Semi-relativistic QED

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1 Pauli-Fierz model

In this paper we are concerned with the so-called semi-relativistic Pauli-Fierz model in quantum electrodynamics, which is abbreviated as SRPF model. SRPF model is a relativistic version of the so-called Pauli-Fierz midel which has been studied so far. Before explaining SRPF model we should review results obtained for the Pauli-Fierz model.

The Pauli-Fierz Hamiltonian is defined by

$$H_{\rm PF} = \frac{1}{2}(p_x - A(x))^2 + V + H_{\rm rad}. \tag{1.1}$$

Below we give definitions of notations appeared in (1.1). Operator $H_{\rm PF}$ is a linear operator defined on the Hilbert space $\mathcal{H}$ given by the tensor product of Hilbert spaces:

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F},$$

where $\mathcal{F}$ denotes the boson Fock space over one particle state space $L^2(\mathbb{R}^3 \times \{1,2\}) = W$,i.e,

$$\mathcal{F} = \oplus_{n=0}^{\infty} [\otimes_s^n W]$$

with symmetric n-fold tensor product $\otimes_s^n$, $p_x = (-i\nabla_1, -i\nabla_2, -i\nabla_3)$ denotes the momentum operator for the matter (= electron) and $V : \mathbb{R}^3 \to \mathbb{R}$ an external potential. $H_{\rm rad}$ is a self-adjoint operator acting on $\mathcal{F}$,which denotes the free field Hamiltonian defined by the second quantization of the multiplication operator by $|k| = \omega(k)$, i.e.,

$$H_{\rm rad} = d\Gamma(\omega).$$

Finally in order to define the quantized radiation field $A_\mu(x)$, we introduce the annihilation operator and the creation operator. For $f \in W$, let $a^\dagger(f) : \mathcal{F} \to \mathcal{F}$ be the creation operator defined by

$$\left(a^\dagger(f)\Phi\right)^{(n+1)} = \sqrt{n+1} S_{n+1}\left(f \otimes \Phi^{(n)}\right),$$

where S_{n+1} is the symetrizer on $\otimes^{n+1} W$. Then the annihilation operator $a(f)$ is given by the adjoint of $a^\dagger(\bar{f})$:

$$a(f) = (a^\dagger(\bar{f}))^*.$$

It is satisfied that $[a(f), a^\dagger(g)] = (\bar{f}, g)$. Then for each $x \in \mathbb{R}^3$, quantized radiation field $A_\mu(x)$ is given by

$$A_\mu(x) = a^\dagger(f_x^\mu) + a(\tilde{f}_x^\mu),$$

where $f_x^\mu(k,j) = \frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} e_\mu^j(k) e^{-ikx}$, $\tilde{f}_x^\mu(k,j) = \frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}} e_\mu^j(k) e^{ikx}$ and $(e^1(k), e^2(k), k/|k|)$ is a right-hand system in $\mathbb{R}^3$ for each $k \in \mathbb{R}^3$. Suppose that $\tilde{\hat{\varphi}} = \hat{\varphi} = \bar{\hat{\varphi}}$ and $\sqrt{\omega}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^3)$ throughout this paper. Let V be relatively bounded with respect to $-\Delta$ with a relative bound strictly smaller than one. Then $H_{\rm PF}$ is self-aidjoint on $D(-\Delta) \cap D(H_{\rm rad})$ and bounded from below. Moreover it is essentially self-adjoint on any core of $-\Delta + H_{\rm rad}$. The spectral properties of $H_{\rm PF}$ have been studied in the last two decades, in particular special attentions have been payed for studying the so-called ground state. In general, eigenvectors associated with the bottom of the spectrum of self-adjoint operator K is called the ground state of K. We note that the existence of ground states does not necessarily hold true. Under some condition it is proven that $H_{\rm PF}$ has the unique ground state. This fact is not trivial due to the zero spectral gap, i.e., the bottom of the spectrum of $H_{\rm PF}$ is the edge of the continuous spectrum.

2 Semi-relativistic Pauli-Fierz model and Feynman-Kac formula

As is seen above $H_{\rm PF}$ can be regarded as the minimal coupling of the decoupled Hamiltonian $-\frac{1}{2}\Delta + V + H_{\rm rad}$ by $A(x)$. The SRPF Hamiltonian is defined by the Schrödinger operator $-\frac{1}{2}\Delta + V$ replaced by the semi-relativistic Schrödinger operator $\sqrt{-\Delta + M^2}$. We give the definition of SRPF Hamiltonian. It is however not straightforward to define SRPF Hamiltonian as a self-adjoint operator due to non-local kinetic term. The lemma below is a key fact to define SRPF Hamiltonian.

Lemma 2.1 ([5]) *Suppose that $\omega^{3/2}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$. Then $(p_x - A(x))^2 + M^2$ is essentially self-adjoint on $D(\Delta) \cap C^\infty(N)$, where N denotes the number operator and $C^\infty(N) = \cap_{k=1}^\infty D(N^k)$.*

We denote the closure of $(p_x - A(x))^2 + M^2 \lceil_{D(\Delta)\cap C^\infty(N)}$ by simply the same notation $(p_x - A(x))^2 + M^2$, which is self-adjoint. Hence we can define the self-adjoint operator

$$T = \sqrt{(p_x - A(x))^2 + M^2}$$

by the spectral resolution of $(p_x - A(x))^2 + M^2$.

Definition 2.2 *Suppose that $\omega^{3/2}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$. Then $H_{\rm SRPF}$ is defined by*

$$H_{\rm SRPF} = T \dotplus H_f \dotplus V,$$

where $\dotplus$ denotes the quadratic form sum. From now on we write H for $H_{\rm SRPF}$ for notational convenience.

In order to construct a functional integral representation (Feynman-Kac type formula) we prepare probabilistic notations. Let $(B_t)_{t\in\mathbb{R}}$ be 3-dimensional Brownian motion on a probability space (Ω_p, B_p, P^x) and $(T_t)_{t\geq 0}$ be a subordinator defined on a probability space (Ω_ν, B_ν, ν) such that

$$\mathbb{E}\left[e^{-uT_t}\right] = e^{-t(\sqrt{2u+M^2}-M)} foru \geq 0.$$

We are concerned with H_{SRPF} by means of a functional integral. Let $f, g \in L^2(\mathbb{R}^d)$ and $X_t = B_{T_t}$ for $t \geq 0$. Then it is well known that

$$\int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{\mathrm{P}\times\nu}\left[\overline{f(X_0)} g(X_t)\right] = (f, e^{-th} g),$$

where h denotes the semi-relativistic Schrödinger operator:

$$h = \sqrt{-\Delta + M^2} - M.$$

Furthermore we can see that

$$\int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{\mathrm{P}\times\nu}\left[\overline{f(X_0)} g(X_t) e^{-\int_0^{T_t} V(B_s)ds}\right] = \left(f, e^{-t(h+V)} g\right)$$

Let us consider the field part. Let (Q, μ) be a probability space and $(\phi(f), f \in \oplus^3 L^2(\mathbb{R}^3))$ a Gaussian random variable indexed by $f \in \oplus^3 L^2(\mathbb{R}^3)$ such that the mean is zero and the covariance is given by

$$\mathbb{E}_\mu\left[\phi(f)\phi(g)\right] = \frac{1}{2}(\hat{f}, D\hat{g}),$$

where $D = D(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}\right)_{1\leq\mu,v\leq 3}$ is 3×3matrix. Also we define a probability space (Q_E, μ_E) and the Gaussian random variable $(\phi_E(f), f \in \oplus^3 L^2(\mathbb{R}^4))$ such that the mean is zero and the covariance is given by

$$\mathbb{E}_{\mu_E}\left[\phi_E(f)\phi_E(g)\right] = \frac{1}{2}(\hat{f}, D \otimes \mathbb{1}\hat{g}).$$

It is well-known that $\mathcal{F} \cong L^2(Q, d\mu)$ and $A_\mu(x) \cong \phi\left(\oplus^3_{\nu=1}\delta_{\mu\nu}\tilde{\varphi}(\cdot - x)\right)$, where $\tilde{\varphi} = (\tilde{\varphi}/\sqrt{\omega})^\vee$. Moreover there exists a family of isometries $(J_t)_{t\in\mathbb{R}}$ such that $J_t : L^2(Q) \to L^2(Q_E)$ with

$$J_t^* J_s = e^{-|t-s|\overline{H_{\mathrm{rad}}}},$$

where $\overline{H_{\mathrm{rad}}}$ denotes the free field Hamiltonian in $L^2(Q)$, which is unitary equivalent to H_{rad} in $\mathcal{F}$. Then we define

$$\tilde{H} = \sqrt{\left(p_x - \tilde{A}(x)\right)^2 + M^2} \dotplus V \dotplus \overline{H_{\mathrm{rad}}}$$

in $L^2(\mathbb{R}^3) \otimes L^2(Q)$, where $\tilde{A}_\mu(x) = \phi\left(\oplus^3_{\nu=1}\delta_{\mu\nu}\tilde{\varphi}(\cdot - x)\right)$. It is seen that $H \cong \widetilde{H}$. Under this identification we consider $\widetilde{H}$ instead of H in what follows.

Theorem 2.3 ([7]) *It follows that*

$$(F, e^{-t\widetilde{H}}G) = \int_{\mathbb{R}^3} dx \mathbb{E}^{x,0}_{\mathrm{P}\times\mathrm{V}} \left[\left(J_0 F(B_0), e^{-i\phi_E(K_t)} J_t G(B_{T_t}) \right) e^{-\int_0^{T_t} V(B_s)ds} \right],$$

where

$$K_t = \oplus_{\mu=1}^3 \int_0^{T_t} j_{T_s^*} \tilde{\varphi}(\cdot - B_s) ds$$

with $T_s^* = \inf\{t \geq 0 | T_t = s\}$*, and* $j_t : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^4)$ *is defined by*

$$\widehat{j_t f}(k_0 k) = \frac{e^{-itk_0}}{\sqrt{\pi}} \frac{\sqrt{\omega(k)}}{\sqrt{\omega(k)^2 + |k_0|^2}} \hat{f}(k), \quad (k, k_0) \in \mathbb{R}^3 \times \mathbb{R}.$$

A crucial point of the functional integral representation is that an interaction term is put together as $e^{-i\phi_E(K_t)}$. The immediate corollary is to specify the domain of $\widetilde{H}$.

Theorem 2.4 ([7, 2]) *Suppose that* $\omega^{3/2}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$ *and* V *is relatively bounded with respect to* $\sqrt{-\Delta}$ *with a relative bound strictly smaller than one. Then* $\widetilde{H}$ *is self-adjoint on* $D(\sqrt{-\Delta}) \cap D(\overline{H_{\mathrm{rad}}})$.

Proof We show the outline of the proof. Using the functional integral representation we can show that

$$e^{-t\widetilde{H}} D(\sqrt{-\Delta}) \cap D(\overline{H_{\mathrm{rad}}}) \subset D(\sqrt{-\Delta}) \cap D(\overline{H_{\mathrm{rad}}})$$

which yields that $\widetilde{H}_0$ is essentially self-adjoint on $D(\sqrt{-\Delta}) \cap D(\overline{H_{\mathrm{rad}}})$. Next we can show the bound

$$\|\sqrt{-\Delta}F\| + \|\overline{H_{\mathrm{rad}}}F\| \leq C\|\widetilde{H}_0 F\|, \tag{2.1}$$

where $\widetilde{H}_0$ is $\widetilde{H}$ with V replaced by 0. From (2.1) it follows that $\widetilde{H}_0$ is self-adjoint on $D(\sqrt{-\Delta}) \cap D(\overline{H_{\mathrm{rad}}})$. Furthermore we can see that V is relatively bounded with respect to $\widetilde{H}_0$. Hence $\widetilde{H}$ is self-also adjoint on $D(\sqrt{-\Delta}) \cap D(\overline{H_{\mathrm{rad}}})$ by the Kato-Rellich theorem.

3 Existence and uniqueness of ground state

In this section we review the existence and the uniqueness of the ground state of $\widetilde{H}$. Let us assume that $M > 0$. In this case the existence of ground state has been shown in e.g., [8, 9] for $M > 0$ and $m \geq 0$. Now we suppose that $M = 0$. In this case Hamiltonian under consideration is of the form.

$$|p_x - A(x)| + V + \overline{H_{\mathrm{rad}}}.$$

Hence the kinetic term is not smooth function of $\frac{1}{2}(p_x - A(x))^2$, which is a serious disadvantage to show the existence of the ground state. In [2] it is shown that

$$\sigma(H) = \{E\} \cup [E + m, \infty)$$

under the assumption $\omega(k) = \sqrt{|k|^2 + m^2}$. The most singular case is $m = M = 0$, but the existence of the ground state is established in [3].

Theorem 3.1 ([3]) *Suppose that* $\omega^{3/2}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$, $\lim_{|x|\to\infty} V(x) = \infty$ *and* $\|\nabla V\|_\infty < \infty$. *Then* $\widetilde{H}$ *has the ground state.*

Proof We introduce an artificial boson mass $m > 0$. In this case the existence of normalized ground state Ψ_m is established. It is enough to show that the weak limit of Ψ_m,

$$w - \lim_{m\to 0} \Psi_m = \Psi_0,$$

is non-zero. In order to avoid infrared divergence we introduce $\widetilde{H}_R$ which is defined by $\widetilde{H}$ with $\tilde{\varphi}(\cdot - x)$ replaced by $\tilde{\varphi}(\cdot - x) - \tilde{\varphi}(\cdot)$. Note that $\widetilde{H}_R \cong \widetilde{H}$. By an application of asymptotic annihilation operator:

$$a_\infty(f) = \lim_{t\to\infty} e^{it\widetilde{H}_R} e^{-it\overline{H_{\rm rad}}} a(f) e^{it\overline{H_{\rm rad}}} e^{-it\widetilde{H}_R}$$

we can see that $a_\infty(f)\Psi_m = 0$. By the Cook method argument, we then have

$$a(f)\Psi_m = -\int_{\mathbb{R}^3} f(k)\left(\widetilde{H}_R - E + \omega(k)\right)^{-1} c_j(k)\langle x\rangle^2 \Psi_m dk,$$

where $c_j(k)$ is a bounded operator for each $k \in \mathbb{R}^3$, and $\langle x\rangle = \sqrt{|x|^2+1}$. Let N be the number operator in $\mathcal{F}$. From this formula we can see that

1. $\|N\Psi_m\| \leq C\|\langle x\rangle^2\Psi_m\|$, where C is independent of m.
2. $\sup_m \|\Psi_m^{(n)}\|_{W^{1,P}(\Omega)} < \infty$ *for* $1 \leq \forall p < 2$ and any compact set $\Omega \subset \mathbb{R}^{3n}$.
3. $\|e^{|x|}\Psi_m\| < \infty$.

By (1) - (3) above, we can see that Φ_m *strongly* converges to Φ_0. Thus $\Phi_0 \neq 0$ follows.

Finally we show the uniqueness of ground state.

Corollary 3.2 ([6, 7]) *Let* $t \geq 0$. *Then* $e^{i\frac{\pi}{2}N} e^{-t\widetilde{H}} e^{-i\frac{\pi}{2}N}$ *is positivity improving. In particular when* $\widetilde{H}$ *has a ground state, it is unique up to multiple constants.*

4 Decay of bound states

In this section we discuss a martingale property of $\widetilde{H}$, which can be applied to spatial decay of bound states. Let $h = -\frac{1}{2}\Delta + V$ be a Schrödinger operator with an external potential V. Define

$$X_t(x) = e^{tE} e^{-\int_0^t V(B_s+x)ds} f(B_t + x),$$

where f denotes a bound state such that $hf = Ef$. Then it can be checked that $(X_t(x))_{t\geq 0}$ is a martingale with respect to the natural filtration of Brownian motion. We would like to extend this to quantum field theory. Define

$$H_t(x) = e^{tE} e^{-\int_0^t V(B_{T_s}+x)ds} e^{-i\phi(K_t(x))} J_t \Phi(B_{T_t} + x)$$

where $\widetilde{H}\Phi = E\Phi$.

Theorem 4.1 ([7]) *Let V be relativistic Kato decomposable potential. Then there exists a filtration $(M_t)_{t\geq 0}$ such that $(H_t(x))_{t\geq 0}$ is a martingale, i.e.,*

$$\mathbb{E}^{0,0}\mathbb{E}_{\mu_E}[H_t(x)|M_s] = H_s(x) \ for \ t > s.$$

An application of the martingale property is to show the spatial decay of bound state Φ.

Corollary 4.2 *Let τ be a stopping time with respect to $(M_t)_{t\geq 0}$. Then*

$$\|\Phi_b(x)\|_{L^2(Q)} \leq \|\Phi_b\| \mathbb{E}^0 \left[e^{-\int_0^{t\wedge\tau}(V(Z_r+x)-E)dr} \right]$$

where $Z_t = B_{T_t}$.

Proof By Theorem 4.1 we see that $(J_0\Phi \cdot H_t(x))_{t\geq 0}$ is an $L^2(Q_E)$-valued martingale. Hence $(J_0\Phi \cdot H_{t\wedge\tau}(x))_{t\geq 0}$ is also martingale which implies that

$$\mathbb{E}^{0,0}\mathbb{E}_{\mu_E}\left[J_0\Phi \cdot H_t(x)\right] = \mathbb{E}^{0,0}\mathbb{E}_{\mu_E}\left[J_0\Phi \cdot H_{t\wedge\tau}(x)\right].$$

We have

$$\|\Phi_b(x)\|_{L^2(Q)} = \sup_{\substack{\Phi\in L^2(Q)\\ \|\Phi\|=1}} \mathbb{E}^{0,0}\mathbb{E}_{\mu_E}\left[J_0\Phi \cdot H_t(x)\right]$$
$$\leq \sup_{\substack{\Phi\in L^2(Q)\\ \|\Phi\|=1}} \mathbb{E}^{0,0}\mathbb{E}_{\mu_E}\left[J_0\Phi \cdot H_{t\wedge\tau}(x)\right] \leq \|\Phi\| \mathbb{E}^0 \left[e^{-\int_0^{t\wedge\tau} V(Z_r+x)-Edr} \right].$$

□

Spatial decay properties can be derived immediately from the lemma above.

Corollary 4.3 ([7]) *Let V be relativistic Kato decomposable.*

1. *Suppose that* $\lim_{|x|\to\infty} V(x) + E < 0$.
   $(m > 0)$ *there exists constant C such that*

$$\|\Phi_b(x)\| \leq Ce^{-c|x|}\|\Phi_b\|,$$

   $(m = 0)$ *there exists constant Csuch that*

$$\|\Phi_b(x)\| \leq \frac{C}{1+|x|^4}\|\Phi_b\|.$$

2. Suppose that $\lim_{|x|\to\infty} V(x) = \infty$. Then there exist constants c and C such that

$$\|\Phi_b(x)\| \le Ce^{-c|x|}.$$

Proof (1)We take $\tau_R = \inf\{s||z_s + x| < R\}$, which is a stopping time, and corollary follows from Corollary 4.2. (2)We take $\tau_R = \inf\{s||z_s| > R\}$ which is also a stopping time and the corollary follows from Corollary 4.2

5 Path measures associated with the ground state

For SRPF Hamiltonian we can consider the path measure associated with the ground state. The path measure of this kind is useful to study ground state expectation with respect to observables for the Nelson model and spin-boson model [1, 4] in scalar quantum field theory. Unfortunately the path measure of SRPF Hamiltonian can not be applied as those models do. In this section we only show the existence of the path measure for SRPF Hamiltonian. Let $\phi \in L^2(\mathbb{R}^3)$ be positive. Define the family of probability measures $\mu_T, T > 0$, by

$$\mu_T(A) = \frac{1}{Z_T}\int_{\mathbb{R}^3} dx \mathbb{E}^{0,0}_{\mathbb{P}\times\nu}\left[\mathbb{1}_A \phi(B_{-T_t})\phi(B_{T_t})e^{-\int_{-t}^t V(B_{T_s})ds}e^{-\frac{1}{2}\xi}\right],$$

where $\xi = \frac{1}{2}(K_t, DK_t)$ and $A \in \mathscr{G}$. Here $\mathscr{G} = \cup_{s\ge 0}\mathcal{F}_{[-s,s]}$ and $\mathcal{F}_{[-s,s]} = \sigma(Z_r : r \in [-s,s])$.

$$Z_r = B_{T_r} = \begin{cases} B_{T_r} & r > 0 \\ B_{-T_{-r}} & r < 0. \end{cases}$$

ξ plays a role of a pair interaction in a Gibbs measure, but we do not mention it here. Let $\mathfrak{X} = \Omega_p \times \Omega_\nu$.

Theorem 5.1 *There exists a probability measure $\mu_\infty(A)$ on $(\mathfrak{X}, \sigma(\mathscr{G}))$ such that*

$$\lim_{T\to\infty}\mu_T(A) = \mu_\infty(A) \quad \forall A \in \mathscr{G}.$$

Proof The main idea of the proof is to show that

$$\mu_T(A) = e^{2Es}\int_{\mathbb{R}^3} dx \mathbb{E}\left[\mathbb{1}_A\left(\frac{\phi_{T-s}(Z_{-s})}{\|\phi\|}, J_{[-s,s]}\frac{\phi_{T-s}(Z_s)}{\|\phi_T\|}\right)\right],$$

where $\phi_s = e^{-s(\widetilde{H}-E)}\phi \otimes \mathbb{1}$ and

$$J_{[-t,t]} = J^*_{-t}e^{-\int_{-t}^t V(z_s)ds}e^{-i\phi_E(K_t)}J_t.$$

Since $\widetilde{H}$ has the ground state, we can see that $\frac{\phi_{T-s}}{\|\phi_T\|} \to e^{sE}\Psi_g$ as $T \to \infty$. Hence we have

$$\lim_{T\to\infty}\mu_T(A) = e^{2Es}\int_{\mathbb{R}^3} dx \mathbb{E}^x\left[\mathbb{1}_A\left(\Psi_g(Z_{-s}), J_{[-s,s]}\Psi_g(Z_s)\right)\right]$$

and the right hand side above has the extension to the probability measure in $(\mathfrak{X}, \sigma(\mathscr{G}))$
□

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