Central Limit Theorem and Renormalization Group Methods in Some Models in Mathematical Physics

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ABSTRACT

Models we can solve are very much restricted. Either they have small coupling constants or they have very special structures like infinitely many conserved quantities. But in models like 4D lattice gauge field theory, 2D Heisenberg model (with large β) and NVS equation, we soon encounter serious difficulties.

The RG method is believed to be free from these difficulties, but still depends on perturbations, namely Gaussian integrals (two-point functions) around the origin or around points which form a flow. This is possible only when the functional integral is saturated or exhausted by a sum of Gaussian integrals.

What can we do in the models mentioned above which are highly non-linear and far from Gaussian?

Assume that ξ is a random Gaussian variable with mean zero and covariance Γ. Then $q =: \xi^2 :_\Gamma = \xi^2 - \Gamma$ is again approximately a Gaussian random variable. Thus the integral by ξ is replaced by a Gaussian integral by q, and so on. In some cases, this simplifies the integral.

This intuitive image means that some non-linear, non-Gaussian models can be described by sum of (approximately) Gaussian systems by representing ξ^4 by $: \xi^2 :^2$. This idea may be a variation of the old idea of Simon and Griffith who represented ϕ^4 model by the Ising spins.

For example, in the case of 2D O(N) sigma model, denoting the (N-component) block spin at order n by $\phi_n(x) \in R^N$, the core part of the RG recursion reads

$$\int \exp\left[-\sum_{\xi\in\Box} \frac{g_n}{4N} (: \phi_{n+1}^2(x+\xi/L) :_{G_{n+1}} + q(x+\xi/L))^2 - \frac{1}{N} \sum q^2(x+\xi/L) \right]$$
$$\times \prod dq(x+\xi/L) = \exp\left[-\frac{g_{n+1}}{4N} (: \phi_{n+1}^2(x) :_{G_{n+1}} :)^2 \right]$$

where $\Box$ is the box of size $L \times L$ centred at the origin and $q(x) =: z^2(x) : +\phi(x)) \cdot z(x)$ are Gaussian random variables derived by the above argument. The point is that the effective coupling constants converges to a constant $g^* > 0$. (g_n can increase exponentially, but this does not happen!)

This idea can be used to establish non-existence of phase transitions in 2D O(N) sigma (Heisenberg) model with large N. Though we are now trying our idea to the long-standing unsolved problems like 4D lattice gauge field model, it is too early to say anything about these models.