On Kato's Inequality for the relativistic Schrödinger Operators with magenetic fields*

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This lecture deals with whether *Kato's inequality* holds for the *magnetic* relativistic Schrödinger operator H_A with vector potential A(x) and mass $m \geq 0$ associated with the classical relativistic Hamiltonian symbol $\sqrt{(\xi - A(x))^2 + m^2}$ such as

$$\mathrm{Re}[(\mathrm{sgn}\, u)H_A u] \geq \sqrt{-\Delta + m^2}\, |u|, \tag{1}$$

in the distribution sense, for u is in $L^2(\mathbf{R}^d)$ with $H_A u$ in $L^1_{\mathrm{loc}}(\mathbf{R}^d)$.

In the literature there are three magnetic relativistic Schrödinger operators associated with the classical symbol (1) (e.g. [I 12], [I 13]). The first two $H_A^{(1)}$ and $H_A^{(2)}$ are to be defined as pseudo-differential operators: for $f \in C_0^\infty(\mathbf{R}^d)$,

$$(H_A^{(1)} f)(x) := \frac{1}{(2\pi)^d} \iint_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y)\cdot\xi} \sqrt{\Big(\xi - A\big(\frac{x+y}{2}\big)\Big)^2 + m^2}\, f(y)dyd\xi, \tag{2}$$

$$(H_A^{(2)} f)(x) := \frac{1}{(2\pi)^d} \iint_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y)\cdot\xi} \sqrt{\Big(\xi - \int_0^1 A((1-\theta)x + \theta y)d\theta\Big)^2 + m^2}\, f(y)dyd\xi. \tag{3}$$

The third $H_A^{(3)}$ is defined as the square root of the nonnegative selfadjoint (nonrelativistic Schrödinger) operator $(-i\nabla - A(x))^2 + m^2$ in $L^2(\mathbf{R}^d)$:

$$H_A^{(3)} := \sqrt{(-i\nabla - A(x))^2 + m^2}. \tag{4}$$

$H_A^{(1)}$ is the so-called *Weyl* pseudo-differential operator ([ITa 86], [I 89]). $H_A^{(2)}$ is a modification of $H_A^{(1)}$ given in [IfMP 07], and $H_A^{(3)}$ used in [LSei 10] to discuss *relativistic stability of matter*.

All these three operators are nonlocal operators, and, under suitable condtion on $A(x)$, become selfadjoint. For $A = 0$ we put $H_0 = \sqrt{-\Delta + m^2}$, where $-\Delta$ is the *minus-signed* Laplacian in $\mathbf{R}^d$. $H_A^{(2)}$ and $H_A^{(3)}$ are gauge-covariant, but not $H_A^{(1)}$.

Inequality (1) for $H_A^{(1)}$ has been shown in [I 89], [ITs 76], and similarly will be for $H_A^{(2)}$.

For $H_A^{(3)}$, we assume that $d \geq 2$, as in case $d = 1$ any magnetic vector potential can be removed by a gauge tranformation. We want to show

Theorem 1 (Kato's inequality). *Let $m \geq 0$ and assume that $A \in [L^2_{\mathrm{loc}}(\mathbf{R}^d)]^d$. Then if u is in $L^2(\mathbf{R}^d)$ with $H_A^{(3)} u$ in $L^1_{\mathrm{loc}}(\mathbf{R}^d)$, then the distributional inequality holds:*

$$\mathrm{Re}[(\mathrm{sgn}\, u)H_A^{(3)} u] \geq \sqrt{-\Delta + m^2}\, |u|, \tag{5}$$

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or

$$\mathrm{Re}[(\mathrm{sgn}\, u)H_A^{(3)}u] \geq [\sqrt{-\Delta+m^2}-m]\,|u|. \tag{6}$$

Here $(\mathrm{sgn}\, u)(x) := \overline{u(x)}/|u(x)|, \text{if } u(x) \neq 0; \quad = 0, \text{if } u(x) = 0.$

From Theorem 1 follows the following corollary.

Corollary (Diamagnetic inequality) *Let* $m \geq 0$ *and assume that* $A \in [L^2_{\mathrm{loc}}(\mathbf{R}^d)]^d$. *Then* $f,\, g \in L^2(\mathbf{R}^d)$

$$|(f, e^{-t[H_A^{(3)}-m]}g)| \leq (|f|, e^{-t[H_0-m]}|g|). \tag{7}$$

Once Theorem 1 is established, we can apply it to show the following theorem on essential selfadjointness of the relativistic Schrödinger operator with both vector and scalar potentials $A(x)$ and $V(x)$:

$$H := H_A^{(3)} + V. \tag{8}$$

Theorem 2. *Let* $m \geq 0$ *and assume that* $A \in [L^2_{\mathrm{loc}}(\mathbf{R}^d)]^d$. *If* $V(x)$ *is in* $L^2_{\mathrm{loc}}(\mathbf{R}^d)$ *with* $V(x) \geq 0$ *a.e., then* $H = H_A^{(3)} + V$ *is essentially selfadjoint on* $C_0^\infty(\mathbf{R}^d)$ *and its unique selfadjoint extension is bounded below by* m.

The characteristic feature is that, unlike $H_A^{(1)}$ and $H_A^{(2)}$, $H_A^{(3)}$ is, since being defined as an operator square root (4), neither an integral operator nor a pseudo-differential operator associated with a certain tractable symbol. $H_A^{(3)}$ is, under the condition of the theorem, essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$ so that $H_A^{(3)}$ has domain containing $C_0^\infty(\mathbf{R}^d)$ as an operator core, but one does seem unabale to determine the domain of $H_A^{(3)}$. So the crucial point is in how to derive regularity of the weak solution $u \in L^2(\mathbf{R}^d)$ of equation

$$H_A^{(3)}u \equiv \sqrt{(-i\nabla - A(x))^2 + m^2}\, u = f, \qquad \text{for given } f \in L^1_{\mathrm{loc}}(\mathbf{R}^d).$$

We shall show inequality (5)/(6), modifying the method used in the case ([I 89], [ITs 92]) for the Weyl pseudo-differential operator $H_A^{(1)}$, basically along the idea of Kato's original proof for the magnetic nonrelativistic Schrödinger operator $\frac{1}{2}(-i\nabla - A(x))^2$ in [K 72]. However, the present case seems to be not so simple as to need much further modification within "operator theory *plus alpha*".

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