
&\le c(p)^k\|V\|_p^k\int_0^t ds_1\cdots\int_{s_{k-1}}^t ds_k s_1^{-d/(2p)}(s_2-s_1)^{-d/(2p)}\cdots(s_k-s_{k-1})^{-d/(2p)} \\
&=\frac{(c(p)\|V\|_p t^\varepsilon\Gamma(\varepsilon))^k}{\Gamma(1+k\varepsilon)},
\end{aligned}
$$

where $\varepsilon = 1-\frac{d}{2p} > 0$. Then it can be derived that

$$
\sup_{x\in\mathbb{R}^d}\mathbb{E}^x[e^{\int_0^t V(B_s)ds}] \le m_\varepsilon\left(c(p)\|V\|_p\Gamma(\varepsilon)t^\varepsilon\right) \le k_\varepsilon e^{c(p)^{1/\varepsilon}\|V\|_p^{1/\varepsilon}\Gamma(\varepsilon)^{1/\varepsilon}t}.
$$

Then (29) is proven. Let $p=1$. Then $d=1$ and it follows directly that

$$
\mathbb{E}^x\left[V(B_t)\right]ds \le (2\pi t)^{-1/2}\|V\|_1 = c(1)t^{-1/2}\|V\|_1.
$$

Hence (29) is proven in a similar way to the case of $p>1$. □

We introduce Assumption 4.1.

Assumption 4.1 *The following conditions hold:*

(1) $\varphi\in\mathscr{S}'(\mathbb{R}^d)$, $\overline{\hat\varphi(k)}=\hat\varphi(-k)$ *and* $\hat\varphi/\omega, \hat\varphi/\sqrt{\omega}\in L^2(\mathbb{R}^d)$.
(2) V *is Kato-decomposable.*

Suppose Assumption 4.1 and define the family of operators

$$
(T_tF)(x) = \mathbb{E}^x[e^{-\int_0^t V(B_r)dr}\mathrm{I}_{[0,t]}F(B_t)].
$$

Lemma 4.8 *T_t is bounded.*

PROOF. Let $F\in\mathscr{H}_\mathrm{N}$. Since $\|\mathrm{I}_{[0,t]}F(B_t)\|\le 2e^{E(t)}\|F(B_t)\|$, where $E(t)$ is given by

$$
E(t) = \frac{t}{2}\|\hat\varphi/\sqrt{\omega}\|^2 + 2t(1\vee t)(\|\hat\varphi/\sqrt{\omega}\|^2+\|\hat\varphi/\omega\|^2),
$$

from

$$
\|T_tF\|^2_{\mathscr{H}_\mathrm{N}} = \int_{\mathbb{R}^d}dx\|\mathbb{E}^x[e^{-\int_0^t V(B_r)dr}\mathrm{I}_{[0,t]}F(B_t)]\|^2_{L^2(\mathscr{Q})}
$$

it follows that

$$
\|T_tF\|^2_{\mathscr{H}_\mathrm{N}} \le 2\int_{\mathbb{R}^d}dx\mathbb{E}[e^{-2\int_0^t V(B_r+x)dr}]\mathbb{E}[\|F(B_t+x)\|^2]e^{2E(t)}.
$$

Since V is Kato-decomposable, we have $\sup_{x\in\mathbb{R}^d}\mathbb{E}[e^{-2\int_0^t V(B_r+x)dr}]=C<\infty$, and thus $\|T_tF\|^2_{\mathscr{H}_\mathrm{N}}\le 4Ce^{2E(t)}\|F\|^2_{\mathscr{H}_\mathrm{N}}$ follows. □

We note that if infrared regular condition $\int_{\mathbb{R}^d}|\hat\varphi(k)|^2/\omega(k)^3dk<\infty$ holds, then $\|\mathrm{I}_{[0,t]}\|\le 2e^{E(t)}$ can be replaced with

$$\|\mathrm{I}_{[0,t]}\| \leq 2e^{tE(\hat{\varphi})}. \tag{32}$$

In what follows we show that $\{T_t : t \geq 0\}$ is a symmetric C_0-semigroup. To do that we introduce the time shift operator u_t on $L^2(\mathbb{R}^d)$ by

$$u_t f(x) = f(x_0 - t, \mathbf{x}), \quad x = (x_0, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d.$$

It is straightforward that $u_t^* = u_{-t}$ and $u_t^* u_t = 1$. We denote the second quantization of u_t by $U_t = \Gamma_{\mathrm{E}}(u_t)$ which acts on $L^2(\mathscr{Q}_{\mathrm{E}})$ and is a unitary map. We can see that $U_t \mathrm{I}_s = \mathrm{I}_{s+t}$. We set

$$K_t = \int_0^t \delta_s \otimes \varphi(\cdot - B_s) ds.$$

Lemma 4.9 *$T_s T_t = T_{s+t}$ holds for $s, t \geq 0$.*

PROOF. By the definition of T_t we have

$$T_s T_t F = \mathbb{E}^x [e^{-\int_0^s V(B_r) dr} \mathrm{I}_{[0,s]} \mathbb{E}^{B_s} [e^{-\int_0^t V(B_r) dr} \mathrm{I}_{[0,t]} F(B_t)]]. \tag{33}$$

By the formulae $\mathrm{I}_s \mathrm{I}_0^* = \mathrm{I}_s \mathrm{I}_s^* U_{-s}^* = \mathrm{E}_s U_{-s}^*$ and $\mathrm{I}_t = U_{-s} \mathrm{I}_{t+s}$, (33) is equal to

$$\mathbb{E}^x [e^{-\int_0^s V(B_r) dr} \mathrm{I}_0^* e^{-\phi_{\mathrm{E}}(K_s)} \mathrm{E}_s \mathbb{E}^{B_s} [e^{-\int_0^t V(B_r) dr} U_{-s}^* e^{-\phi_{\mathrm{E}}(K_t)} U_{-s} \mathrm{I}_{t+s} F(B_t)]]. \tag{34}$$

Since U_s is unitary, we have $U_{-s}^* e^{-\phi_{\mathrm{E}}(K_t)} U_{-s} = e^{-\phi_{\mathrm{E}}(u_{-s}^* K_t)}$ as an operator. The test function of the exponent $u_{-s}^* K_t$ is given by

$$u_{-s}^* K_t = \int_0^t \delta_{r+s} \otimes \tilde{\varphi}(\cdot - B_r) dr.$$

Moreover by the Markov property of E_t, $t \in \mathbb{R}$, we may neglect E_s in (34), and by the Markov property of $(B_t)_{t \geq 0}$ we have

$$\begin{aligned} T_s T_t F &= \mathbb{E}^x [e^{-\int_0^s V(B_r) dr} \mathrm{I}_0^* e^{-\phi_{\mathrm{E}}(K_s)} \mathbb{E}^x [e^{-\int_s^{s+t} V(B_r) dr} e^{-\phi_{\mathrm{E}}(K_s^{s+t})} \mathrm{I}_{s+t} F(B_{s+t}) | \mathscr{F}_s]] \\ &= \mathbb{E}^x [e^{-\int_0^{s+t} V(B_r) dr} \mathrm{I}_0^* e^{-\phi_{\mathrm{E}}(K_{s+t})} \mathrm{I}_{s+t} F(B_{s+t})] = T_{s+t} F, \end{aligned}$$

where $K_s^{s+t} = \int_s^{s+t} \delta_r \otimes \tilde{\varphi}(\cdot - B_r) dr$ and $(\mathscr{F}_t)_{t \geq 0}$ denotes the natural filtration of $(B_t)_{t \geq 0}$. □

The strong continuity of the map $t \mapsto T_t$ on $\mathscr{H}_{\mathrm{N}}$ can be checked, while $T_0 = 1$ is trivial.

Theorem 4.10 *Semigroup $\{T_t : t \geq 0\}$ is a symmetric C_0-semigroup.*

PROOF. Since it was shown that $\{T_t : t \geq 0\}$ is a C_0-semigroup, it is enough to show that T_t is symmetric, i.e., $(F, T_t G) = (T_t F, G)$. Let $R = \Gamma_{\mathrm{E}}(r)$ be the second quantization of the reflection r, and $U_t = \Gamma_{\mathrm{E}}(u_t)$. Then we have

$$(F, T_t G) = \int_{\mathbb{R}^d} dx \mathbb{E}^x \left[e^{-\int_0^t V(B_s) ds} (\mathrm{I}_t F(B_0), e^{-\phi_{\mathrm{E}}(u_t r K_t)} \mathrm{I}_0 G(B_t)) \right],$$

where $u_t r K_t = \int_0^t \delta_{t-s} \otimes \tilde{\varphi}(\cdot - B_s) ds$. Noticing that $\dot{B}_s = B_{t-s} - B_t \stackrel{\mathrm{d}}{=} B_s$ for $0 \le s \le t$. Thus we can replace B_s with $\dot{B}_s$. Thus

$$(F, T_t G) = \int_{\mathbb{R}^d} dx \mathbb{E}[e^{-\int_0^t V(\dot{B}_s + x) ds} (\mathrm{I}_t F(\dot{B}_0 + x), e^{-\phi_{\mathrm{E}}(u_t r \dot{K}_t)} \mathrm{I}_0 G(\dot{B}_t + x))].$$

Here $\dot{K}_t = \int_0^t \delta_s \otimes \tilde{\varphi}(\cdot - \dot{B}_s) ds$. Exchanging $\int_{\mathbb{R}^d} dx$ and $\int_{\mathscr{X}} d\mathscr{W}^0$, and changing variable $-B_t + x$ to y, we have

$$(F, T_t G) = \int_{\mathbb{R}^d} dy \mathbb{E}[e^{-\int_0^t V(B_{t-s} + y) ds} (\mathrm{I}_t F(B_t + y), e^{-\phi_{\mathrm{E}}(\tilde{K}_t)} \mathrm{I}_0 G(B_0 + y))].$$

Here $\tilde{K}_t = \int_0^t \delta_{t-s} \otimes \tilde{\varphi}(\cdot - B_{t-s} - y) ds$. Thus we conclude that $(F, T_t G) = (T_t F, G)$ and the theorem follows. □

By Theorem 4.10 there exists a self-adjoint operator H_{Kato} such that $T_t = e^{-tH_{\mathrm{Kato}}}$ for $t \ge 0$.

Definition 4.3 (Nelson Hamiltonian with Kato-class potential) *Let V be a Kato - decomposable potential. Then we call the self-adjoint operator H_{Kato} Nelson Hamiltonian with Kato-class potential V.*

In what follows notational simplicity we also write H for H_{Kato}. I.e., the Nelson Hamiltonian written by H but with Kato-decomposable potential is understood as H_{Kato}.

4.3 Martingales

Let us consider the Schrödinger operator $H_{\mathrm{p}} = -\frac{1}{2}\Delta + V$. Let f be an eigenvector of H_{p}; $H_{\mathrm{p}} f = E_{\mathrm{p}} f$. Then we define the random process

$$h_t(x) = e^{tE_{\mathrm{p}}} e^{-\int_0^t V(B_r + x) dr} f(B_t + x), \quad t \ge 0.$$

Note that $\mathbb{E}[h_t(x)] = f(x)$ for all $t \ge 0$. We see that the random process $(h_t(x))_{t \ge 0}$ is martingale with respect to $(\mathscr{F}_t)_{t \ge 0}$. We extend this to the Nelson Hamiltonian. Suppose that E is an eigenvalue associated with a bound state Φ;

$$H\Phi = E\Phi.$$

Define

$$H_t(x) = e^{tE} e^{-\int_0^t V(B_r + x) dr} e^{-\phi_{\mathrm{E}}(\int_0^t \delta_r \otimes \varphi(\cdot - x - B_r) dr)} \mathrm{I}_t \Phi(B_t + x). \tag{35}$$

Then $(H_t(x))_{t \ge 0}$ is a random process on $(\mathscr{X} \times \mathscr{Q}_{\mathrm{E}}, \mathscr{B}(\mathscr{X}) \times \Sigma_{\mathrm{E}}, \mathscr{W} \times \mu_{\mathrm{E}})$ for each $x \in \mathbb{R}^d$. By the functional integral representation we have

$$(\Psi, \Phi) = (\Psi, e^{-t(H-E)} \Phi) = \int_{\mathbb{R}^d} dx \mathbb{E}_{\mu_{\mathrm{E}}}[\bar{\Psi}(x) \mathbb{E}[\mathrm{I}_0^* H_t(x)]].$$

Hence it follows that $\Phi(x) = (e^{-t(H-E)}\Phi)(x) = \mathbb{E}[\mathrm{I}_0^* H_t(x)]$. Define the filtration $(\mathscr{M}_t)_{t\geq 0}$ by

$$\mathscr{M}_t = \mathscr{F}_t \times \Sigma_{(-\infty,t]}, \quad t \geq 0.$$

Theorem 4.11 *Suppose Assumption 4.1. Then $(H_t(x))_{t\geq 0}$ is martingale with respect to $(\mathscr{M}_t)_{t\geq 0}$.*

PROOF. Let us set

$$K(0,s) = e^{-\int_0^s V(B_r+x)dr} e^{-\phi_{\mathrm{E}}(\int_0^s \delta_r \otimes \varphi(\cdot - x - B_r)dr)}.$$

We have

$$\mathbb{E}_{\mu_{\mathrm{E}}} \mathbb{E}\left[H_t(x)|\mathscr{M}_s\right] = e^{tE} K(0,s) \mathbb{E}_{\mu_{\mathrm{E}}} \mathbb{E}\left[K(s,t)\Phi(B_t + x)|\mathscr{M}_s\right]$$

We compute the conditional expectation of the right-hand side above. From Markov property of $(B_t)_{t\geq 0}$ it follows that

$$\begin{aligned}
&\mathbb{E}_{\mu_{\mathrm{E}}} \mathbb{E}\left[K(s,t)\mathrm{I}_t \Phi(B_t + x)|\mathscr{M}_s\right] \\
&= \mathbb{E}_{\mu_{\mathrm{E}}}\left[\mathbb{E}^{B_s}\left[e^{-\int_0^{t-s} V(B_r+x)dr} e^{-\phi_{\mathrm{E}}(\int_s^t \delta_r \otimes \varphi(\cdot - x - B_{r-s})dr)} \mathrm{I}_t \Phi(B_{t-s} + x)|\Sigma_{(-\infty,s]}\right]\right].
\end{aligned}$$

From the Markov property of projection $\mathrm{E}_s = \mathrm{I}_s \mathrm{I}_s^*$ it furthermore follows that

$$\begin{aligned}
&= \mathbb{E}_{\mu_{\mathrm{E}}}\left[\mathbb{E}^{B_s}\left[e^{-\int_0^{t-s} V(B_r+x)dr} e^{-\phi_{\mathrm{E}}(\int_s^t \delta_r \otimes \varphi(\cdot - x - B_{r-s})dr)} \mathrm{I}_t \Phi(B_{t-s} + x)|\Sigma_s\right]\right] \\
&= \mathrm{E}_s \mathbb{E}^{B_s}\left[e^{-\int_0^{t-s} V(B_r+x)dr} e^{-\phi_{\mathrm{E}}(\int_s^t \delta_r \otimes \varphi(\cdot - x - B_{r-s})dr)} \mathrm{I}_t \Phi(B_{t-s} + x)\right].
\end{aligned}$$

Let $U_s = \Gamma(u_s)$, where $u_s : \mathscr{E} \to \mathscr{E}$ denotes the time-shift operator defined by $u_s f(t,x) = f(t+s,x)$. Since the shift operator and the projection are related to $E_s = \mathrm{I}_s \mathrm{I}_0^* U_{-s}$, we have

$$\begin{aligned}
&= \mathrm{I}_s \mathrm{I}_0^* U_{-s} \mathbb{E}^{B_s}\left[e^{-\int_0^{t-s} V(B_r+x)dr} e^{-\phi_{\mathrm{E}}(\int_s^t \delta_r \otimes \varphi(\cdot - x - B_{r-s})dr)} \mathrm{I}_t \Phi(B_{t-s} + x)\right] \\
&= \mathrm{I}_s \mathrm{I}_0^* \mathbb{E}^{B_s}\left[e^{-\int_0^{t-s} V(B_r+x)dr} e^{-\phi_{\mathrm{E}}(\int_s^t \delta_{r-s} \otimes \varphi(\cdot - x - B_{r-s})dr)} \mathrm{I}_{t-s} \Phi(B_{t-s} + x)\right] \\
&= \mathrm{I}_s \mathrm{I}_0^* \mathbb{E}^{B_s}\left[K(0,t-s)\mathrm{I}_{t-s} \Phi(B_{t-s} + x)\right].
\end{aligned}$$

Hence we conclude that

$$\mathbb{E}_{\mu_{\mathrm{E}}} \mathbb{E}\left[H_t(x)|\mathscr{M}_s\right] = e^{sE} K(0,s) e^{-(t-s)(H-E)} \Phi(B_s + x) = H_s(x).$$

Then $(H_t(x))_{t\geq 0}$ is martingale. □

Let τ be a stopping time with respect to $\mathscr{M}_t$. Then $H_{t\wedge\tau}(x)$ is also martingale. In particular $\mathbb{E}_{\mu_{\mathrm{E}}} \mathbb{E}[H_t(x)] = \mathbb{E}_{\mu_{\mathrm{E}}} \mathbb{E}[H_{t\wedge\tau}(x)]$.

4.4 Main theorem

In tis section we assume that Φ is a bound state of H: $H\Phi = E\Phi$.

Lemma 4.12 *Suppose Assumption 4.1. Then* $\|\Phi(\cdot)\|_{L^2(\mathscr{Q})} \in L^\infty(\mathbb{R}^d)$.

PROOF. It follows that $\Phi(x) = \mathbb{E}_{\mu_\mathrm{E}}\mathbb{E}[\mathrm{I}_0^* H_t(x)]$ for an arbitrary $t > 0$. Since we can see that $\mathrm{I}_{[0,t]}$ is bounded and $\|\mathrm{I}_{[0,t]}\| \le 2e^{E(t)}$, we obtain

$$\|\Phi(x)\| \le 2e^{tE}e^{E(t)}\left(\mathbb{E}^x[e^{-2\int_0^t V(B_r)dr}]\right)^{1/2}\left(\mathbb{E}^x[\|\Phi(B_t)\|^2]\right)^{1/2}.$$

Since $\sup_{x\in\mathbb{R}^d}\mathbb{E}^x[e^{-2\int_0^t V(B_r)dr}] < \infty$ and $\mathbb{E}^x[\|\Phi(B_t)\|^2] \le C\|\Phi\|$, the lemma follows. □

Now we state the main theorem in this article.

Theorem 4.13 (Point-wise exponential decay) *Suppose Assumption 4.1 and infrared regular condition* $\int_{\mathbb{R}^d}|\hat{\varphi}(k)|^2/\omega(k)^3 dk < \infty$. *Assume either (1) or (2):*

(1)$\lim_{|x|\to\infty} V(x) = \infty$,
(2)$\lim_{|x|\to\infty} V_-(x) + E + E(\hat{\varphi}) < 0$.

Then there exist constants c and C such that

$$\|\Phi(x)\|_{L^2(\mathscr{Q})} \le Ce^{-c|x|}.$$

PROOF. Suppose (1). Let $\tau_R = \inf\{t||B_t| > R\}$. Then τ_R is a stopping time with respect to the natural filtration of the Brownian motion $(B_t)_{t\ge 0}$. Let

$$W_R(x) = \inf\{V(y)||x-y| < R\}.$$

Note that $W_R(x) \le V(x+y)$ for $|y| < R$, and we see that

$$X_R(x) = W_R(x) - E - E(\hat{\varphi}) \to \infty \quad (|x| \to \infty).$$

Let $\Psi \in L^2(Q)$. Then $\mathrm{I}_0\Psi \cdot H_t(x)$ is also martingale. Actually we can see that

$$\mathbb{E}_{\mu_\mathrm{E}}\mathbb{E}[\mathrm{I}_0\Psi \cdot H_t(x)|\mathscr{M}_s] = \mathrm{I}_0\Psi \cdot \mathbb{E}_{\mu_\mathrm{E}}\mathbb{E}[H_t(x)|\mathscr{M}_s] = \mathrm{I}_0\Psi \cdot H_s(x).$$

Hence it follows that

$$(\mathrm{I}_0\Psi, \Phi(x))_{L^2(Q)} = \mathbb{E}_{\mu_\mathrm{E}}\mathbb{E}[\mathrm{I}_0\Psi \cdot H_0(x)] = \mathbb{E}_{\mu_\mathrm{E}}\mathbb{E}[\mathrm{I}_0\Psi \cdot H_{t\wedge\tau_R}(x)].$$

Boud $\|\mathrm{I}_{[0,t]}\| \le 2e^{tE(\hat{\varphi})}$ is derived from the infrared regular condition and the exponent is linear in t. Then

$$\|\Phi(x)\| = \sup_{\Psi\in L^2(\mathscr{Q}),\|\Psi\|=1} |(\mathrm{I}_0\Psi,\Phi(x))| = |\mathbb{E}_{\mu_{\mathrm{E}}}\mathbb{E}[\mathrm{I}_0\Psi\cdot H_{t\wedge\tau_R}(x)]|$$
$$\le \|\mathbb{E}[\mathrm{I}_0^* H_{t\wedge\tau_R}(x)]\| \le 2\mathbb{E}[e^{-\int_0^{t\wedge\tau_R}(V(B_r+x)-E-E(\hat\varphi))dr}]\sup_{x\in\mathbb{R}^d}\|\Psi_{\mathrm{g}}(x)\|.$$

Then it is enough to estimate $\mathbb{E}[e^{-\int_0^{t\wedge\tau_R}(V(B_r+x)-E-E(\hat\varphi))dr}]$. Let

$$\mathbb{E}[e^{-\int_0^{t\wedge\tau_R}(V(B_r+x)-E-E(\hat\varphi))dr}] \le \mathbb{E}[1\!\!1_{\{\tau_R<t\}}e^{-(t\wedge\tau_R)X_R(x)}] + \mathbb{E}[1\!\!1_{\{\tau_R\ge t\}}e^{-(t\wedge\tau_R)X_R(x)}].$$

It is trivial to see that $\mathbb{E}[1\!\!1_{\{\tau_R\ge t\}}e^{-(t\wedge\tau_R)X_R(x)}] \le e^{-tX_R(x)}$. We also see that

$$\mathbb{E}[1\!\!1_{\{\tau_R<t\}}e^{-(t\wedge\tau_R)X_R(x)}] \le \mathbb{E}[1\!\!1_{\{|B_t|\ge R\}}] = \frac{S_{d-1}}{(2\pi)^{d/2}}\int_{R/\sqrt{t}}^\infty e^{-r^2/2}r^{d-1}dr \le c_1 e^{-c_2R^2/t}.$$

Let $R = p|x|$ $(0<p<1)$ and $t=\delta|x|$, where we assume that δ is a positive constant. Since $X_{p|x|}(x)\to\infty$ as $|x|\to\infty$, we have

$$\mathbb{E}[e^{-(t\wedge\tau_R)X_R(x)}] \le e^{-\delta|x|c} + c_1 e^{-c_2(p^2/\delta)|x|}.$$

Then the theorem follows. Next suppose (2). Let $\tau_R(x) = \inf\{t\ge 0||B_t+x|\le R\}$. Similarly to (1) it is enough to estimate $\mathbb{E}[e^{\int_0^{t\wedge\tau_R(x)}(V_-(B_r+x)+E+E(\hat\varphi))dr}]$. By the assumption on V_-, there exist $\varepsilon>0$ and $R>0$ such that for all $|x|>R$ it holds that $V_-(x)+E+E(\hat\varphi) < -\varepsilon < 0$. By the definition of stopping time $|B_r+x|\ge R$ for $r\le t\wedge\tau_R(x)$. Then the integrand in $\int_0^{t\wedge\tau_R}(V_-(B_r+x)+E+E(\hat\varphi))dr$ is less than $-\varepsilon$. Hence

$$\mathbb{E}[e^{\int_0^{t\wedge\tau_R(x)}(V_-(B_r+x)+E+E(\hat\varphi))dr}] \le \mathbb{E}[e^{-(t\wedge\tau_R(x))\varepsilon}]$$
$$= \mathbb{E}[1\!\!1_{\{t<\tau_R(x)\}}e^{-t\varepsilon}] + \mathbb{E}[1\!\!1_{\{t\ge\tau_R(x)\}}e^{-\tau_R(x)\varepsilon}] \le e^{-t\varepsilon} + \mathbb{E}^x[1\!\!1_{\{t\ge\tau_R(0)\}}].$$

We have $\mathbb{E}^x[1\!\!1_{\{t\ge\tau_R(0)\}}] = (2\pi)^{-d/2}\int_{|u|\le R/t} e^{-\frac12(|u|^2-2\frac{u\cdot x}{\sqrt{|x|}}+\frac{|x|^2}{t})}du$. Set $t=R=|x|$. Then

$$\mathbb{E}^x[1\!\!1_{\{t\ge\tau_R(0)\}}] \le (2\pi)^{-d/2}e^{\sqrt{|x|}-\frac12|x|}\int_{|u|\le 1} e^{-\frac12|u|^2}du \le c_1e^{-c_2|x|}$$

and

$$\mathbb{E}[e^{\int_0^{t\wedge\tau_R(x)}(V_-(B_r+x)+E+E(\hat\varphi))dr}] \le e^{-\varepsilon|x|} + c_1e^{-c_2|x|}.$$

Thus the proof is complete. □

5 Proof of Proposition 4.2

PROOF. For $\varepsilon \geq 0$ let $\hat{H_{\rm I}}^\varepsilon = \rho_\varepsilon(\hat{H}_{\rm I})$, where $\rho_\varepsilon(X) = X + \varepsilon X^2$. Then $\hat{H_{\rm I}}^\varepsilon$ is bounded below for $\varepsilon > 0$. We can also see that $e^{-tH^\varepsilon} \to e^{-tH}$ strongly as $\varepsilon \downarrow 0$. We shall construct a functional integral representation of $(F, e^{-tH^\varepsilon}G)$ for $\varepsilon > 0$ and by a limiting argument we construct that of $(F, e^{-tH}G)$ for $\varepsilon = 0$. Assume that $V \in C_0^\infty(\mathbb{R}^d)$. Let $h = (-1/2)\Delta$. By the Trotter-Kato product formula [12, 13] and $e^{-|s-t|\hat{H}_{\rm f}} = {\rm I}_s^*{\rm I}_t$, we have

$$(F, e^{-tH^\varepsilon}G) = \lim_{n\to\infty}\left({\rm I}_0 F, \left(\prod_{j=0}^{n-1} {\rm I}_{\frac{jt}{n}} e^{-\frac{t}{n}\hat{H_{\rm I}}^\varepsilon} e^{-\frac{t}{n}h} e^{-\frac{t}{n}V} {\rm I}^*_{\frac{jt}{n}}\right) {\rm I}_t G\right).$$

Here $\prod_{j=1}^n t_j = t_1 \cdots t_n$. Using the identity ${\rm I}_s e^{-\hat{H_{\rm I}}^\varepsilon} {\rm I}_s = E_s e^{-\hat{H_{\rm I}}^\varepsilon(s)} E_s$ for $s \in \mathbb{R}$, where

$$\hat{H_{\rm I}}^\varepsilon(s) = \int_{\mathbb{R}^d}^{\otimes} \rho_\varepsilon\left(\phi_{\rm E}(\delta_s \otimes \varphi(\cdot - x))\right) dx$$

and $E_s = {\rm I}_s {\rm I}_s^*$ is a projection, we can see that

$$(F, e^{-tH^\varepsilon}G) = \lim_{n\to\infty}\left({\rm I}_0 F, \left(\prod_{j=0}^{n-1} E_{\frac{jt}{n}} e^{-\frac{t}{n}\hat{H_{\rm I}}^\varepsilon(\frac{jt}{n})} e^{-\frac{t}{n}h} e^{-\frac{t}{n}V} E_{\frac{jt}{n}}\right) {\rm I}_t G\right).$$

By the Markov property of E_s's we can neglect E_s's. Then

$$(F, e^{-tH^\varepsilon}G) = \lim_{n\to\infty}\left({\rm I}_0 F, \left(\prod_{j=0}^{n-1} e^{-\frac{t}{n}\hat{H_{\rm I}}^\varepsilon(\frac{jt}{n})} e^{-\frac{t}{n}h} e^{-\frac{t}{n}V}\right) {\rm I}_t G\right).$$

The right-hand side above can be represented in terms of the Wiener measure as

$$(F, e^{-tH^\varepsilon}G) = \lim_{n\to\infty}\int_{\mathbb{R}^d} dx \mathbb{E}^x\left[e^{-\sum_{j=0}^{n-1} V(B_{\frac{jt}{n}})}\left({\rm I}_0 F(B_0) e^{-\sum_{j=0}^{n-1}\frac{t}{n}\hat{H_{\rm I}}^\varepsilon(\frac{jt}{n})} {\rm I}_t G(B_t)\right)\right].$$

Note that $s \mapsto \delta_s \otimes \varphi(\cdot - B_s)$ is strongly continuous as a map $\mathbb{R} \to \mathscr{E}$, almost surely. Hence $s \mapsto \phi_{\rm E}(\delta_s \otimes \varphi(\cdot - B_s))$ is also strongly continuous as a map $\mathbb{R} \to L^2(\mathscr{Q}_E)$. Then we can compute the limit and the result is

$$(F, e^{-tH^\varepsilon}G) = \int_{\mathbb{R}^d} dx \mathbb{E}^x\left[e^{-\int_0^t V(B_s)ds}\left({\rm I}_0 F(B_0), e^{-\varepsilon Q_t - \phi_{\rm E}(\int_0^t \delta_s \otimes \varphi(\cdot - B_s) ds)} {\rm I}_t G(B_t)\right)\right].$$

Here $Q_t = \int_0^t \phi_{\rm E}(\delta_s \otimes \varphi(\cdot - B_s))^2 ds$. Take $\varepsilon \downarrow 0$ on both sides above we have

$$(F(B_0), e^{-tH}G) = \int_{\mathbb{R}^d} dx \mathbb{E}^x\left[e^{-\int_0^t V(B_s)ds}\left({\rm I}_0 F(B_0), e^{-\phi_{\rm E}(\int_0^t \delta_s \otimes \varphi(\cdot - B_s) ds)} {\rm I}_t G(B_t)\right)\right].$$

Then the theorem follows for $V \in C_0^\infty(\mathbb{R}^d)$. By a simple limiting argument we can prove (17) for $V \in \mathscr{R}$. □

Acknowledgements The author thanks Oliver Matte for useful discussions and comments for bounds in Section 2.2, and also thanks organizers: Kenji Yajima, Arne Jensen, Hisashi Okamoto, Yoshio Tsutsumi, Shu Nakamura, Keiichi Kato, Norikazu Saito, and Fumihiko Nakano for inviting him to "Tosio Kato Centennial Conference" held in the university of Tokyo at September 4-8 of 2017.

Finally this work is financially supported by JSPS KAKENHI 16H03942, CREST JPMJCR14D6 and JSPS open partnership joint research with Denmark 1007867.

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