Spectral analysis of atoms interacting with a quantized radiation field

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Dedicated to the memory of Tosio Kato

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1 Introduction

The purpose of this talk is a review of a recent progress of the spectral analysis of a model in nonrelativistic quantum electrodynamics. A nonrelativistic electron minimally coupled (i.e., by replacing the particle momentum $\mathbf{p}$ by the covariant one $\mathbf{p} - \alpha A$) [1] with the transverse degrees of freedom of a massless quantized Maxwell field is described by "the Pauli-Fierz model"[170], which successfully gave an interpretation of "the Lamb shift" in [37, 145, 199]. In particular the ground states of the Pauli-Fierz model will be our primary concern here. The general references (books) of this talk are [22, 52, 88, 97, 99, 138, 175, 176, 177, 178, 185, 188, 139, 200].

1.1 The history of quantum filed models

We will review a history of the Pauli-Fierz type models, e.g, the Nelson model [164], spin-boson models (e.g., [149]), polaron models (e.g., [81]).

In 1937, F.Bloch and A.Nordsieck [42, 41] investigated a radiation field interacting with a classical current, and shown that the mean number of emitted quanta is infinite by an infrared divergence.

In 1938, W.Pauli and M.Fierz [170] introduced the Pauli-Fierz model.

In 1947, H.A.Bethe [37] theoretically interpreted the Lamb shift.

In 1948, T.A.Welton [199] gave an intuitive explanation of the Lamb shift.

In 1949, Z.Koba [145] extended Welton's result [199] to a relativistic model.

In 1950, R.Feynman [70] applied a path integral to a mathematical formulation of quantum electrodynamics.

In 1952, O.Miyatake [161] and L.van Hove [119] found that the ground state of a Hamiltonian in a Fock space weakly converges to zero as a cutoff is removed [2] .

In 1955, R.Feynman [71] applied a path integral to estimate the ground state energy of a polaron model.

[1] See e.g., [28, 29, 30, 78, 153, 157] for a classical Pauli operator $(\mathbf{p} - A)^2 + V + \sigma \cdot B$.

[2] Some relation between the van Hove-Miyatake phenomenon and an infrared divergence is discussed by H.Ezawa [64]. The Hida space $(\mathcal{S})^*$ ([98]) is a dual space of a subspace $(\mathcal{S})$ in a Fock space. The van Hove-Miyatake phenomenon is investigated by J.Potthoff and L.Streit [171] in the Hida space. However the phenomenon still survives in the Hida space.

In 1958, E.Lieb and K.Yamazaki gave estimates of the ground state energy and some ground state expectation values of a polaron model in [156].

In 1962, D.Shale [182] obtained a mathematical manner to study both of the infrared and ultraviolet divergences.

In 1963, Y.Kato and N.Mugibayashi [143] constructed asymptotic fields and were concerned with the spectrum of a Hamiltonian. E.Nelson [162, 163] examined Feynman's result [70] in a simple model but with a mathematical rigorous manner.[3]

In 1964, E.Nelson [164] introduced a model of nonrelativistic quantum particles linearly coupled with scalar bosons, so called "the Nelson model", and he renormalized its Hamiltonian.

In 1968-1969, R.Høegh-Krohn applied the Kato-Mugibayashi scattering theory [143] to the Nelson model in [120]-[122], and extended the work to general models in [123]-[125].

In 1968-1972, J.Glimm and A.Jaffe analyzed the ground state properties of a quantum field model ($\lambda\phi^4$-model) from the point of view of *the constructive quantum field theory* in the series of papers [84]-[87] [4] (see books e.g., [22, 88, 185]).

In 1969, P.Blanchard [40] were concerned with asymptotics of the Pauli-Fierz model with *the dipole approximation* and discussed an infrared divergence.

In 1970, I.Segal [180, 181] proved the essential self-adjointness and the indecomposability of a quantum field Hamiltonian. J.P. Eckmann [60] renormalized the Nelson model with relativistic kinematics (Eckmann's model).

In 1971, J.Cannon [44] studied the quantum field theoretical property (Wightman functionals,etc.) of the Nelson model. L.Gross [90] proved the existence and uniqueness of the ground state of relativistic and nonrelativistic polaron models for *zero* total-momentum.

In 1972, L.Gross [91] studied the massive Nelson model with relativistic kinematics (Eckmann's model) and constructed a Hilbert space on which a self-adjoint operator without an ultraviolet cutoff acts. S.Albeverio [2, 3] was concerned with the scattering theory of Eckmann's model.

In 1973, E. Nelson [165, 166] constructed a quantum field from a Markov field.

[3] E.Nelson established imaginary-time path integrals. T.Ichinose and H.Tamura [132, 133] constructed a distribution-valued countably additive measure presenting a real-time evolution of a Dirac Hamiltonian in two space-time dimensions.

[4] From the point of view of the constructive quantum field theory, K.R. Ito [134, 135], and D.Brydges, J.Fröhlich and E.Seiler [43] considered QED in two space-time dimensions.

Hilbert space	$L^2(\mathbf{R}^d) \otimes \underbrace{\mathcal{F} \otimes \cdots \otimes \mathcal{F}}_{d-1}$
Decoupled Hamiltonian	$(-(1/2)\Delta + V) \otimes \mathbf{1} + \mathbf{1} \otimes H_{\rm f}$
Free Hamiltonian	$H_{\rm f} = \int \omega(k) a^{r\dagger}(k) a^r(k) dk$
Dispersion relation	$\omega(k) = \lvert k \rvert$
Quantized field	$A_\mu(\hat\lambda, x) = (1/\sqrt{2}) \int a^{r\dagger}(k)\hat\lambda(-k) e^r_\mu(k) e^{-ikx} + a^r(k)\hat\lambda(k) e^r_\mu(k) e^{ikx} dk$
Canonical pair	$\Pi_\mu(\hat\lambda, x) = i(1/\sqrt{2}) \int a^{r\dagger}(k)\hat\lambda(-k) e^r_\mu(k) e^{-ikx} - a^r(k)\hat\lambda(k) e^r_\mu(k) e^{ikx} dk$
CCR	$[A_\mu(\hat\lambda, x), \Pi_\nu(\hat\rho, x)] = i(\overline{d_{\mu\nu}\hat\lambda}, \hat\rho), \quad d_{\mu\nu}(k) = 1 - k_\mu k_\nu/\lvert k \rvert^2$
Total Hamiltonian	$(1/2)\left(\mathbf{p} \otimes \mathbf{1} - \alpha A(\hat\lambda, x)\right)^2 + V \otimes \mathbf{1} + \mathbf{1} \otimes H_{\rm f}$
Self-adjointness	Essentially self-adjoint on $D(\Delta) \cap D(H_{\rm f})$ for all $\alpha \in \mathbf{R}$
Ground state $\Psi_{\rm g}$	Exists for $\lvert\alpha\rvert \ll 1$ and is unique
Particle-localization	$\lVert \Psi_{\rm g}(x) \rVert \le D e^{-\delta \lvert x \rvert}$
Boson-localization	$(\Psi_{\rm g}, e^{\beta N} \Psi_{\rm g}) < \infty$?, $\quad \beta > 0$
Finite-time Gibbs meas.	$f(X_0) f(X_{2t}) e^{-\int_0^{2t} V(X_s) ds} e^{-(\alpha^2/4) \int_0^{2t} d\mathbf{b}_\mu(s) \int_0^{2t} d\mathbf{b}_\nu(s') W_{\mu\nu}(X_s - X_{s'}, s-s')} dX$
Pair potential	$W_{\mu\nu}(X,t) = \int_{\mathbf{R}^d} d_{\mu\nu}(k) \lvert\hat\lambda(k)\rvert^2 e^{ikX} e^{-\lvert t\rvert \omega(k)} dk$
Infinite-time Gibbs meas.	Exist
Diamagnetic inequality	$\lvert \left(\Psi, e^{-tH}\Phi\right) \rvert \le \left(\lVert\Psi\rVert, e^{-t(-(1/2)\Delta + V)} \lVert\Phi\rVert\right)$
Stability	$\inf \sigma(-(1/2)\Delta + V) \le \inf \sigma(H)$

Table 1: The spinless one-particle Pauli-Fierz model

Hilbert space	$L^2(\mathbf{R}^d)\otimes\mathcal{F}$
Decoupled Hamiltonian	$(-(1/2)\Delta+V)\otimes\mathbf{1}+\mathbf{1}\otimes H_{\mathrm{f}}^{\mathrm{N}}$
Free Hamiltonian	$H_{\mathrm{f}}^{\mathrm{N}}:=\int\omega(k)a^\dagger(k)a(k)dk$
Dispersion relation	$\omega(k)=\sqrt{\|k\|^2+m^2}$, $m\geq 0$
Quantized field	$\phi(\hat\lambda,x)=(1/\sqrt2)\int a^\dagger(k)\hat\lambda(-k)e^{-ikx}+a(k)\hat\lambda(k)e^{ikx}dk$
Canonical pair	$\pi(\hat\lambda,x)=i(1/\sqrt2)\int a^\dagger(k)\hat\lambda(-k)e^{-ikx}-a(k)\hat\lambda(k)e^{ikx}dk$
CCR	$[\phi(\hat\lambda),\pi(\hat\rho)]=(\overline{\hat\lambda},\hat\rho)$
Total Hamiltonian	$(-(1/2)\Delta+V)\otimes\mathbf{1}+\alpha\phi(\hat\lambda,x)+\mathbf{1}\otimes H_{\mathrm{f}}^{\mathrm{N}}$
Self-adjointness	Self-adjoint on $D(\Delta)\cap D(H_{\mathrm{f}})$ for all $\alpha\in\mathbf{R}$
Ground state Ψ_{g}	Exists for all $\alpha\in\mathbf{R}$ and is unique
Particle localization	$\\|\Psi_{\mathrm{g}}(x)\\|\leq De^{-\delta\|x\|}$
Boson localization	$(\Psi_{\mathrm{g}},e^{\beta N}\Psi_{\mathrm{g}})<\infty$ for all $\beta\in\mathbf{R}$
Finite-time Gibbs meas.	$f(X_0)f(X_{2t})e^{-\int_0^t V(X_s)ds}e^{(\alpha^2/4)\int_0^{2t}ds\int_0^{2t}ds' W(X_s-X_{s'},s-s')}dX$
Pair potential	$W(X,t)=\int_{\mathbf{R}^d}\|\hat\lambda(k)\|^2e^{ikX}e^{-\|t\|\omega(k)}dk$
Infinite-time Gibbs meas.	Exist
Diamagnetic inequality	$\|\left(\Psi,e^{-tH}\Phi\right)\|\leq\left(\\|\Psi\\|,e^{-t(-1/2\Delta+V-\alpha^2\\|\hat\lambda/\sqrt\omega\\|^2)}\\|\Phi\\|\right)$
Stability	$\inf\sigma(-(1/2)\Delta+V)\leq\inf\sigma(H)+(\alpha^2/2)\\|\hat\lambda/\sqrt\omega\\|^2$

Table 2: The one-particle Nelson model

The Pauli-Fierz polaron model	$H(p) = (1/2)\left(p - \mathbf{P}_{\rm f} - \alpha A(\hat{\lambda}, 0)\right)^2 + H_{\rm f}, \quad p \in \mathbf{R}^d$
Field momentum	$\mathbf{P}_{\rm f} = \int k a^{r\dagger}(k) a^r(k) dk$
The Nelson polaron model	$H(p) = (1/2)(p - \mathbf{P}_{\rm f}^{\rm N})^2 + \alpha\phi(\hat{\lambda}, 0) + H_{\rm f}^{\rm N}, \quad p \in \mathbf{R}^d$
Field momentum	$\mathbf{P}_{\rm f}^{\rm N} = \int k a^\dagger(k) a(k) dk$

Table 3: Polaron models

In 1973-1974, J.Fröhlich investigated an infrared divergence of a polaron model in [74, 75]. He also shown the existence and uniqueness of the ground state of a polaron model *without* an ultraviolet cutoff for sufficiently small total momentum.

In 1976, K.Rzazewski and W.Zakowicz [179] solved an initial value problem of the Pauli-Fierz model with the dipole approximation and an x^2-potential.

In 1978-1980, J.Fröhlich and Y.M.Park [79, 80] opened a problem on the analysis of nonrelativistic quantum electrodynamics.

In 1980, A.Grossmann and A.Tip [93] studied a resonance of a single mode Pauli-Fierz model with the dipole approximation and an x^2-potential.

In 1981-2000(!), A.Arai gave a firm mathematical base on the Pauli-Fierz model. The first mathematical rigorous results on the model were, as far as we know, due to A.Arai. He investigated the model with *the dipole approximation* in the series of papers [7]-[18], and shown that the model was exactly solvable, i.e., he obtained the self-adjointness of the Hamiltonian, the existence and uniqueness of its ground state, asymptotic completeness, the instability of its embedded eigenvalues (resonance), scaling limits, and long-time behaviors of a two-point function, etc.

In 1983, M.D.Donsker and S.R.S.Varadhan [58] obtained, independently of the existence of the ground states, asymptotics of the ground state energy of a polaron model as the coupling constant tends to infinity, by means of a large deviation theory of path integrals.

In 1985, T.Okamoto and K.Yajima [167] shown the existence of a resonance of the *massive* Pauli-Fierz model in terms of a complex scaling technique ([5]).

In 1986, H.Spohn proved the existence of the ground state [193] and its localization [192] of a polaron model for *arbitrary* values of total momentum for one or two dimensions. He also considered an effective mass in [197].

In 1989, H.Spohn [195] investigated the ground state properties of a spin-boson model, in which he proved the existence of the ground states of the spin-boson model and shown its localization [5]. The work has been continued by H.Spohn, R.Stückl and W.Wreszinski in [198] to generalized versions: "J-spin boson models".

In 1995, M.Hübner and H.Spohn [128, 129] studied a resonance of the spin-boson model with a help of a modification of a positive commutator method. For the Pauli-Fierz model with a *confined* external potential and *sufficiently small* coupling constants, V.Bach, J.Fröhlich and I.E.Sigal [31] proved the existence of a ground state, its particle localization, and the existence of resonance poles, by means of a renormalization group method. The full papers [32, 33] were published in 1998.

In 1996 A.Arai and M.Hirokawa proved the existence of the ground state of a spin-boson model for *sufficiently small* coupling constants in [23], and extended this to a generalized version in [24][6].

In 1996-1997, C.Fefferman, J.Fröhlich and J.M.Graf [72, 73] considered the stability of the Pauli-Fierz model and gave a lower bound of its ground state energy.

In 1997, H.Spohn [194] shown the asymptotic completeness of the Pauli-Fierz model with the dipole approximation and *non* x^2-potentials. E.Lieb and L.E.Thomas [155] gave an alternative simple proof of the asymptotics of the ground state energy of a polaron model given by Donsker and Varadhan [58].

In 1998, H.Spohn [197] proved the existence of the ground state of the Nelson model for *arbitrary* coupling constants by a functional integral method. After [197], C.Gérard [83] proved the same thing as that of [197] with some generalization in an entirely different way. V.Bach, J.Fröhlich and I.E.Sigal [34] proved the existence of the ground states of the Pauli-Fierz model *without* an infrared cutoff and with Coulomb potentials (cf. F.Hiroshima [109, 113]), and they shown that the spectrum of the model was *purely* absolutely continuous except in small neighborhood of the ground state energy and the ionization thresholds. See also [35].

In 1999, E.Lieb and M.Loss [154] contributed to estimate both of upper and lower bounds of the ground state energy of the Pauli-Fierz model. R. Minlos and H.Spohn [160] proved the absence of the ground states of the Nelson model *with* an

[5] The problem whether the ground state of the spin-boson model exists or not in the original Hilbert space had not yet been solved in [195]. H.Spohn actually shown the existence of its ground states in the Hilbert space in his "unpublished note" dated 26 June 1989!

[6] GSB-model.

infrared divergence[7].

In 2000, A.Arai [19] proved *independently of the existence of the ground states* that the essential spectrum of the Pauli-Fierz model coincided with its spectrum.[8] F.Hiroshima proved the essential self-adjointness of the Pauli-Fierz model for *arbitrary* coupling constants in [112], and he also shown the uniqueness of its ground state in [110]. V.Betz, F.Hiroshima, J.Lőrinczi, R.Minlos and H.Spohn [111, 38] constructed an infinite-time Gibbs measure associated with the Nelson model and shown the boson localization of its ground state for *arbitrary* coupling constants. F.Hiroshima and H. Spohn [117] shown a binding through an interaction between a particle and a quantum field for the Pauli-Fierz model with the dipole approximation and *shallow* potentials[9]. Recently, M.Griesemer, E.Lieb and M.Loss [89] address that the ground state of the Pauli-Fierz model exists for *arbitrary* coupling constants!

2 The Pauli-Fierz model

2.1 A Fock-Cook representation

We start with introducing some basic facts of a quantum field often used in this talk. We define the Boson Fock space over $L^2(\mathbf{R}^d)$ by

$$\mathcal{F} := \mathcal{F}(L^2(\mathbf{R}^d)) := \oplus_{n=0}^{\infty} \left(\otimes_s^n L^2(\mathbf{R}^d) \right),$$

where $\otimes_s^0 L^2(\mathbf{R}^d) := \mathbf{C}$ and $\otimes_s^n L^2(\mathbf{R}^d)$ denotes the symmetric tensor product of $L^2(\mathbf{R}^d)$, i.e., $f \in \otimes_s^n L^2(\mathbf{R}^d)$ if and only if $f \in L^2(\underbrace{\mathbf{R}^d \times \cdots \times \mathbf{R}^d}_{n})$ and

$$f(k_1, \cdots, k_i, \cdots, k_j, \cdots, k_n) = f(k_1, \cdots, k_j, \cdots, k_i, \cdots, k_n), \quad 1 \le i, j \le n.$$

The creation operator $a^\dagger(f)$ and the annihilation operator $a(f)$ smeared by $f \in L^2(\mathbf{R}^d)$ are defined by, for $\Psi = \oplus_{n=0}^{\infty} \Psi^{(n)} \in \mathcal{F}$,

$$\left(a^\dagger(f)\Psi \right)^{(n)} (k_1, \cdots, k_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(k_j) \Psi^{(n-1)}(k_1, \cdots, \widehat{k_j}, \cdots, k_n),$$

[7] A.Arai and M.Hirokawa [25] gave a sufficient condition for the existence of the ground states in some domain with an infrared divergence for a GSB model. See also A.Arai, M.Hirokawa and F.Hiroshima [27].

[8] Asymptotic completeness for the *massive* Nelson model is established in [55, 6].

[9] A.Arai and M.Hirokawa [26] found a non-perturbative ground state of the Wigner-Weisskopf model for *large* coupling constants. Also see [101].

$$\left(a(f)\Psi\right)^{(n)}(k_1,\cdots,k_n)=\sqrt{n+1}\sum_{j=1}^{n}\int_{\mathbf{R}^d} f(k)\Psi^{(n+1)}(k_1,\cdots,\overset{j}{\hat{k}},\cdots,k_n)dk,$$

where $\hat{}$ denotes neglecting the term[10]. Let $\Omega_{\rm b} := 1\oplus 0\oplus 0\cdots\in\mathcal{F}$ be the bare vacuum. It is well known that

$$\mathcal{F}_0 := \mathcal{L}\left\{a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega_{\rm b},\Omega_{\rm b}\,\middle|\, f_j\in L^2(\mathbf{R}^d), j=1,...,n, n\in\mathbf{N}\right\}$$

is dense in $\mathcal{F}$, where $\mathcal{L}\{\cdots\}$ denotes the finite linear hull of vectors in $\{\cdots\}$. Moreover $\mathcal{F}_0$ is an invariant subspace of $a^\sharp = a^\dagger$ or a. $a^\sharp$ obeys the canonical commutation relations on $\mathcal{F}_0$, i.e.,

$$[a(f),a^\dagger(g)] = (\overline{f},g)_{L^2(\mathbf{R}^d)},\quad [a^\sharp(f),a^\sharp(g)]=0,$$

where $(f,g)_K$ (resp. $\|f\|_K$) denotes the scalar product (resp. the norm) of Hilbert space K. We omit K in $(f,g)_K$ unless no confusion may arise. Note that $(f,g)_K$ is linear in g and antilinear in f. $a^\sharp$ satisfies that

$$(a(f)\Psi,\Phi)_{\mathcal{F}} = (\Psi,a^\dagger(\overline{f})\Phi)_{\mathcal{F}}$$

for $\Psi,\Phi\in\mathcal{F}_0$. We define

$$\mathcal{F}_{\rm EM} := \underbrace{\mathcal{F}\otimes\cdots\otimes\mathcal{F}}_{d-1},\quad \mathcal{F}_{\rm EM0} := \underbrace{\mathcal{F}_0\hat{\otimes}\cdots\hat{\otimes}\mathcal{F}_0}_{d-1},$$

where $\hat{\otimes}$ denotes an algebraic tensor product, and $a^{r\sharp}:\mathcal{F}_{\rm EM}\to\mathcal{F}_{\rm EM}$ is defined by

$$a^{r\sharp}(f) := \underbrace{\mathbf{1}\otimes\cdots\otimes\overbrace{a^\sharp(f)}^{r}\otimes\cdots\otimes\mathbf{1}}_{d-1},\quad r=1,...,d-1.$$

It obeys that, on $\mathcal{F}_{\rm EM0}$,

$$[a^{r\dagger}(f),a^{s\dagger}(g)] = \delta_{rs}(\overline{f},g),\quad [a^{r\sharp}(f),a^{s\sharp}(g)]=0.$$

We denote by the same symbol $a^\sharp$ its closed extension. The vectors

$$e^r(k) := (e_1^r(k),\cdots,e_d^r(k)),\quad r=1,...,d-1,$$

are $d-1$ possible orthonormal polarization vectors perpendicular to k, i.e.,

$$e^r(k)\cdot e^s(k)=\delta_{rs},\quad e^r(k)\cdot k=0,\quad \text{a.e.}k\in\mathbf{R}^d.$$

[10]Formally we write $a^\sharp(f)=\int a^\sharp(k)f(k)dk$.

Note that[11]

$$d_{\mu\nu}(k) := e^r_\mu(k)e^r_\nu(k) = \delta_{\mu\nu} - (k_\mu k_\nu)/|k|^2.$$

We define a quantized radiation field $A_\mu(\hat{\lambda})$ by

$$A_\mu(\hat{\lambda}) := A_\mu(\hat{\lambda}, x) := \frac{1}{\sqrt{2}}\left\{a^{r\dagger}(e^r_\mu e^{-ikx}\tilde{\hat{\lambda}}) + a^r(e^r_\mu e^{ikx}\hat{\lambda})\right\}, \quad \mu = 1, ..., d,$$

and its canonical pair $\Pi_\mu(\hat{\lambda})$ by

$$\Pi_\mu(\hat{\lambda}) := \Pi(\hat{\lambda}, x) := i\frac{1}{\sqrt{2}}\left\{a^{r\dagger}(e^r_\mu e^{-ikx}\tilde{\hat{\lambda}}) - a^r(e^r_\mu e^{ikx}\hat{\lambda})\right\}, \quad \mu = 1, ..., d,$$

where $\widetilde{g}(k) := g(-k)$ and $\widehat{g}$ denotes the Fourier transform of g. Note that

$$\mathrm{div}A(\hat{\lambda}) = \sum_{\mu=1}^{d}[\mathbf{p}_\mu, A_\mu(\hat{\lambda})] = 0, \quad \text{(the Coulomb gauge)},$$

on some domain. It is checked that [12]

$$[A_\mu(\hat{\lambda}), \Pi_\nu(\hat{\rho})] = i(\overline{d_{\mu\nu}\hat{\lambda}}, \hat{\rho}),$$

$$[A_\mu(\hat{\lambda}), A_\nu(\hat{\rho})] = [\Pi_\mu(\hat{\lambda}), \Pi_\nu(\hat{\rho})] = 0,$$

on $\mathcal{F}_{\mathrm{EM}0}$ and

$$(A_\mu(\hat{\lambda})\Omega_{\mathrm{b}}, A_\nu(\hat{\rho})\Omega_{\mathrm{b}}) = \frac{1}{2}(d_{\mu\nu}\hat{\lambda}, \hat{\rho}) = (\Pi_\mu(\hat{\lambda})\Omega_{\mathrm{b}}, \Pi_\nu(\hat{\rho})\Omega_{\mathrm{b}}).$$

Throughout this talk we assume that

$$\hat{\lambda}(-k) = \overline{\hat{\lambda}(k)}, \tag{2.1}$$

namely, λ is real. This assumption ensures that both of $A_\mu(\hat{\lambda})$ and $\Pi_\nu(\hat{\lambda})$ are symmetric operators.

[11] The summation of repeated indexes are understood (the Einstein rule).

[12] Formally one writes

$$[A_\mu(k), \Pi_\nu(k')] = i(\delta_{\mu\nu} - k_\mu k_\nu/|k|^2)\delta(k - k') \quad \text{or} \quad [A_\mu(x), \Pi_\nu(y)] = i(\delta_{\mu\nu} - \partial_\mu\partial_\nu/|x - y|)\delta(x - y).$$

2.2 The second quantization

Let h be a self-adjoint operator of $L^2(\mathbf{R}^d)$. Define $S_t : \mathcal{F} \to \mathcal{F}$, $t \in \mathbf{R}$, by

$$S_t a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega_{\rm b} := a^\dagger(e^{ith}f_1)\cdots a^\dagger(e^{ith}f_n)\Omega_{\rm b}, \quad S_t\Omega_{\rm b} := \Omega_{\rm b}.$$

It is seen that $\underbrace{S_t \otimes \cdots \otimes S_t}_{d-1}$, $t \in \mathbf{R}$, is a strongly continuous one-parameter unitary group on $\mathcal{F}_{\rm EM}$. Thus there exists a self-adjoint operator $d\Gamma_{\rm b}(h)$ in $\mathcal{F}_{\rm EM}$ such that

$$\underbrace{S_t \otimes \cdots \otimes S_t}_{d} = e^{itd\Gamma_{\rm b}(h)}, \quad t \in \mathbf{R}.$$

We call $d\Gamma_{\rm b}(h)$ "the second quantization" [49] of h. Actually $d\Gamma_{\rm b}(h)$ acts [13] as follows:

$$d\Gamma_{\rm b}(h)\Omega_{\rm b} = 0,$$

$$d\Gamma_{\rm b}(h)a^{\dagger r_1}(f_1)\cdots a^{\dagger r_n}(f_n)\Omega_{\rm b} = \sum_{j=1}^n a^{\dagger r_1}(f_1)\cdots a^{\dagger r_j}(hf_j)\cdots a^{\dagger r_n}(f_n)\Omega_{\rm b}.$$

Let

$$\omega_\mu(k) := \sqrt{|k|^2+\mu^2}, \quad \mu \ge 0,$$

and define the free Hamiltonian in $\mathcal{F}_{\rm EM}$ by [14]

$$H_{\rm b} := d\Gamma_{\rm b}(\omega_\mu).$$

It is known that[15]

$$\sigma(H_{\rm b}) = \{0\} \cup [\mu, \infty), \quad \sigma_{\rm p}(H_{\rm b}) = \{0\}, \quad \sigma_{\rm ess}(H_{\rm b}) = [\mu, \infty),$$

and $\{0\}$ is of multiplicity one and

$$H_{\rm b}\Omega_{\rm b} = 0.$$

In what follows we assume that $\mu = 0$ and set

$$\omega := \omega_0.$$

[13] $d\Gamma_{\rm b}(h) = \oplus_{n=0}^\infty \sum_{j=1}^n \underbrace{\mathbf{1} \otimes \cdots \overbrace{h}^{j} \cdots \otimes \mathbf{1}}_{n}$.

[14] Formally $H_{\rm b}$ is written as $H_{\rm b} = \int \omega_m(k)a^{r\dagger}(k)a^r(k)dk$.

[15] $\sigma(T)$:the spectrum of T, $\sigma_{\rm ess}(T)$:the essential spectrum of T, $\sigma_{\rm disc}(T)$:the discrete spectrum of T, $\sigma_{\rm p}(T)$:the point spectrum of T, $\sigma_{\rm ac}(T)$:the absolutely continuous spectrum of T.

It is a direct calculation that

$$e^{itd\Gamma_{\rm b}(h)}a^{r\dagger}(f)e^{-itd\Gamma_{\rm b}(h)} = a^{r\dagger}(e^{ith}f),$$

$$e^{itd\Gamma_{\rm b}(h)}a^{r}(f)e^{-itd\Gamma_{\rm b}(h)} = a^{r}(e^{-ith}f).$$

In particular, for $N_{\rm b} := d\Gamma_{\rm b}(\mathbf{1})$,

$$e^{i\pi/2N_{\rm b}}a^{r\dagger}(f)e^{-i\pi/2N_{\rm b}} = ia^{r\dagger}(f), \tag{2.2}$$

$$e^{i\pi/2d\Gamma_{\rm b}(h)}a^{r}(f)e^{-i\pi/2d\Gamma_{\rm b}(h)} = -ia^{r}(f). \tag{2.3}$$

Operator $N_{\rm b}$ is called the number operator. From (2.2) and (2.3) it follows that

$$e^{i\pi/2N_{\rm b}}A_\mu(\hat{\lambda})e^{-i\pi/2N_{\rm b}} = \Pi_\mu(\hat{\lambda}), \quad \mu = 1, ..., d. \tag{2.4}$$

For later convenience, we introduce some fundamental inequalities:

$$\|a^{r\dagger}(f)\Psi\| \le \|f\|\|\Psi\| + \|f/\sqrt{\omega}\|\|H_{\rm b}^{1/2}\Psi\|, \tag{2.5}$$

$$\|a^{r}(f)\Psi\| \le \|f/\sqrt{\omega}\|\|H_{\rm b}^{1/2}\Psi\|, \tag{2.6}$$

for [16] $\Psi \in D(H_{\rm b}^{1/2})$ and

$$\|a^{r\sharp}(f)a^{r\sharp}(f)\Psi\| \le (\|f/\sqrt{\omega}\| + \|f\|)(\|f\sqrt{\omega}\| + \|f\| + \|\sqrt{\omega}f\| + \|\omega f\|)\|(H_{\rm b} + \mathbf{1})\Psi\|, \tag{2.7}$$

for $\Psi \in D(H_{\rm b})$ ([13]). Moreover

$$\|a^{r\dagger}(f)\Psi\| \le \|f\|(\|\Psi\| + \|N_{\rm b}^{1/2}\Psi\|), \tag{2.8}$$

$$\|a^{r}(f)\Psi\| \le \|f\|\|N_{\rm b}^{1/2}\Psi\|, \tag{2.9}$$

for $\Psi \in D(N_{\rm b}^{1/2})$.

2.3 The definition of the Pauli-Fierz Hamiltonian

Let

$$\mathcal{H}_{\rm b} := L^2(\mathbf{R}^d) \otimes \mathcal{F} \cong \int_{\mathbf{R}^d}^{\oplus} \mathcal{F} dx.$$

Here $L^2(\mathbf{R}^d)$ accommodates the state space of the electron moving in d-dimensional space and $\mathcal{F}$ that of bosons (photons). Define

$$A_\mu := \int_{\mathbf{R}^d}^{\oplus} A_\mu(\hat{\lambda}, x)dx, \quad \mu = 1, ..., d.$$

[16] $D(T)$ denotes the domain of T.

The Pauli-Fierz Hamiltonian $H_{\rm PF}$ is defined as a densely defined symmetric operator acting in $\mathcal{H}_{\rm b}$ by

$$H_{\rm PF} := \frac{1}{2M}(\mathbf{p}\otimes\mathbf{1}-\alpha A)^2 + V\otimes\mathbf{1}+\mathbf{1}\otimes H_{\rm b},$$

where M is the mass of the electron, α a coupling constant[17]and we work with a unit $\hbar = c = 1$ [18]. For simplicity we set $M = 1$. $\hat\lambda$ serves as *an ultraviolet cutoff.* A physically reasonable choice of λ is

$$\hat\lambda(k) = \hat\rho(k)/\sqrt{(2\pi)^d\omega(k)},$$

where ρ is a charge density, i.e.,

$$\alpha = -\int_{\mathbf{R}^d}\rho(x)dx, \quad \rho(x)\geq 0. \tag{2.10}$$

In particular for $d = 3$,

$$\int_{\mathbf{R}^3}\frac{\hat\lambda(k)^2}{\omega(k)^2}dk < \infty \tag{2.11}$$

implies that

$$0 = \sqrt{(2\pi)^3}\hat\rho(0) = \int_{\mathbf{R}^3}\rho(x)dx = -\alpha.$$

We call (2.11) *infrared cutoff condition.* Throughout this talk we do *not* impose (2.10).

[17]Physically $\alpha = -\sqrt{1/137}$ with a unit $\hbar = c = 1$

[18] Actually $H_{\rm PF}$ is a Hamiltonian reduced by "the one-particle sector". Define the antisymmetric Fock space by $\mathcal{F}_{\rm as} := \oplus_{n=0}^\infty\left(\otimes_{as}^n L^2(\mathbf{R}^d)\right)$, where $\otimes_{as}^n L^2(\mathbf{R}^d)$ denotes the n-fold antisymmetric tensor product of $L^2(\mathbf{R}^d)$. Set $\mathcal{H}_{\rm T} := \mathcal{F}_{\rm as}\otimes\mathcal{F}_{\rm EM}$. Then

$$\mathcal{H}_{\rm T} = \oplus_{Z=0}^\infty\mathcal{H}^{\rm Z}, \quad \mathcal{H}^{\rm Z} := (\otimes_{as}^Z L^2(\mathbf{R}^d))\otimes\mathcal{F}\cong L^2_{as}(\mathbf{R}^{dZ})\otimes\mathcal{F}_{\rm EM}.$$

Let $\Psi(x)$ and $\Psi^\dagger(x)$ be formal kernels of the annihilation operator and the creation operator in $\mathcal{F}_{\rm as}$, respectively, i.e., anticommutation relations $\{\Psi(x),\Psi^\dagger(y)\} = \delta(x-y)$ holds. The total Hamiltonian H is defined on $\mathcal{H}_{\rm T}$ by

$$\begin{aligned} H := &\frac{1}{2}\int\Psi^\dagger(x)\left(\mathbf{p}-\alpha A(\hat\lambda,x)\right)^2\Psi(x)dx \\ &+\int\omega(k)a^{r\dagger}(k)a^r(k)dk + \alpha^2\int\Psi^\dagger(x)\Psi^\dagger(y)V(x-y)\Psi(x)\Psi(y)dxdy,\end{aligned}$$

where $V(x) = -1/(4\pi|x|)$. Thus it follows that

$$H\lceil_{\mathcal{H}^1} = H_{\rm PF},$$

$$H\lceil_{\mathcal{H}^Z} = \frac{1}{2}\sum_{j=1}^Z\left(\mathbf{p}_j-\alpha A(\hat\lambda,x_j)\right)^2 + H_{\rm f} - \alpha^2\sum_{i\neq j}^Z\frac{1}{4\pi|x_i-x_j|}, \quad Z\geq 2.$$

When $Z\geq 2$, a longitudinal interaction (a Coulomb potential) does appear.

2.4 Self-adjointness for $|\alpha| \ll 1$

We abbreviate $\mathbf{1} \otimes X$ and $X \otimes \mathbf{1}$ by X unless no confusion arise. The Pauli-Fierz Hamiltonian is written as

$$H_{\mathrm{PF}} = H_{\mathrm{p}} + H_{\mathrm{b}} + \alpha H_{\mathrm{I}},$$

where

$$H_{\mathrm{p}} := -\Delta/2 + V, \quad H_{\mathrm{I}} := -\mathbf{p}A + \alpha A^2.$$

Assume that

$$\|\Delta f\| \le a\|H_{\mathrm{p}} f\| + b\|f\| \tag{2.12}$$

for $f \in D(H_{\mathrm{p}})$ with some constants a and b. Let $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \sqrt{\omega}\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$. Then, by the fundamental inequalities (2.5), (2.6) and (2.7), we easily have

$$\|\mathbf{p}A\Psi\| \le C_1\|(H_{\mathrm{p}} + H_{\mathrm{b}} + \mathbf{1})\Psi\|, \tag{2.13}$$

$$\|A^2\Psi\| \le C_2\|(H_{\mathrm{b}} + \mathbf{1})\Psi\| \tag{2.14}$$

with some constants C_1 and C_2 for $\Psi \in D(H_{\mathrm{p}}) \cap D(H_{\mathrm{b}})$.

Proposition 2.1 ([167]) *Let $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \sqrt{\omega}\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$ and $|\alpha|$ be sufficiently small. Assume (2.12). Then H_{PF} is self-adjoint on $D(H_{\mathrm{p}}) \cap D(H_{\mathrm{b}})$, bounded below, and essentially self-adjoint on any core of $H_{\mathrm{p}} + H_{\mathrm{b}}$.*

Proof: By virtue of (2.13) and (2.14), we have

$$\|H_{\mathrm{I}}\Psi\| \le C'\|(H_{\mathrm{p}} + H_{\mathrm{b}})\Psi\| + C''\|\Psi\|$$

with some constants C' and C''. The proposition follows from the Kato-Rellich theorem and the fact that $D(H_{\mathrm{p}} + H_{\mathrm{b}}) = D(H_{\mathrm{p}}) \cap D(H_{\mathrm{b}})$. QED

2.5 Problems of embedded eigenvalues and binding through a coupling

Here we state the purpose of this talk. The decoupled Hamiltonian ($\alpha = 0$) is denoted by

$$H_{\mathrm{d}} := H_{\mathrm{p}} + H_{\mathrm{b}}.$$

First we let

$$\sigma(H_{\rm p}) = \{E_j\}_{j=0}^N \cup [\Sigma, \infty), \quad E_0 \le E_1 \le \cdots < \Sigma.$$

Then

$$\sigma(H_{\rm d}) = [E_0, \infty), \quad \sigma_{\rm p}(H_{\rm d}) = \{E_j\}_{j=0}^N.$$

Thus all the point spectra of $H_{\rm d}$ are embedded in the continuous spectrum. We can say that the spectral analysis of $H = H_{\rm d} + \alpha H_{\rm I}$ is a problem of a perturbation of embedded point spectra. We will see that, under some condition, the point spectrum E_0 survives after adding the perturbation $\alpha H_{\rm I}$. See Section 6.

Secondly we assume that

$$\sigma(H_{\rm p}) = [0, \infty), \quad \sigma_{\rm p}(H_{\rm p}) = \emptyset.$$

Then

$$\sigma(H_{\rm d}) = [0, \infty), \quad \sigma_{\rm p}(H_{\rm d}) = \emptyset.$$

Our question is as follows: does there exist the ground state of $H = H_{\rm d} + \alpha H_{\rm I}$ for some $\alpha > 0$? The answer is YES. As heuristic level one argues that the coupling to the radiation field amounts to renormalizing a bare mass M to an "effective" mass $M(\alpha^2)$ with $M(\alpha^2)$ increasing in α^2.Thus effectively instead of $H_{\rm p} = -\Delta/(2M) + V$ we should consider

$$-\Delta/(2M(\alpha^2)) + V. \tag{2.15}$$

Hence a bound state can be produced through a coupling α sufficiently large. Most likely (2.15) has no sharp mathematical meaning. However we will see an associated phenomenon of the Pauli-Fierz model in Section 8.

3 A Schrödinger representation

3.1 The simultaneous diagonalization of the quantized radiation field

In order to obtain a functional integral representation of a heat semigroup, we shall take a Schrödinger representation of the quantized radiation field $A(\hat\lambda)$. Note that

$$(A_\mu(\hat\lambda)\Omega_{\rm b}, A_\nu(\hat\rho)\Omega_{\rm b}) = \frac{1}{2}(d_{\mu\nu}\hat\lambda, \hat\rho),$$

$$[A_\mu(\hat\lambda), A_\nu(\hat\rho)] = 0.$$

Define a quadratic form on $\oplus^d L^2(\mathbf{R}^d)$ by

$$q(f,g) := \frac{1}{2}(d_{\mu\nu}\hat{f}_\mu, \hat{g}_\nu), \quad f, g \in \oplus^d L^2(\mathbf{R}^d).$$

In particular we set $q(f,f) := q(f)$. Let (Q,ν) be a probability measure space and $\phi(f)$ a Gaussian random process on (Q,ν)[19] indexed by *real* $f \in \oplus^d L^2(\mathbf{R}^d)$ with a covariance

$$\int_Q \phi(f)\phi(g)d\nu(\phi) = \frac{1}{2}q(f,g).$$

Note that

$$\int_Q e^{\alpha\phi(f)}d\nu(\phi) = e^{(\alpha^2/2)q(f)}, \quad \alpha \in \mathbf{C}.$$

We set for $f \in \oplus^d L^2(\mathbf{R}^d)$

$$\phi(f) := \phi(\Re f) + i\phi(\Im f).$$

Let Ω be the identity function in $L^2(Q)$. Set

$$L_0^2(Q) := \{ :\phi(f_1)\cdots\phi(f_n):, \Omega | f_j \in \oplus^d L^2(\mathbf{R}^d), j = 1, ..., n, n \in \mathbf{N}\},$$

where the wick product $:\phi(f_1)\cdots\phi(f_n):$ is recursively defined by

$$:\phi(f): = \phi(f),$$

$$:\phi(f)\phi(f_1)\cdots\phi(f_n): := \phi(f) :\phi(f_1)\cdots\phi(f_n):$$
$$-\frac{1}{2}\sum_{j=1}^n (\overline{f}, f_j) :\phi(f_1)\cdots\widehat{\phi(f_j)}\cdots\phi(f_n): .$$

It is known that $L_0^2(Q)$ is dense in $L^2(Q)$ and

$$(:\phi(f_1)\cdots\phi(f_n):, :\phi(g_1)\cdots\phi(g_m):)_{L^2(Q)} = \delta_{nm}\sum_{\pi\in\mathcal{G}_n} q(f_1, g_{\pi(1)})\cdots q(f_n, g_{\pi(n)}),$$

[19] Actually we can construct the Gaussian measure ν on "the Schwartz distribution space of transverse vector potentials" ([73, 105])

$$\mathcal{S}^T := \left\{ \Psi \in \underbrace{\mathcal{S}'_{\rm real}(\mathbf{R}^d)\times\cdots\times\mathcal{S}'_{\rm real}(\mathbf{R}^d)}_{d} \middle| {\rm div}\Psi = 0 \right\}$$

by the Minlos theorem (e.g., [97]).

where $\mathcal{G}_n$ denotes the set of the nth-degree permutations. Let $T : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ be a contractive operator. We define a contractive operator [20] $\Gamma(T) : L^2(Q) \to L^2(Q)$ by

$$\Gamma(T)\ {:}\phi(f_1)\cdots\phi(f_n){:} := {:}\phi([T]f_1)\cdots\phi([T]f_n){:}\,,$$

$$\Gamma(T)\Omega := \Omega,$$

where $[T] := \underbrace{T \oplus \cdots \oplus T}_{d}$. Let h be a self-adjoint operator of $L^2(\mathbf{R}^d)$. Then $\Gamma(e^{ith})$ is a strongly continuous one-parameter unitary group in t. Thus there exists a self-adjoint operator $d\Gamma(h)$ of $L^2(Q)$ such that

$$\Gamma(e^{ith}) = e^{itd\Gamma(h)}, \quad t \in \mathbf{R}.$$

The number operator in $L^2(Q)$ is defined by

$$N := d\Gamma(\mathbf{1}),$$

and the canonical pair of $\phi(\lambda)$ by

$$\pi(\lambda) := e^{i\pi N/2}\phi(\lambda)e^{-i\pi N/2}.$$

Let

$$\widehat{\omega} := \omega(-i\nabla)$$

and we define the free Hamiltonian of $L^2(Q)$ by

$$H_{\mathrm{f}} := d\Gamma(\widehat{\omega}).$$

Set

$$\mathbf{A}_\mu(\lambda) := \phi(\underbrace{0 \oplus \cdots \overbrace{\lambda}^{\mu} \cdots \oplus 0}_{d}), \quad \mu = 1, ..., d.$$

Proposition 3.1 ([105]) *There exists a unitary operator $\theta : \mathcal{F} \to L^2(Q)$ such that (1) $\theta\Omega_{\mathrm{b}} = \Omega$; (2) $\theta^{-1}H_{\mathrm{b}}\theta = H_{\mathrm{f}}$; (3) $\theta^{-1}\mathbf{A}_\mu(\lambda(\cdot - x))\theta = A_\mu(\hat{\lambda}, x)$ for each $x \in \mathbf{R}^d$.*

Let $\mathcal{H}$ be a Hilbert space defined by[21]

$$\mathcal{H} := L^2(\mathbf{R}^d) \otimes L^2(Q) \cong \int_{\mathbf{R}^d}^{\oplus} L^2(Q)dx.$$

[20] Γ is a functor from the set of contractive operators on $L^2(\mathbf{R}^d)$ to that on $L^2(Q)$. See [164].

[21] $\mathcal{H}$ is the set of $L^2(Q)$-valued L^2-functions on $\mathbf{R}^d$. Thus, for $F \in \mathcal{H}$, $F(x) \in L^2(Q)$ a.e. $x \in \mathbf{R}^d$ and $\int_{\mathbf{R}^d} \|F(x)\|^2_{L^2(Q)}dx = \|F\|^2_{\mathcal{H}}$.

Set

$$\mathbf{A}_\mu := \int_{\mathbf{R}^d}^{\oplus} \mathbf{A}_\mu(\lambda(\cdot - x))dx.$$

The Pauli-Fierz Hamiltonian in a Schödinger representation is defined by

$$H := \frac{1}{2}\left(\mathbf{p}\otimes\mathbf{1} - \alpha\mathbf{A}\right)^2 + V\otimes\mathbf{1} + \mathbf{1}\otimes H_{\mathrm{f}}.$$

Let

$$\Theta := \int_{\mathbf{R}^d}^{\oplus} \theta dx.$$

From Proposition 3.1 it follows that on a dense domain

$$\Theta^{-1}H\Theta = H_{\mathrm{PF}}.$$

3.2 Ergodic properties of the decoupled Hamiltonian

Let (M, m) be a σ-finite measure space. We say that $\Psi \in L^2(M, dm)$ is positive if $\Psi \geq 0$ $(\Psi \neq 0)$ for a.e. M. We also say that operator A of $L^2(M, dm)$ is "positivity preserving" (simply we say PP) if $(\Psi, A\Phi)_{L^2(M,dm)} \geq 0$ for all positive Ψ, Φ, moreover, "positivity improving" (simply we say PI) if $(\Psi, A\Phi)_{L^2(M,dm)} > 0$ for all positive Ψ, Φ. Let K be a nonnegative self-adjoint operator in $L^2(M, dm)$. It is well known that if e^{-tK} is PI, then the ground state of K is unique and strictly positive.

Let T be a contractive operator of $L^2(Q)$. It is established (e.g.,[88, 185]) that $\Gamma(T)$ is PP and that $\Gamma(T)$ is PI if $\|T\| < 1$.

Proposition 3.2 ([68, 69, 183]) *$e^{-tH_{\mathrm{f}}}$ is PI for all $t > 0$ in $L^2(Q)$.*

Define a set V_0 of external potentials V by

V_0: $V = V_+ - V_-$ such that $V_\pm \geq 0$, $V_+ \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ and V_- is infinitesimally small with respect to the Laplacian in the sense of form.

Proposition 3.3 ([188]) *Let $V \in V_0$. Then $e^{-tH_{\mathrm{p}}}$ is PI for all $t > 0$ in $L^2(\mathbf{R}^d)$.*

Proposition 3.4 ([110]) *Let $V \in V_0$. Then $e^{-t(H_{\mathrm{p}}\otimes\mathbf{1}+\mathbf{1}\otimes H_{\mathrm{f}})}$ is PI for all $t > 0$ in $\mathcal{H}$.*

Proposition 3.4 does not directly follows from Propositions 3.2 and 3.3. It is seen that $e^{-t(H_{\mathrm{p}}\otimes\mathbf{1}+\mathbf{1}\otimes H_{\mathrm{f}})} = e^{-t(H_{\mathrm{p}}\otimes\mathbf{1})}e^{-t(\mathbf{1}\otimes H_{\mathrm{f}})}$, however, both of $e^{-t(H_{\mathrm{p}}\otimes\mathbf{1})}$ and $e^{-t(\mathbf{1}\otimes H_{\mathrm{f}})}$ are *not* PI, which are PP in $\mathcal{H}$.

By Proposition 3.4, $H_{\mathrm{p}} + H_{\mathrm{f}}$ has a strictly positive unique ground state $\phi_{\mathrm{p}}\otimes\Omega$, where ϕ_{p} denotes the ground state of H_{p}.

4 Functional integral representations

In this section we assume that $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \sqrt{\omega}\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$, $|\alpha| \ll 1$, V is relatively bounded with respect to the Laplacian. Set

$$H = H_0 + H_{\rm f} + V,$$

where

$$H_0 := \frac{1}{2}\left(\mathbf{p} - \alpha\mathbf{A}\right)^2.$$

We want to construct a functional integral representation of the form

$$\left(\Phi, e^{-\beta_0 K} e^{-t_1 H} f_1 e^{-\beta_1 K} e^{-(t_2-t_1)H} f_2 \cdots f_{m-1} e^{-\beta_{m-1}K} e^{-(t_m-t_{m-1})H}\Psi\right)_{\mathcal{H}},$$

where $f_j \in L^\infty(\mathbf{R}^d)$, $j = 1, ..., m-1$, K is a nonnegative self-adjoint operator.

4.1 A decomposition of $e^{-td\Gamma(h(-i\nabla))}$ and Gaussian random processes

For $f, g \in \oplus^d L^2(\mathbf{R}^{d+1})$, we define

$$q_0(f,g) := \int_{\mathbf{R}^{d+1}} d_{\mu\nu}(k)\overline{\hat{f}_\mu}(k,k_0)\hat{g}_\nu(k,k_0)dkdk_0.$$

Let (Q_0, ν_0) denote a probability measure space and $\phi_0(f)$ be a Gaussian random process indexed by *real* $f \in \oplus^d L^2(\mathbf{R}^{d+1})$ with a covariance

$$\int_{Q_0} \phi_0(f)\phi_0(g)\nu_0(d\phi_0) = \frac{1}{2}q_0(f,g).$$

For $f \in \oplus^d L^2(\mathbf{R}^{d+1})$, we define

$$\phi_0(f) = \phi_0(\Re f) + i\phi_0(\Im f).$$

Let Ω_0 be the identity function in $L^2(Q_0)$. Let $j_t : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^{d+1})$ be defined by

$$\widehat{j_t f}(k,k_0) = \frac{e^{-itk_0}}{\sqrt{2\pi}}\sqrt{\frac{\omega(k)}{\omega(k)^2 + |k_0|^2}}\hat{f}(k), \quad (k,k_0) \in \mathbf{R}^d \times \mathbf{R}, \quad t \in \mathbf{R}.$$

It is immediate to see that

$$(j_t f, j_s g)_{L^2(\mathbf{R}^{d+1})} = \frac{1}{2}(\hat{f}, e^{-|t-s|\omega}\hat{g})_{L^2(\mathbf{R}^d)},$$

namely

$$j_t^* j_s = \frac{1}{2} e^{-|t-s|\omega(-i\nabla)}, \quad t, s \in \mathbf{R}. \tag{4.1}$$

Let $J_t : L^2(Q) \to L^2(Q_0)$ be defined by

$$J_t \ :\phi(f_1)\cdots\phi(f_n): := :\phi_0([j_t]f_1)\cdots\phi_0([j_t]f_n):, \quad J_t\Omega = \Omega_0.$$

It is easily seen that, by (4.1), J_t extends to an isometry of $L^2(Q)$ to $L^2(Q_0)$ such that

$$J_t^* J_s = e^{-|t-s|H_{\mathrm{f}}}, \quad t, s \in \mathbf{R}. \tag{4.2}$$

In addition to ϕ_0, we need another Gaussian random process. For $f, g \in \oplus^d L^2(\mathbf{R}^{d+2})$, we define

$$q_1(f, g) := \int_{\mathbf{R}^{d+2}} d_{\mu\nu}(k) \overline{\hat{f}_\mu}(k, k_0, k_1) \hat{g}_\nu(k, k_0, k_1) dk dk_0 dk_1.$$

Let (Q_1, ν_1) denote a probability measure space and $\phi_1(f)$ be a Gaussian random process indexed by real $f \in \oplus^d L^2(\mathbf{R}^{d+2})$ with a covariance

$$\int_{Q_1} \phi_1(f)\phi_1(g)\nu_1(d\phi_1) = \frac{1}{2} q_1(f, g).$$

For $f \in \oplus^d L^2(\mathbf{R}^{d+2})$, we define

$$\phi_1(f) = \phi_1(\Re f) + i\phi_1(\Im f).$$

Let Ω_1 be the identity function in $L^2(Q_1)$. Let h be a nonnegative multiplication operator of $L^2(\mathbf{R}^d)$. We define $\xi_t : L^2(\mathbf{R}^{d+1}) \to L^2(\mathbf{R}^{d+2})$ by

$$\widehat{\xi_t f}(k, k_0, k_1) = \frac{e^{-itk_1}}{\sqrt{\pi}} \sqrt{\frac{h(k)}{h(k)^2 + |k_1|^2}} \hat{f}(k, k_0) \quad (k, k_0, k_1) \in \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}, \quad t \in \mathbf{R}.$$

Similarly to (4.1) we have

$$\xi_t^* \xi_s = \frac{1}{2} e^{-|t-s|(h(-i\nabla)\otimes \mathbf{1})} \tag{4.3}$$

under identification $L^2(\mathbf{R}^{d+1}) \cong L^2(\mathbf{R}^d) \otimes L^2(\mathbf{R})$. Define $\Xi_t : L^2(Q_0) \to L^2(Q_1)$ by

$$\Xi_t \ :\phi_0(f_1)\cdots\phi_0(f_n): := :\phi_1([\xi_t]f_1)\cdots\phi_1([\xi_t]f_n):, \quad \Xi_t\Omega_1 = \Omega_0.$$

From (4.3) it follows that

$$\Xi_t^* \Xi_s = e^{-|t-s|d\Gamma(h(-i\nabla)\otimes \mathbf{1})}. \tag{4.4}$$

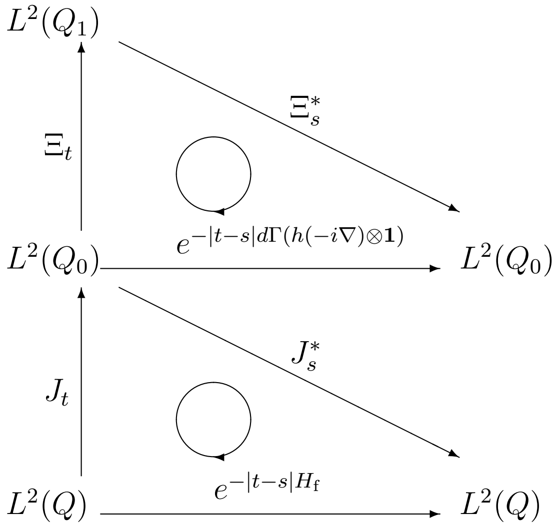

Figure 1: (4.2) and (4.4)

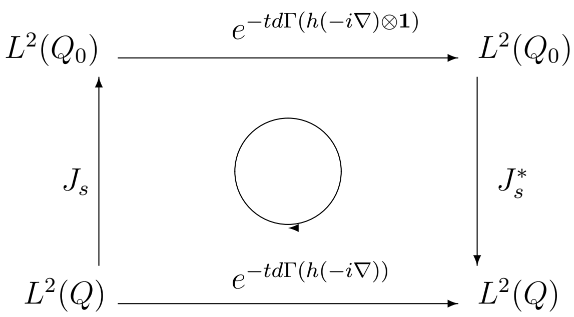

Figure 2: (4.5)

From the definitions of J_t and Ξ_s, we see that

$$J_s e^{-td\Gamma(h(-i\nabla))} = e^{-td\Gamma(h(-i\nabla)\otimes\mathbf{1})} J_s. \tag{4.5}$$

We define the canonical pairs of $\phi_0(f)$ and $\phi_1(g)$ by

$$\pi_0(f) := e^{i\pi N_0/2}\phi_0(f)e^{-i\pi N_0/2},$$

$$\pi_1(g) := e^{i\pi N_1/2}\phi_1(g)e^{-i\pi N_1/2},$$

respectively, where N_0 and N_1 are the number operators in $L^2(Q_0)$ and $L^2(Q_1)$, respectively.

4.2 Functional integrals

Let $\mathbf{b}(t) := \{\mathbf{b}_\mu(t)\}$ be the d-dimensional Brownian motion starting at the origin on the probability measure space $(\mathrm{C}([0,\infty);\mathbf{R}^d), d\mathbf{b})$. Let $X_s := \mathbf{b}(s) + x$ be the Wiener path and $dP := dx \otimes d\mathbf{b}$ on $\mathcal{W} := \mathbf{R}^d \times \mathrm{C}([0,\infty);\mathbf{R}^d)$.

We define the subspace of coherent states in $L^2(Q)$ by

$$L^2_{\mathrm{C}}(Q) := \{F(\phi(f_1),\cdots,\phi(f_n)) | F \in \mathcal{S}(\mathbf{R}^n), f_j \in \oplus^d L^2(\mathbf{R}^d), j = 1,...,n, n \in \mathbf{N}\},$$

where $\mathcal{S}(\mathbf{R}^n)$ denotes the set of Schwartz test functions on $\mathbf{R}^n$.

Theorem 4.1 (Functional integral representation I [105, 111]) *Let h be a nonnegative multiplication operator of $L^2(\mathbf{R}^d)$ and set $K := d\Gamma(h(-i\nabla))$. Let $0 \le t_1 \le t_2 \le \cdots \le t_m$, and $0 \le \tau_1 \le \tau_2 \le \cdots \le \tau_m$. We assume that $F_0, F_m \in \mathcal{H}$, $F_1,\cdots,F_{m-1} \in L^2_{\mathrm{C}}(Q)\widehat{\otimes}L^\infty(\mathbf{R}^d)$. Set $\widehat{F}_j := \Xi_{\tau_j} J_{t_j} F_j$. Then*

$$\left(F_0, e^{-\tau_1 K}e^{-t_1 H}F_1 e^{-(\tau_2-\tau_1)K}e^{-(t_2-t_1)H}F_2\cdots F_{m-1}e^{-(\tau_m-\tau_{m-1})K}e^{-(t_m-t_{m-1})H}F_m\right)$$
$$= \int_{\mathcal{W}} dP e^{-\int_0^t V(X_s)ds}\left(\widehat{F}_0(X_0), e^{i\alpha\phi_1(\mathbf{L}(X))}\widehat{F}_{t_1}(X_{t_1})\cdots\widehat{F}_{t_m}(X_{t_m})\right)_{L^2(Q_1)},$$

where

$$\mathbf{L}(X) := \oplus_{\mu=1}^d \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \xi_{\tau_j} j_s \lambda(\cdot - X_s) d\mathbf{b}_\mu(s) \in \oplus^d L^2(\mathbf{R}^{d+2}),$$

and $\int_T^S \cdots d\mathbf{b}_\mu(s)$ denotes $L^2(\mathbf{R}^{d+2})$-valued [22] *stochastic integrals.*[23]

[22] $\lambda(\cdot - X_s) \in L^2(\mathbf{R}^d)$, $j_s\lambda(\cdot - X_s) \in L^2(\mathbf{R}^{d+1})$, $\xi_{\tau_j} j_s\lambda(\cdot - X_s) \in L^2(\mathbf{R}^{d+2})$.

[23] Let $F : \mathbf{R}\times\mathbf{R}^d \to K$, where K is a Hilbert space. Then K-valued stochastic integral is defined by

$$\int_0^t F(s,\mathbf{b}(s))d\mathbf{b}_\mu(s) := s-\lim_{n\to\infty}\sum_{k=1}^{2^n} F\left(\frac{k-1}{2^n}t, \mathbf{b}\left(\frac{k-1}{2^n}t\right)\right)\left\{\mathbf{b}_\mu\left(\frac{k}{2^n}t\right) - \mathbf{b}_\mu\left(\frac{k-1}{2^n}t\right)\right\}$$

in $L^2(\mathrm{C}(\mathbf{R};\mathbf{R}^d);K)$. See [188].

Proof: For instance we set $V = 0$. By the Trotter-Kato product formula [142] we have

$$e^{-tH} = s - \lim_{n\to\infty} \left(e^{-t/nH_0} e^{-t/nH_{\rm f}}\right)^n.$$

Put $a_n := t_n - t_{n-1}$ and $b_n := \tau_n - \tau_{n-1}$. Thus

$$\lim_{n\to\infty} \Big(F_0, e^{-b_1 K} \left(e^{-a_1/nH_0} e^{-a_1/nH_{\rm f}}\right)^n F_1 e^{-b_2 K} \left(e^{-a_1/nH_0} e^{-a_1/nH_{\rm f}}\right)^n F_2 e^{-b_3 K} \cdots$$

$$\cdots F_{m-1} e^{-b_m K} \left(e^{-a_m/nH_0} e^{-a_m/nH_{\rm f}}\right)^n F_m\Big)$$

Since [105, 188]

$$e^{-tH_0} = s - \lim_{n\to\infty} \left(Q_{t/2^n}\right)^{2^n},$$

where $Q_s : \mathcal{H} \to \mathcal{H}$ is defined by, for $F(\cdot) \in \mathcal{H}$,

$$Q_s F(x) := \int_{\mathbf{R}^d} p_s(x-y) e^{(i\alpha/2)\phi\left(\oplus_{\mu=1}^d (\lambda(\cdot - x) + \lambda(\cdot - y))\cdot(x_\mu - y_\mu)\right)} F(y) dy, \tag{4.6}$$

$$Q_0 F(x) := F(x), \tag{4.7}$$

where $p_t(x) := (2\pi t)^{-d/2} \exp\left(-|x|^2/2t\right)$. Using the facts that

$$e^{-tH_{\rm f}} = J_T^* J_{T+t},$$

$$J_s e^{\phi(f)} J_s^* = (J_s J_s^*) e^{\phi_0(j_s f)} (J_s J_s^*)$$

as an operator, and the Markov property[24] of $J_s J_s^*$ [88, 185], we have

$$= \left(J_0 F_0, \left(e^{-b_1 K}\right) e^{i\alpha\phi_0(t_0,t_1)} \left(J_{t_1} F_1\right) \left(e^{-b_2 K}\right) e^{i\alpha\phi_0(t_1,t_2)} \cdots\right.$$

$$\left.\cdots \left(J_{t_{m-1}} F_{m-1}\right) \left(e^{-b_m K}\right) e^{i\alpha\phi_0(t_{m-1},t_m)} J_{t_m} F_m\right),$$

where

$$\phi_0(t_a, t_b) := \phi_0\left(\oplus_{\mu=1}^d \int_{t_a}^{t_b} j_s \lambda(\cdot - X_s) d\mathbf{b}_\mu(s)\right).$$

Using also that

$$\Xi_{T+t}^* \Xi_T = e^{-tK},$$

$$\Xi_t e^{\phi_0(f)} \Xi_t^* = (\Xi_t \Xi_t^*) e^{\phi_0(\xi_t f)} (\Xi_t \Xi_t^*)$$

as an operator, we get the desired results by the Markov property of $\Xi_t \Xi_t^*$. QED

[24] Let $E_s := J_s J_s^*$. Thus E_s is a projection of $L^2(Q_0)$. Define $Q_{[a,b]} := \mathcal{L}\{F \in L^2(Q_0) | F \in {\rm Ran} E_s, s \in [a,b]\}$. Let $\Sigma_{[a,b]}$ be the smallest σ-field generated by $Q_{[a,b]}$. Let Ψ be measurable with respected to $\Sigma_{[a,b]}$ and Ψ with respect to $\Sigma_{[c,d]}$, where $a \le b \le c \le d$. Then, for $b \le s \le c$, $(\Psi, E_s \Phi) = (\Psi, \Phi)$.

Corollary 4.2 *Let $F, G \in \mathcal{H}$. Then*

$$(F, e^{-tH}G) = \int_{\mathcal{W}} dP e^{-\int_0^t V(X_s)ds} (J_0 F(X_0), e^{i\alpha\phi_0(\mathbf{K}_t(X))} J_t G(X_t))_{L^2(Q_0)},$$

where

$$\mathbf{K}_t(X) := \oplus_{\mu=1}^d \int_0^t j_s \lambda(\cdot - X_s) d\mathbf{b}_\mu(s).$$

In particular, for $f \in L^2(\mathbf{R}^d)$,

$$(f \otimes \Omega, e^{-tH} f \otimes \Omega) = \int_{\mathcal{W}} dP e^{-\int_0^t V(X_s)ds} \overline{f(X_0)} f(X_t) e^{-(\alpha^2/2) q_0(\mathbf{K}_t(X))}.$$

We immediately see a Kato-type inequality ([140]) [25]

Corollary 4.3 (Diamagnetic inequality [103, 105]) *Let $F, G \in \mathcal{H}$. Then*

$$\left|\left(F, e^{-tH} G\right)\right| \le \left(|F|, e^{-t(H_\mathrm{p}+H_\mathrm{f})}|G|\right).$$

In particular

$$\inf \sigma(H_\mathrm{p}) \le \inf \sigma(H).$$

Proof: Note that $|J_t G| = J_t |G|$, since J_t is PP, and that $\inf \sigma(H_\mathrm{p} + H_\mathrm{f}) = \inf \sigma(H_\mathrm{p})$. Thus corollary follows directly from Corollary 4.2. QED

Corollary 4.4 *Let $f \in L^2(\mathbf{R}^d)$. Then*

$$(f \otimes \Omega, e^{-tH} e^{-sK} e^{-tH} f \otimes \Omega)$$

$$= \int_{\mathcal{W}} dP e^{-\int_0^t V(X_s)ds} \overline{f(X_0)} f(X_{2t}) e^{-(\alpha^2/2) q_0(\mathbf{K}_{2t}) + (\alpha^2/2) F(X)}, \tag{4.8}$$

where

$$F(X) := 2q_1 \left(\oplus_{\mu=1}^d \int_0^t \xi_0 j_s \lambda(\cdot - X_s) d\mathbf{b}_\mu(s), \oplus_{\mu=1}^d \int_t^{2t} \xi_t j_{s'} \lambda(\cdot - X_{s'}) d\mathbf{b}_\mu(s') \right).$$

Proof: By Theorem 4.1 we have

$$\text{L.H.S. } (4.8) = \int_{\mathcal{W}} dP e^{-\int_0^{2t} V(X_s)ds} \overline{f}(X_0) f(X_{2t}) \left(\Omega_1, e^{i\alpha\phi_1(W)}\Omega_1\right)_{L^2(Q_1)}$$

$$= \int_{\mathcal{W}} dP e^{-\int_0^{2t} V(X_s)ds} \overline{f}(X_0) f(X_{2t}) e^{-(\alpha^2/2) q_1(W)},$$

[25] The Kato inequality is studied and applied in e.g., [57, 63, 96, 103, 105, 130, 131, 140, 141, 150, 189, 186] etc.

where

$$W = \oplus_{\mu=1}^d \left(\int_0^t \xi_0 j_s \lambda(\cdot - X_s) d\mathbf{b}_\mu(s) + \int_t^{2t} \xi_t j_s \lambda(\cdot - X_s) d\mathbf{b}_\mu(s) \right).$$

Since

$$q_1(W) = q_0(\mathbf{K}_{2t}) - F(X),$$

we get the desired result. QED

Remark 4.5 *Formally we see that*

$$F(X) = \int_0^t d\mathbf{b}_\mu(s) \int_t^{2t} d\mathbf{b}_\nu(s') \int_{\mathbf{R}^d} \left(1 - e^{-th(k)}\right) d_{\mu\nu}(k) e^{-|s-s'|\omega(k)} |\hat{\lambda}(k)|^2 e^{ik(X_s - X_{s'})} dk,$$

$$q_0(\mathbf{K}_t(X)) = \int_0^t d\mathbf{b}_\mu(s) \int_0^t d\mathbf{b}_\nu(s') \int_{\mathbf{R}^d} d_{\mu\nu}(k) e^{-|s-s'|\omega(k)} e^{ik(X_s - X_{s'})} |\hat{\lambda}(k)|^2 dk.$$

This formal expression appears in [94, 110, 70, 194].

5 Essential self-adjointness for arbitrary $\alpha \in \mathbf{R}$

5.1 Translation invariance and invariant domains

We redefine $Q_s : \mathcal{H} \to \mathcal{H}$ for *arbitrary* $\alpha \in \mathbf{R}$ by

$$Q_s F(x) := \int_{\mathbf{R}^d} p_s(x-y) e^{(i\alpha/2)\phi\left(\oplus_{\mu=1}^d (\lambda(\cdot - x) + \lambda(\cdot - y)) \cdot (x_\mu - y_\mu)\right)} F(y) dy, \quad s > 0, \tag{5.1}$$

$$Q_0 F(x) := F(x). \tag{5.2}$$

Let

$$S(t) := s - \lim_{n \to \infty} \left(Q_{t/2^n}\right)^{2^n}.$$

Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$. Thus by a direct calculation we see that $S(t)$ exists and

$$(F, S(t)G) = \int_{\mathcal{W}} dP \left(G(X_0), e^{i\alpha\phi(\mathbf{Z}(X))} G(X_t)\right),$$

where

$$\mathbf{Z}(X) := \oplus_{\mu=1}^d \int_0^t \lambda(\cdot - X_s) d\mathbf{b}_\mu(s) \in \oplus^d L^2(\mathbf{R}^d).$$

By the definition of Q_s we immediately see that

$$(F, S(t)S(s)G) = (F, S(s+t)G),$$

$$\lim_{t\to\infty}(F, S(t)G) = (F, S(0)G) = (F, G).$$

Hence $S(t)$, $t \geq 0$, is a strongly continuous one-parameter semigroup in t. Thus there exists a nonnegative self-adjoint operator $\widehat{H}_0$ in $L^2(Q)$ such that

$$S(t) = e^{-t\widehat{H}_0}.$$

Lemma 5.1 *Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$. Then, for all $\alpha \in \mathbf{R}$,*

$$H_0\lceil_{D(\Delta)\cap D(N)} \subset \widehat{H}_0.$$

Proof: For $F \in C_0^\infty(\mathbf{R}^d)\widehat{\otimes}L_0^2(Q)$ and $G \in \mathcal{H}$, we have [48, 105]

$$\left(G, \frac{1}{t}\left(e^{-t\widehat{H}_0} - \mathbf{1}\right) F\right)_{\mathcal{H}} = -\int_0^1 ds \left(e^{-t\widehat{H}_0} G, H_0 F\right)_{\mathcal{H}}. \tag{5.3}$$

Since

$$\|H_0F\| \leq C\left(\|\Delta F\| + \|NF\| + \|F\|\right)$$

with some constant C, by a limiting argument we extend (5.3) to $F \in D(\Delta)\cap D(N)$. Take $G \in D(\widehat{H}_0)$. We have

$$-\left(\widehat{H}_0 G, F\right) = \lim_{t\to\infty}\left(G, \frac{1}{t}\left(e^{-t\widehat{H}_0} - \mathbf{1}\right) F\right) = -\int_0^1 ds\,(G, H_0F) = -(G, H_0F).$$

Then $\left(\widehat{H}_0 G, F\right) = (G, H_0F)$, which yields that $F \in D(\widehat{H}_0)$ and $\widehat{H}_0 F = H_0 F$. Hence lemma follows. QED

Lemma 5.2 *Let $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \sqrt{\omega}\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$. Then we have, for all $\alpha \in \mathbf{R}$, that*

$$H_0\lceil_{D(\Delta)\cap D(H_{\mathrm{f}})} \subset \widehat{H}_0. \tag{5.4}$$

Proof: Since (5.3) extends to $F \in D(\Delta)\cap D(H_{\mathrm{f}})$, lemma follows in the similar way as that of Lemma 5.1. QED

We define

$$\widehat{H} := \widehat{H}_0 \dot{+} H_{\mathrm{f}}.$$

Let $V = 0$. We note that, for $\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$,

$$H\lceil_{D(\Delta)\cap D(N)\cap D(H_{\mathrm{f}})} \subset \widehat{H}, \tag{5.5}$$

moreover for $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \sqrt{\omega}\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$,

$$H\lceil_{D(\Delta)\cap D(H_{\rm f})} \subset \widehat{H}. \tag{5.6}$$

Similarly to the proof of Theorem 4.1 we have

$$\left(F, e^{-t\widehat{H}}G\right) = \int_{\mathcal{W}} dP\left(J_0 F(X_0), e^{i\alpha\phi_0(\mathbf{K}_t(X))} J_t G(X_t)\right). \tag{5.7}$$

In particular, for a.e. $(x, \phi) \in \mathbf{R}^d \times Q$,

$$\left(e^{-t\widehat{H}}F\right)(\phi, x) = \mathbf{E}\mathbf{J}_t G(X_t),$$

where $\mathbf{E}$ denotes the expectation value with respect to $d\mathbf{b}$ and

$$\mathbf{J}_t := \mathbf{J}_t(X) := J_0^* e^{i\alpha\phi_0(\mathbf{K}_t(X))} J_t.$$

The following Burkholder type inequality [138, p.166] is useful to estimate stochastic integrals.

Lemma 5.3 *Let $\omega^{k/2}\hat{\lambda} \in L^2(\mathbf{R}^d)$, $k = 0, 1, ..., n$. Then*

$$\mathbf{E}\left\|(\widehat{\omega}\otimes\mathbf{1})^{k/2}\int_0^t j_s\lambda(\cdot - X_s)d\mathbf{b}_\mu(s)\right\|^{2m}_{L^2(\mathbf{R}^{d+1})} \le \frac{(2m)!}{2^m} t^m \|\omega^{k/2}\hat{\lambda}\|^{2m}_{L^2(\mathbf{R}^d)}.$$

Proof: See [112, Theorem 4.6]. QED

Lemma 5.4 *(1) Let $\hat{\lambda}, \omega^n\hat{\lambda} \in L^2(\mathbf{R}^d)$ and $G \in D(H_{\rm f}^n)$, $n = 1, 2$. Then*

$$e^{-t\widehat{H}}G \in D(H_{\rm f}^2).$$

(2) Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$, and $G \in D(N^k)$. Then

$$e^{-t\widehat{H}}G \in D(N^k).$$

Proof: We prove (1). (2) is proved similarly. It is enough to prove both of

$$\left(e^{-t\widehat{H}}G\right)(x) \in D(H_{\rm f}^2), \quad a.e. x \in \mathbf{R}^d, \tag{5.8}$$

and

$$\int_{\mathbf{R}^d} \|H_{\rm f}^2 e^{-t\widehat{H}}G(x)\|^2_{L^2(Q)} dx < \infty. \tag{5.9}$$

It is immediately seen that J_t (resp.J_t^*) maps $D(H_{\rm f}^2)$ (resp.$D(d\Gamma(\widehat{\omega}\otimes\mathbf{1})^2)$) to $D(d\Gamma(\widehat{\omega}\otimes\mathbf{1})^2)$ (resp. $D(H_{\rm f}^2)$), and that $e^{i\alpha\phi_0(\mathbf{K}_t(X))}$ leaves $D(d\Gamma(\widehat{\omega}\otimes\mathbf{1})^2)$ invariant. Then we have for $\Psi\in D(H_{\rm f}^2)$,

$$H_{\rm f}^2\mathbf{J}_t\Psi = J_t^* e^{i\alpha\phi_0(\mathbf{K}_t(X))}S(X)^2J_0\Psi,\quad a.e.(x,\mathbf{b})\in\mathcal{W},$$

where

$$S(X) := d\Gamma(\widehat{\omega}\otimes\mathbf{1}) + \alpha\pi_0([\widehat{\omega}\otimes\mathbf{1}]\mathbf{K}_t(X)) + (\alpha^2/2)q_0([\widehat{\omega}\otimes\mathbf{1}]\mathbf{K}_t(X),\mathbf{K}_t(X)).$$

Using Burkholder inequality (5.3), and fundamental inequalities (2.5),(2.6) and (2.7), we have

$$\mathbf{E}\|H_{\rm f}^2\mathbf{J}_tG(X)\|_{L^2(Q)} \le C\|(H_{\rm f}+\mathbf{1})^2G(x)\|_{L^2(Q)}$$

with some constant C. Since $\left(e^{-t\widehat{H}}F\right)(\phi,x) = \mathbf{E}\mathbf{J}_tG(X_t)$,

$$\|H_{\rm f}^2e^{-t\widehat{H}}G\|_{\mathcal{H}} \le C\|(H_{\rm f}+\mathbf{1})^2G\|_{\mathcal{H}}.$$

Hence lemma follows. QED

We define the total momentum $\mathbf{P}_\mu$ by

$$\mathbf{P}_\mu := \mathbf{p}_\mu\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{P}_{{\rm f},\mu},\quad \mu=1,...,d,$$

where

$$\mathbf{P}_{{\rm f},\mu} := d\Gamma(-i\nabla_\mu).$$

By the definitions of $e^{-t\widehat{H}}$ we see that (translation invariance)[26]

$$e^{-is\mathbf{P}_\mu}e^{-t\widehat{H}} = e^{-t\widehat{H}}e^{-is\mathbf{P}_\mu},\quad t\ge 0,\quad s\in\mathbf{R}. \tag{5.10}$$

Lemma 5.5 *Let* $\hat{\lambda},\omega\hat{\lambda},\omega^2\hat{\lambda}\in L^2(\mathbf{R}^d)$ *and*

$$D_\mu := D(\mathbf{p}_\mu^2)\cap D(H_{\rm f}\mathbf{p}_\mu)\cap D(H_{\rm f}^2),\quad \mu=1,...,d.$$

Then, for all $t\ge 0$,

$$e^{-t\widehat{H}}D_\mu \subset D_\mu,\quad \mu=1,...,d.$$

[26]Let $V=0$. It follows that on some domain $[\mathbf{P}_\mu, H]=0$, $\mu=1,...,d$.

Proof: By translation invariance (5.10), it follows that, for $\Psi \in D(\mathbf{P}_\mu)$, $e^{-t\widehat{H}}\Psi \in D(\mathbf{P}_\mu)$ and

$$\mathbf{P}_\mu e^{-t\widehat{H}}\Psi = e^{-t\widehat{H}}\mathbf{P}_\mu\Psi. \tag{5.11}$$

Note that

$$D(H_{\mathrm{f}}^n) \subset D(\mathbf{P}_{\mathrm{f},\mu}^n), \quad n = 1, 2.$$

Let $G \in D_\mu$. Thus $\mathbf{P}_\mu G \in D(\mathbf{P}_\mu)$, and (5.11) implies that

$$e^{-t\widehat{H}}G \in D(\mathbf{P}_\mu^2).$$

By Lemma 5.4, we have

$$e^{-t\widehat{H}}G \in D(H_{\mathrm{f}}^2) \subset D(\mathbf{P}_{\mathrm{f},\mu}^2).$$

It is easily checked that

$$e^{-t\widehat{H}}G \in D(\mathbf{P}_\mu\mathbf{P}_{\mathrm{f},\mu}) \cap D(\mathbf{P}_{\mathrm{f},\mu}\mathbf{P}_\mu).$$

From

$$D(\mathbf{p}_\mu^2) \supset D(\mathbf{P}_\mu^2) \cap D(\mathbf{P}_\mu\mathbf{P}_{\mathrm{f},\mu}) \cap D(\mathbf{P}_{\mathrm{f},\mu}\mathbf{P}_\mu) \cap D(\mathbf{P}_{\mathrm{f},\mu}^2),$$

it follows that

$$e^{-t\widehat{H}}G \in D(\mathbf{p}_\mu^2).$$

Since $e^{-t\widehat{H}}G \in D(H_{\mathrm{f}}\mathbf{p}_\mu)$ is easily seen, we get $e^{-t\widehat{H}}G \in D_\mu$. QED

5.2 Essential self-adjointness

Theorem 5.6 ([112]) *Let V be a relatively bounded with respect to the Laplacian with a sufficiently small relative bound ε. Set*

$$S_{\mathrm{ess}} := C^\infty(N) \cap D(H_{\mathrm{f}}^2) \bigcap_{\mu=1}^d \left\{ D(\mathbf{p}_\mu^2) \cap D(H_{\mathrm{f}}\mathbf{p}_\mu) \right\}.$$

We assume that $\hat{\lambda}, \omega\hat{\lambda}, \omega^2\hat{\lambda} \in L^2(\mathbf{R}^d)$. Then H is essentially self-adjoint on S_{ess} and bounded below. In particular $D(\Delta) \cap D(N) \cap D(H_{\mathrm{f}})$ is a core of H.

Proof: We have $S_{\mathrm{ess}} \subset D(\Delta) \cap D(H_{\mathrm{f}}) \cap D(N) \subset D(\widehat{H})$. Moreover S_{ess} is invariant subspace of $e^{-t\widehat{H}}$ by Lemma 5.5. Since $\widehat{H}\lceil_{D(\Delta)\cap D(H_{\mathrm{f}})\cap D(N)} \subset H$ for $V = 0$ by (5.5), we obtain that H for $V = 0$ is essentially self-adjoint on S_{ess}. By a diamagnetic inequality (Corollary 4.3), V is also relatively bounded with respect to H with a relative bound $< \varepsilon$ [105, 188]. Hence theorem follows from the Kato-Rellich theorem. QED

Under the assumptions of Theorem 5.6, note that it is *not* clear that

$$D(H) \supset D(\Delta) \cap D(H_{\rm f}).$$

Corollary 5.7 *In addition to the assumptions of Theorem 5.6, we assume that* $\hat{\lambda}/\sqrt{\omega}, \sqrt{\omega}\hat{\lambda} \in L^2(\mathbf{R}^d)$*. Then H is essentially self-adjoint on*

$$S'_{\rm ess} := D(H_{\rm f}^2) \bigcap_{\mu=1}^{d} \left\{ D(\mathbf{p}_\mu^2) \cap D(H_{\rm f}\mathbf{p}_\mu) \right\}$$

and bounded below. In particular $D(\Delta) \cap D(H_{\rm f})$ is a core of H.

Proof: Since $\widehat{H}\lceil_{D(\Delta)\cap D(H_{\rm f})} \subset H$ by (5.4) for $V = 0$, corollary holds. QED

Corollary 5.8 (Functional integral representations II) *Let* $\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$ *. Then, for all $\alpha \in \mathbf{R}$ and $V \in V_0$, $H := \widehat{H} \dot{+} V_+ \dot{-} V_-$ is well defined and, for which the functional integral representation in Theorem 4.1 holds true.*

Proof: Let $V = 0$. Then the corollary is clear by (5.7). By a diamagnetic inequality (4.3), we see that V_- is also relatively form bounded with respect to $\widehat{H}$. Thus $\widehat{H} \dot{+} V_+ \dot{-} V_-$ is well defined. By a limiting argument (4.1) holds for $\widehat{H} \dot{+} V_+ \dot{-} V_-$. QED

6 Ground states

Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$ and $V \in V_0$ unless otherwise stated throughout this section. We redefine the Pauli-Fierz Hamiltonian by

$$H := \widehat{H} \dot{+} V_+ \dot{-} V_-.$$

Let $E := \inf \sigma(H)$ and $E_{\rm p} := \inf \sigma(H_{\rm p})$.

6.1 Ergodic properties and the uniqueness of the ground state

Let

$$U := \exp\left(i\frac{\pi}{2}N\right).$$

Note that

$$J_t e^{iaN} = e^{iaN_0} J_t, \quad a \in \mathbf{R}.$$

Thus we have

$$\Big(F, U^{-1}e^{-tH}UG\Big) = \int_{\mathcal{W}} dP e^{-\int_0^t V(X_s)ds} \Big(F(X_0), J_0^* e^{i\alpha\pi_0(\mathbf{K}_t(X))} J_t G(X_t)\Big).$$

The purpose of this subsection is to prove that $U^{-1}e^{-tH}U$ is PI for all $t \geq 0$.

Lemma 6.1 *Let $F \in L^2(Q)$ be a positive. Then there exists a positive sequence $F_n \in L^2_{\mathrm{C}}(Q)$ such that $s-\lim_{n\to\infty} F_n = F$.*

Proof: See [126, Theorem 3.2] and [88, 185]. QED

Lemma 6.2 *Let $f \in \oplus^d L^2(\mathbf{R}^d)$. Then $e^{i\pi(f)}$ is PP in $L^2(Q)$ for all $t \in \mathbf{R}$.*

Proof: Let $F := \int f(t) e^{i\sum_{j=1}^N t_j \phi(f_j)} dt$ and $G := \int g(t) e^{i\sum_{j=1}^M t_j\phi(g_j)} dt$ with f, g the Fourier transform of positive Schwartz test functions. By the Weyl relation:

$$e^{i\pi(f)} e^{i\phi(g)} = e^{iq(f,g)} e^{i\phi(g)} e^{i\pi(f)} \tag{6.1}$$

and

$$e^{i\pi(f)}\Omega = e^{-(1/2)q(f)} e^{-\phi(f)}\Omega, \tag{6.2}$$

we have

$$\Big(F, e^{i\pi(f)}G\Big) = \int dt \int ds \overline{f(t)} g(s) \Big(e^{i\sum_{j=1}^N t_j\phi(f_j)}\Omega, e^{i\pi(f)} e^{i\sum_{j=1}^M s_j \phi(g_j)}\Omega\Big) \geq 0. \tag{6.3}$$

From Lemma 6.1, (6.3) follows for arbitrary positive F, G in $L^2(Q)$. QED

Lemma 6.3 *Let $f \in \oplus^d L^2(\mathbf{R}^{d+1})$. Then we have*

$$J_0^* e^{i\pi_0(f)} J_t = e^{-(1/2)\big(q_0(f)+q([j_0^*]f)\big)} \overline{J_0^* e^{-\phi_0(f)} J_t e^{i\pi([j_0^*]f)} e^{\phi([j_t^*]f)} \lceil_{L^2_{\mathrm{C}}(Q)}},$$

where $\overline{A}$ denotes the closed extension of A.

Proof: Note that

$$q_0([j_t]f,g)=q(f,[j_t^*]g),\quad f\in\oplus^d L^2(\mathbf{R}^d),\quad g\in\oplus^d L^2(\mathbf{R}^{d+1}).$$

Let $G\in L^2_{\mathrm{C}}(Q)$ be such that

$$G(\phi(f_1),\cdots,\phi(f_n))=\int_{\mathbf{R}^n} g(t)e^{i\phi(\sum_{j=1}^n t_jf_j)}dt,\quad g\in\mathcal{S}(\mathbf{R}^n).$$

By (6.1) and (6.2), we have

$$e^{i\pi_0(f)}J_tG\Omega=e^{i\pi_0(f)}G(\phi_0(j_tf_1),\cdots,\phi_0(j_tf_n))\Omega_0$$

$$=G(\phi_0(j_tf_1)+q_0([j_t]f_1,f),\cdots,\phi_0(j_tf_n)+q_0([j_t]f_n,f))e^{i\pi_0(f)}\Omega_0$$

$$=e^{-(1/2)q_0(f)}e^{-\phi_0(f)}G(\phi_0(j_tf_1)+q(f_1,[j_t^*]f),\cdots,\phi_0(j_tf_n)+q(f_n,[j_t^*]f))\Omega_0$$

$$=e^{-(1/2)q_0(f)}e^{-\phi_0(f)}j_te^{i\pi([j_0^*]f)}Ge^{-i\pi([j_t^*]f)}\Omega$$

$$=e^{-(1/2)(q_0(f)+q([j_t^*]f))}e^{-\phi_0(f)}J_te^{i\pi([j_0^*]f)}e^{\phi([j_0^*]f)}G$$

Since $L^2_{\mathrm{C}}(Q)$ is dense, lemma follows.[27] QED

Let $f\in\oplus^d L^2(\mathbf{R}^{d+1})$ and define a *bounded* operator on $L^2(Q)$ by

$$Q_M:=J_0^*\left(e^{-\phi_0(f)}\right)_M J_t,$$

where

$$\left(e^{-\phi_0(f)}\right)_M:=\begin{cases} e^{-\phi_0(f)}, & e^{-\phi_0(f)}<M,\\ M, & e^{-\phi_0(f)}\geq M.\end{cases}$$

Lemma 6.4 *We see that Q_M is PI for all $t\in\mathbf{R}$.*

Proof: Let θ_1,θ_2 be positive. It is known that $(\theta_1,Q_M\theta_2)\geq 0$. Hence it is enough to prove that

$$(\theta_1,Q_M\theta_2)\neq 0. \tag{6.4}$$

Assume that $(\theta_1,P_M\theta_2)=0$. Since J_t and J_0 are PP, we have

$$\left\{\mathrm{supp}\left(e^{-\phi_0(f)}\right)_M J_t\theta_2\right\}\bigcap\{\mathrm{supp}J_0\theta_1\}=\emptyset.$$

Moreover $\left(e^{-\phi_0(f)}\right)_M\neq 0$ a.e., since $\int_{Q_0}|\phi_0(f)|^2\nu_0(d\phi_0)<\infty$. Hence

$$\mathrm{supp}J_t\theta_2\cap\mathrm{supp}J_0\theta_1=\emptyset,$$

[27] We feel that $e^{i\pi_0(f)}$ is a shift operator in the space $L^2(Q_0)$ of the infinite degrees of freedom. Intuitively $\phi_0(f)\sim x$, $\pi_0(f)\sim\mathbf{p}$, $U\sim$ the Fourier transformation, in $L^2(\mathbf{R}^d)$.

which deduces that

$$0 = (J_0\theta_1, J_t\theta_2) = \left(\theta_1, e^{-tH_{\rm f}}\theta_2\right). \tag{6.5}$$

Since $e^{-tH_{\rm f}}$ is PI by Proposition 3.2,

$$\left(\theta_1, e^{-tH_{\rm f}}\theta_2\right) > 0.$$

Thus we have a contradiction with (6.5). Thus (6.4) follows. QED

Lemma 6.5 *Let $f \in \oplus^d L^2(\mathbf{R}^{d+1})$. Then $J_0^* e^{i\pi_0(f)} J_t$ is PI for all $t \in \mathbf{R}$.*

Proof: Let

$$\mathbf{P}_M := e^{-(1/2)\left(q_0(f)+q([j_0^*]f)\right)} J_0^* \left(e^{-\phi_0(f)}\right)_M J_t e^{i\pi([j_0^*]f)} \left(e^{\phi([j_t^*]f)}\right)_M.$$

Note that $\mathbf{P}_M$ is PI by Lemmas 6.2 and 6.4. For positive $F \in L^2_{\rm C}(Q)$,

$$\mathbf{P}_M F \le e^{-(1/2)\left(q_0(f)+q([j_0^*]f)\right)} J_0^* e^{-\phi_0(f)} J_t e^{i\pi([j_0^*]f)} e^{\phi([j_t^*]f)} F = J_0^* e^{i\pi_0(f)} J_t F.$$

Thus, by a limiting argument, for arbitrary positive $F \in L^2(Q)$, we have

$$\mathbf{P}_M F \le J_0^* e^{i\pi_0(f)} J_t F.$$

Since $\mathbf{P}_M F > 0$, lemma holds. QED

Theorem 6.6 ([110]) *We see that $U^{-1} e^{-tH} U$ is PI for all $t \ge 0$.*

Proof: Let $F = F(x, \phi)$ and $G = G(x, \phi)$ be positive in $\mathcal{H}$. Define

$$D_F := \{x \in \mathbf{R}^d | F(x, \cdot) \not\equiv 0\}, \quad D_G := \{x \in \mathbf{R}^d | G(x, \cdot) \not\equiv 0\},$$

and

$$D_{FG} := \{x + \mathbf{b}(\cdot) \in \mathcal{W} | x + \mathbf{b}(t) \in D_F, x \in D_G\}.$$

It is checked that

$$\int_{D_{FG}} dP = \int_{D_G} dx \int_{\mathrm{C}([0,\infty);\mathbf{R}^d)} \mathbf{1}_{\{b \in \mathrm{C}(\mathbf{R};\mathbf{R}^d) | x + b(t) \in D_F\}} d\mathbf{b}$$

$$= \int_{D_G} dx \int_{D_F} p_t(x-y) dy > 0.$$

Thus $F(X_0,\cdot)\not\equiv 0$ and $G(X_t,\cdot)\not\equiv 0$ on D_{FG}. Since $J_0^* e^{i\alpha\pi_0(\mathbf{K}_t(X))}J_t$ is PI on $L^2(Q)$, we have

$$\Big(F, U^{-1}e^{-tH}UG\Big) = \int_{\mathcal{W}} dP e^{-\int_0^t V(X_s)ds}\Big(F(X_0), J_0^* e^{i\alpha\pi_0(\mathbf{K}_t(X))}J_t G(X_t)\Big)$$

$$\geq \int_{D_{FG}} dP e^{-\int_0^t V(X_s)ds}\Big(F(X_0), J_0^* e^{i\alpha\pi_0(\mathbf{K}_t(X))}J_t G(X_t)\Big) > 0.$$

We get the desired results. QED

Corollary 6.7 *Let $\Psi_{\rm g}$ be a ground state of H. Then it is unique and $U\Psi_{\rm g}$ is strictly positive.*

6.2 The particle-localization of ground states

Let $\Psi_{\rm g}$ be the ground state of H. In this subsection we shall show an exponential decay[28] of $\|\Psi_{\rm g}(x)\|_{L^2(Q)}$. We introduce classes of external potentials V: Let Δ be the cube with the unit side centered about the origin in $\mathbf{R}^d$. We say that $V\in L^p_{\rm u}(\mathbf{R}^d)$ [188] if

$$\|f\|^p_{L^p_{\rm u}(\mathbf{R}^d)} := \sup_{x\in\mathbf{R}^d}\int_\Delta |f(x+y)|^p dy < \infty.$$

We define sets $V_{\rm bound}$ and $V_{\rm exp}$ of external potentials by

$V_{\rm bound}$: $V=V_+-V_-$, such that $V_\pm\geq 0$, $V_+\in L^1_{\rm loc}(\mathbf{R}^d)$ and $V_- = \sum_{\rm j=1}^J W_j$ such that $\sup_{z_j\in\mathbf{R}^{d-\mu_j}}\|W_j(\cdot,z_j)\|_{L^p_{\rm u}(\mathbf{R}^{\mu_j})}<\infty$ for some μ_j, $j=1,...,J$.

$V_{\rm exp}$: $V=Z+W$, such that $Z\in L^1_{\rm loc}(\mathbf{R}^d)$, $Z>-\infty$, and $W>0$, $W\in L^p(\mathbf{R}^d)$ for some $p>\max\{1,d/2\}$.

It is immediate that $V_{\rm exp}\cup V_{\rm bound}\subset V_0$.

Lemma 6.8 *Let $V\in V_{\rm bound}$. Then*

$$\sup_{x\in\mathbf{R}^d}\|\Psi_{\rm g}(x)\|_{L^2(Q)}<\infty. \tag{6.6}$$

Proof: $\Psi_{\rm g} = e^{tE}e^{-tH}\Psi_{\rm g}$. Thus we have

$$\Psi_{\rm g} = e^{tE}\mathbf{E}e^{-\int_0^t V(X_s)ds}\mathbf{J}_t\Psi_{\rm g}(X_t).$$

[28]See for classical cases [46, 52, 56, 61]

Hence

$$\|\Psi_{\rm g}(x)\| \le e^{tE}\mathbf{E}e^{-\int_0^t V(X_s)ds}\|\Psi_{\rm g}(X_t)\| = e^{tE}e^{-tH_{\rm P}}\|\Psi_{\rm g}(\cdot)\|. \tag{6.7}$$

Since $V \in V_{\rm bound}$,

$$\sup_{x\in\mathbf{R}^d}\left|e^{-tH_{\rm P}}\|\Psi_{\rm g}(\cdot)\|\right|(x) < \infty$$

([188, Theorem 25.5, Corollary 25.6]), we get (6.6). QED

Lemma 6.9 *Let $V \in V_{\rm bound}$. Then, for all $f \in C_0^\infty(\mathbf{R}^d)$ and $t > 0$,*

$$\int_{\mathbf{R}^d} f(x)\|\Psi_{\rm g}(x)\|^2 dx \le Ce^{tE}\int_{\mathbf{R}^d} dx|f(x)|\mathbf{E}e^{-\int_0^t V(X_s)ds},$$

where $C := \sup_{x\in\mathbf{R}^d}\|\Psi_{\rm g}(x)\|^2 < \infty$.

Proof: Note that, by Corollary 6.7, $U\Psi_{\rm g} > 0$. Since $f \in L^\infty(\mathbf{R}^d)$, we see that, by Lemma 6.8,

$$\begin{aligned}
&\int_{\mathbf{R}^d} f(x)\|\Psi_{\rm g}(x)\|^2 dx = (fU\Psi_{\rm g}, U\Psi_{\rm g})_{\mathcal{H}} = (f\Psi_{\rm g}, \Psi_{\rm g})_{\mathcal{H}} = e^{tE}(f\Psi_{\rm g}, e^{-tH}\Psi_{\rm g}) \\
&= e^{tE}\int_{\mathcal{W}} dPe^{-\int_0^t V(X_s)ds}f(x)(\Psi_{\rm g}(X_0), \mathbf{J}_t\Psi_{\rm g}(X_t)) \\
&\le e^{tE}\int_{\mathbf{R}^d} dx|f(x)|\mathbf{E}\|\Psi_{\rm g}(x)\|\|\Psi_{\rm g}(X_t)\|e^{-\int_0^t V(X_s)ds} \\
&\le e^{tE}\sup_{x\in\mathbf{R}^d}\|\Psi_{\rm g}(x)\|^2\int_{\mathbf{R}^d} dx|f(x)|\mathbf{E}e^{-\int_0^t V(X_s)ds}.
\end{aligned}$$

Thus lemma follows from Lemma 6.6. QED

The following lemma is known as Carmona's estimate:

Lemma 6.10 ([46]) *Let $V = Z + W \in V_{\rm exp}$. Then for all $t \ge 0$ and $a \ge 0$,*

$$\begin{aligned}
&\mathbf{E}e^{-\int_0^t V(X_s)ds} \le \beta_1 e^{t\beta_2\|W\|_p} \\
&\times\left\{e^{-2tZ^a(x)} + \beta_3\left(\left(a/\sqrt{t}\right)^{\max\{0,d-2\}}+1\right)e^{-2t\inf Z - a^2/2t}\right\}^{1/2}.
\end{aligned} \tag{6.8}$$

where $Z^a(x) := \inf\{Z(y)||y-x| \le a\}$ and $\beta_j, j = 1, 2, 3,$ are positive constants.

Theorem 6.11 *Let $V = Z + W \in V_{\rm bound} \cap V_{\rm exp}$ with Z, W in the definition of $V_{\rm exp}$. Suppose that*

$$Z(x) \ge \gamma|x|^{2m}$$

outside a compact set for some positive constants γ and m. Then for each positive constant δ sufficiently small, there is $D(\delta)$ such that

$$\|\Psi_{\rm g}(x)\| \le D(\delta)\exp\left(-\delta|x|^{m+1}\right). \tag{6.9}$$

Proof: In Lemma 6.10, we set $a = a(x) = \beta_4|x|$ and $t = t(x) = \beta_5|x|$. Then, for $\delta < \min\{2\beta_5(1-\beta_4)^2, \beta_4^2/2\beta_5\}$, there exits $D(\delta)'$ such that

$$Ce^{tE}\mathbf{E}e^{-\int_0^t V(X_s)ds} \le D(\delta)'e^{-\delta|x|^{m+1}}$$

for $|x| > N$ with some sufficiently large N (see [46, Proposition 3.1] for details). By Lemma 6.9 we see that, for all $f \in C_0^\infty(\mathbf{R}^d)$ with $f \ge 0$

$$\int_{\{|x|>N\}} f(x)\left(\|\Psi_{\rm g}(x)\|^2 - D(\delta)'e^{-\delta|x|^{m+1}}\right)dx < 0.$$

Thus (6.9) holds for $|x| > N$. By Lemma 6.8 $\|\Psi_{\rm g}(x)\|$ is bounded. Thus theorem follows. QED

Theorem 6.12 *Let $V = Z + W \in V_{\rm bound} \cap V_{\rm exp}$ with Z, W in the definition of $V_{\rm exp}$. Suppose that*

$$\liminf_{|x|\to\infty} Z(x) > E.$$

Then there exists a positive constant D and δ such that

$$\|\Psi_{\rm g}(x)\| \le De^{-\delta|x|}.$$

Proof: By Lemma 6.10, we prove theorem in a similar way as that of Theorem 6.11 and [46, Proposition 4.1]. Hence we omit it. QED

From Theorems 6.11 and 6.12, it follows that, for V in Theorems 6.11 or/and 6.12,

$$\||x|^k\Psi_{\rm g}\| < \infty$$

for all $k \in \mathbf{N}$. The next corollary tells us a more strong statement.

Corollary 6.13 *Let V be as in Theorems 6.11 or/and 6.12. Then*

$$\||x|^k\Psi_{\rm g}\| \le \sup_{x\in\mathbf{R}^d}\left(\mathbf{E}|x|^{2k}e^{-2\int_0^t V(X_s)ds}e^{2tE}\right)^{1/2}\|\Psi_{\rm g}\|$$

for all $k \ge 0$ and $t \ge 0$.

Proof: By (6.7) we see that

$$\||x|^k\Psi_{\rm g}\|^2 = \int_{\mathbf{R}^d} dx|x|^{2k}\|\Psi_{\rm g}(x)\|^2 \le \int_{\mathbf{R}^d} dx|x|^{2k}\left(e^{tE}\mathbf{E}e^{-\int_0^t V(X_s)ds}\|\Psi_{\rm g}(X_t)\|\right)^2$$
$$\le \int_{\mathbf{R}^d} dx|x|^{2k}\left(\mathbf{E}e^{-2\int_0^t V(X_s)ds}e^{2tE}\right)\left(\mathbf{E}\|\Psi_{\rm g}(X_t)\|^2\right).$$

Thus corollary follows. QED

6.3 The existence of ground states without infrared cutoffs

In this subsection, we take the Fock-Cook representation. The essential idea of a proof of the existence of the ground state of H is due to J.Glimm and A.Jaffe [84] and we learned it by A.Arai and M.Hirokawa [25]. We assume that [29]

$$-\Delta \leq aH_{\rm p} + b$$

with some positive constants a and b and

$$\Sigma - E_{\rm p} > 0, \quad \text{(positive spectral gap)},$$

where $\Sigma := \sigma_{\rm ess}(H_{\rm p})$. Moreover let $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \sqrt{\omega}\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$ and $|\alpha| \ll 1$. Using fundamental estimates (2.5), (2.6) and (2.7), we have

$$(1 - |\alpha|A - |\alpha|B)\, H_{\rm d} + |\alpha|AE_{\rm p} - |\alpha|C \leq H \leq (1 + |\alpha|A + |\alpha|B)\, H_{\rm d} - |\alpha|AE_{\rm p} + |\alpha|C, \tag{6.10}$$

where A, B, C are positive constants. Thus we get

$$|E - E_{\rm p}| \leq |\alpha|D,$$

where $D := |\alpha|BE_{\rm p} + |\alpha|C$. Let

$$\Gamma_a := \{k = (k_1, ..., k_d) \in \mathbf{R}^d | k_\mu = 2\pi n_\mu/a, n_\mu \in \mathbf{Z}, \mu = 1, ..., d\},$$

$$\Gamma(l, a) := [l_1, l_1 + 2\pi/a) \times \cdots [l_d, l_d + 2\pi/a).$$

By the map

$$l_2(\Gamma_a) \ni \{a_l\}_{l\in\Gamma_a} \to (a/2\pi) \sum_{l\in\Gamma_a} a_l \mathbf{1}_{\Gamma(l,a)}(\cdot) \in L^2(\mathbf{R}^d),$$

we identify $l_2(\Gamma_a)$ with a subspace of $L^2(\mathbf{R}^d)$. Define

$$\mathcal{F}^a_{\rm EM} := \mathcal{F}_{\rm EM}(L_2(\Gamma_a)) := \underbrace{\mathcal{F}(l_2(\Gamma_a)) \otimes \cdots \otimes \mathcal{F}(l_2(\Gamma_a))}_{d} \subset \mathcal{F}_{\rm EM}.$$

Set

$$H^{m,a}_{\rm b} := d\Gamma_{\rm b}(\omega(k_a) + m)$$

[29] Let A and B be self-adjoint operators in a Hilbert space K. We say that $A \leq B$ if $D(B) \subset D(A)$ and $(f, Af) \leq (f, Bf)$ for all $f \in D(B)$.

and

$$A_\mu^a := \frac{1}{\sqrt{2}}\left\{a^{r\dagger}\left(\sum_{l\in\Gamma_a}\mathbf{1}_{\Gamma(l,a)}\hat{\lambda}(-l)e^{-ilx}e_\mu^r(l)\right)+a^r\left(\sum_{l\in\Gamma_a}\mathbf{1}_{\Gamma(l,a)}\hat{\lambda}(l)e^{ilx}e_\mu^r(l)\right)\right\},$$

where $k_{a\mu} := k_{a\mu}(k_\mu) := 2\pi n/a$ if $k_\mu \in [2\pi n/a, 2\pi(n+1)/a)$. Note that

$$\sigma(H_{\rm b}^{m,a}) = \sigma_{\rm disc}(H_{\rm b}^{m,a}).$$

Thus a lattice Hamiltonian with an artificial mass m is defined by

$$H_{m,a} := \frac{1}{2}\left(\mathbf{p}-\alpha A^a\right)^2 + V + H_{\rm b}^{m,a}.$$

Lemma 6.14 *$H_{m,a}$ is reduced by $\mathcal{H}_a := L^2(\mathbf{R}^d)\otimes\mathcal{F}_{\rm EM}^a$.*

Proof: See [113] QED

Lemma 6.15 *Let $E_{m,a} := \inf\sigma(H_{m,a})$. Then*

$$H_{m,a}\lceil_{\mathcal{H}_a^\perp} \ge m + E_{m,a}.$$

Proof: For instance we set $l_2 := l_2(\Gamma_a)$. Since $L^2(\mathbf{R}^d) = l_2 \oplus l_2^\perp$, it is seen that

$$\mathcal{F}_{\rm EM} \cong \mathcal{F}_{\rm EM}^a \otimes \mathcal{F}_{\rm EM}(l_2^\perp). \tag{6.11}$$

Let P be the vacuum projection of $\mathcal{F}_{\rm EM}(l_2^\perp)$. Then

$$\mathcal{F}_{\rm EM} \cong \mathcal{F}_{\rm EM}^a \oplus \left(\mathcal{F}_{\rm EM}^a \otimes P^\perp\mathcal{F}_{\rm EM}(l_2^\perp)\right) := \mathcal{F}_{\rm EM}^a \oplus {\mathcal{F}_{\rm EM}^a}^\perp. \tag{6.12}$$

Under identification (6.11) we have

$$H_{m,a} \cong H_{m,a}\lceil_{\mathcal{H}_a}\otimes\mathbf{1} + \mathbf{1}\otimes H_{\rm b}^{m,a}.$$

Then we obtain that

$$\begin{aligned} H_{m,a} &\cong (H_{m,a}\lceil_{\mathcal{H}_a}\otimes P)\oplus\left(H_{m,a}\lceil_{\mathcal{H}_a}\otimes P^\perp\right) + (\mathbf{1}\otimes H_{\rm b}^{m,a}P)\oplus\left(\mathbf{1}\otimes H_{\rm b}^{m,a}P^\perp\right)\\ &\cong (H_{m,a}\lceil_{\mathcal{H}_a}\otimes P)\oplus\left(H_{m,a}\lceil_{\mathcal{H}_a}\otimes P^\perp + \mathbf{1}\otimes H_{\rm b}^{m,a}P^\perp\right)\\ &\cong (H_{m,a}\lceil_{\mathcal{H}_a})\oplus\left(H_{m,a}\lceil_{\mathcal{H}_a}\otimes\mathbf{1} + \mathbf{1}\otimes H_{\rm b}^{m,a}\right)P^\perp. \end{aligned}$$

Hence

$$H_{m,a}\lceil_{\mathcal{H}_a^\perp} \cong \left(H_{m,a}\lceil_{\mathcal{H}_a}\otimes\mathbf{1} + \mathbf{1}\otimes H_{\rm b}^{m,a}\right)P^\perp \ge E_{m,a} + m.$$

Thus we get the desired results. QED

Lemma 6.16 *Let α and m be such that*

$$0 < m < (1 - |\alpha|A - |\alpha|B)(\Sigma - E_{\rm p}) - 2|\alpha|D.$$

Then, for sufficiently large $a > 0$,

$$[E_{m,a}, E_{m,a} + m) \subset \sigma_{\rm disc}(H_{m,a}).$$

Proof: For sufficiently large a, (6.10) holds true with $\omega, \hat\lambda$ replaced by ω_m^a and $(a/2\pi)\sum_{l\in\Gamma_a}\hat\lambda(l)\mathbf{1}_{\Gamma(l,a)}(\cdot)$. Let $E_{\rm p} < \Sigma' < \Sigma$. Let $l := 1 - |\alpha|A - |\alpha|B$ and $\overline{H_{\rm p}} := H_{\rm p} - E_{\rm p}$. We denote by E_A^T the spectral projection of an operator T to a Borel set $A \subset \mathbf{R}$. We have, by (6.10)

$$H_{m,a}\lceil_{\mathcal{H}_a} \geq l H_{\rm b}^{m,a} + l\overline{H_{\rm p}} + E_{\rm p} - |\alpha|D.$$

Hence

$$\begin{aligned} H_{m,a}\lceil_{\mathcal{H}_a} - m - E_{m,a} &\geq l\overline{H_{\rm p}} + \{lH_{\rm b}^{m,a} - (m + E_{m,a} - E_{\rm p} + |\alpha|D)\} \\ \geq l(\Sigma' - E_{\rm p})E^{\overline{H_{\rm p}}}_{[\Sigma'-E_{\rm p},\infty)} &+ (E^{\overline{H_{\rm p}}}_{[0,\Sigma'-E_{\rm p})} \oplus E^{\overline{H_{\rm p}}}_{[\Sigma'-E_{\rm p},\infty)})\{lH_{\rm b}^{m,a} - (m + E_{m,a} - E_{\rm p} + |\alpha|D)\} \\ &= E^{\overline{H_{\rm p}}}_{[0,\Sigma'-E_{\rm p})} \otimes \{lH_{\rm b}^{m,a} - (m + E_{m,a} - E_{\rm p} + |\alpha|D)\} + E^{\overline{H_{\rm p}}}_{[\Sigma'-E_{\rm p},\infty)} \otimes lH_{\rm b}^{m,a} \\ &\quad + \{l(\Sigma' - E_{\rm p}) + E_{\rm p} - |\alpha|D - m - E_{m,a}\}E^{\overline{H_{\rm p}}}_{[\Sigma'-E_{\rm p},\infty)} \\ &\geq E^{\overline{H_{\rm p}}}_{[0,\Sigma'-E_{\rm p})} \otimes E^{H_{\rm b}^{m,a}}_{[0,(|\alpha|D+m+E_{m,a}-E_{\rm p})/l)}. \end{aligned} \tag{6.13}$$

Since the dimension of the range of the right-hand side of (6.13) is finite, that of $E^{H_{m,a}\lceil_{\mathcal{H}_a}}_{[0,m+E_{m,a})}$ is also finite. Thus lemma follows. QED

We define H_m by H with ω replaced by $\omega_m := \omega + m$.

Lemma 6.17 *Let α and m be as in Lemma 6.16. Then H_m has a ground state.*

Proof: Let $E_m := \inf\sigma(H_m)$. Let

$$U_a := \exp\left(ix_\mu \otimes d\Gamma_{\rm b}(k_{a,\mu})\right), \quad U := \exp\left(ix_\mu \otimes d\Gamma_{\rm b}(k_\mu)\right).$$

Then we have

$$U_a H_{m,a} U_a^{-1} = \frac{1}{2}\Big(\mathbf{p}\otimes\mathbf{1} - \mathbf{1}\otimes d\Gamma_{\rm b}(\vec{k_a}) - \alpha\mathbf{1}\otimes A^a(0)\Big)^2 + V\otimes\mathbf{1} + \mathbf{1}\otimes H_{\rm b}^{m,a}.$$

It is a direct calculation to show that $U_a(H_{m,a}-i)^{-1}U_a^{-1}$ uniformly converges to $U(H_m-i)^{-1}U^{-1}$ as $a\to\infty$. Hence by Lemma 6.16, we get that $[E_m, E_m+m)\subset \sigma_{\rm disc}(H_m)$. Thus lemma follows. QED

Let $\Psi_{\rm g}^{(m)}$ be the ground state of H_m. We fix $r=1,...,d-1$ and $f\in L^2(\mathbf{R}^d)$ and set

$$g_\mu := \frac{1}{\sqrt{2}}\tilde{\hat{\lambda}} e_\mu^r e^{-ikx}, \quad G_\mu := (\overline{f}, g_\mu), \quad \mu=1,...,d.$$

Let F be a smooth function on $\mathbf{R}^d$, and l a constant. We have on some domain

$$[a^r(f)+lF, H]$$

$$= -a^r(\omega_m f) + i\alpha G_\mu\left(\partial_\mu - i\alpha A_\mu\right) + l\left\{(1/2)\left(\partial_\mu^2 F\right) + \left(\partial_\mu F\right)\left(\partial_\mu - i\alpha A_\mu\right)\right\}.$$

To neglect both of $G_\mu A_\mu$ and $G_\mu\partial_\mu$, we define $\theta := -i\alpha x\cdot G$. We have

$$[a^r(f)+\theta, H] = -a^r(\omega_m f) - i\alpha\left\{(1/2)\partial_\mu^2(x\cdot G) + x_\nu(\partial_\mu G_\nu)\left(\partial_\mu - (i\alpha)A_\mu\right)\right\}.$$

Since $\partial_\mu^2\left(x\cdot G\right) = 2\left(\partial_\mu G_\mu\right) + x_\nu(\partial_\mu^2 G_\nu)$, finally we have

$$[a^r(f)+\theta, H] = -a^r(\omega_m f) + (-i\alpha)\vartheta, \tag{6.14}$$

where

$$\vartheta := (\partial_\mu G_\mu) + (1/2)x_\nu(\partial_\mu^2 G_\nu) + x_\nu(\partial_\mu G_\nu)(\partial_\mu - i\alpha A_\mu).$$

Lemma 6.18 ([34]) *Let $V\in V_{\rm exp}$. Then there exists a constant $\mathbf{C}$ independent of m such that*

$$\|N_{\rm b}^{1/2}\Psi_{\rm g}^{(m)}\|^2 \le |\alpha|\mathbf{C}\left(\||x|^2\Psi_{\rm g}^{(m)}\|^2 + \||x|\Psi_{\rm g}^{(m)}\|^2 + \|\Psi_{\rm g}^{(m)}\|^2\right).$$

Remark 6.19 *In this lemma we do not assume the infrared cutoff condition: $\hat{\lambda}/\omega\in L^2(\mathbf{R}^d)$.*

Proof: This proof is due to V.Bach, J.Fröhlich and I.E.Sigal [34]. Since

$$((a^r(f)+\theta)\Psi_{\rm g}^{(m)}, (H_m - E)(a^r(f)+\theta)\Psi_{\rm g}^{(m)}) \ge 0,$$

we see that

$$((a^r(f)+\theta)\Psi_{\rm g}^{(m)}, [H_m, a^r(f)+\theta]\Psi_{\rm g}^{(m)}) \ge 0.$$

Thus from (6.14) it follows that

$$(\Psi_{\rm g}^{(m)}, a^{r\dagger}(\omega_m f)a^r(f)\Psi_{\rm g}^{(m)}) \le ((a^r(f)+\theta)\Psi_{\rm g}^{(m)}, (-i\alpha)\vartheta\Psi_{\rm g}^{(m)}) - (\theta\Psi_{\rm g}^{(m)}, a^r(\omega_m f)\Psi_{\rm g}^{(m)}). \tag{6.15}$$

Substituting $f_l/\sqrt{\omega_m}$ for f in (6.15) with $\{f_l\}_{l=1}^\infty$ CONS of $L^2(\mathbf{R}^d)$ and summing up l from one to infinity, we have

$$\begin{aligned}(\Psi_{\rm g}^{(m)}, N\Psi_{\rm g}^{(m)}) &\le (-i\alpha)\Big\{\Big(a^r(ik_\nu g_\nu/\omega)\Psi_{\rm g}^{(m)}, \Psi_{\rm g}^{(m)}\Big)\\ &+(1/2)\Big(a^r(-k_\mu k_\mu g_\nu/\omega)\Psi_{\rm g}^{(m)}, x_\nu\Psi_{\rm g}^{(m)}\Big) + \Big(a^r(ik_\mu g_\nu/\omega)\Psi_{\rm g}^{(m)}, x_\nu(\partial_\mu - i\alpha A_\mu)\Psi_{\rm g}^{(m)}\Big)\Big\}\\ &-\alpha^2\Big\{(g_\nu, ik_\mu g_\mu/\omega_m)\,(x_\nu\Psi_{\rm g}^{(m)}, \Psi_{\rm g}^{(m)}) + (1/2)\Big(g_\nu, -k_\mu^2 g_{\nu'}/\omega_m\Big)\,(x_\nu\Psi_{\rm g}^{(m)}, x_{\nu'}\Psi_{\rm g}^{(m)})\\ &+(g_\nu, ik_\mu g_{\nu'}/\omega_m)(x_\nu\Psi_{\rm g}^{(m)}, x_{\nu'}(\partial_\mu - i\alpha A_\mu)\Psi_{\rm g}^{(m)})\Big\}\end{aligned} \tag{6.16}$$

It is established ([113]) that

$$\|\mathbf{p}_\mu\Psi_{\rm g}^{(m)}\| \le C'\|\Psi_{\rm g}^{(m)}\|, \quad \mu = 1, ..., d,$$

with some constant C'. Note that

$$\|k_\mu k_\nu g_\gamma/\omega_m\| \le \|\omega\hat{\lambda}\|, \quad \|k_\mu g_\nu/\omega_m\| \le \|\hat{\lambda}\|, \quad \mu, \nu, \gamma = 1, ..., d.$$

By inequalities (2.8) and (2.9), there exists constants C'' and C''' independent of $\|\hat{\lambda}/\omega\|$ and m such that

$$\|N_{\rm b}^{1/2}\Psi_{\rm g}^{(m)}\|^2 \le |\alpha|C''\|N^{1/2}\Psi_{\rm g}^{(m)}\| + |\alpha|C'''\left(\||x|^2\Psi_{\rm g}^{(m)}\|^2 + \||x|\Psi_{\rm g}^{(m)}\|^2 + \|\Psi_{\rm g}^{(m)}\|^2\right).$$

Thus lemma follows. QED

Lemma 6.20 *Let* $Q := E_{[E_{\rm p}+\epsilon,\infty)}^{\overline{H_{\rm p}}} \otimes E_{\{0\}}^{H_{\rm b}}$ *with* $\epsilon < \Sigma$. *Then there exists a constant* $\mathbf{D}$ *independent of* m *such that*

$$\|Q\Psi_{\rm g}^{(m)}\| \le |\alpha|\mathbf{D}\|\Psi_{\rm g}^{(m)}\|/(\Sigma - E_{\rm p}).$$

Proof: See [109, 113]. QED

Theorem 6.21 *Suppose that* V *is in Theorems 6.11 and/or 6.12, and* $|\alpha| \ll 1$. *Then the ground states of* H *exists.*

Proof: Let $\Psi_{\rm g}^{(m)}$ be the normalized ground state of H_m. There exists a subsequence m' such that $\Psi_{\rm g}^{(m')}$ weakly converges to a vector Ψ as $m' \to \infty$. If $\Psi \neq 0$, Ψ is the ground state. Let $P := E^{\overline{H_{\rm p}}}_{[E_{\rm p}, E_{\rm p}+\epsilon)} \otimes E^{H_{\rm b}}_{\{0\}}$. Since $P + Q \geq \mathbf{1} - N_{\rm b}$, we have

$$(\Psi_{\rm g}^{(m')}, P\Psi_{\rm g}^{(m')}) \geq \|\Psi_{\rm g}^{(m')}\|^2 - \|Q\Psi_{\rm g}^{(m')}\|^2 - \|N_{\rm b}^{1/2}\Psi_{\rm g}^{(m')}\|^2.$$

From Corollary 6.13 it follows that

$$\||x|^2\Psi_{\rm g}^{(m')}\| + \||x|\Psi_{\rm g}^{(m')}\| \leq C\|\Psi_{\rm g}^{(m')}\|,$$

where C is independent of m. Thus there exists C' independent of m such that

$$\|N_{\rm b}^{1/2}\Psi_{\rm g}^{(m')}\| \leq C'\|\Psi_{\rm g}^{(m')}\|.$$

Since P is a finite rank operator, taking $m' \to \infty$, we get

$$(\Psi, P\Psi) \geq 1 - |\alpha|C' - \alpha^2 \left(\mathbf{D}/(\Sigma - E_{\rm p})\right)^2 > 0.$$

Thus theorem follows. QED

Corollary 6.22 *We assume the same condition as that of Theorem 6.21. Then*

$$s - \lim_{\alpha \to 0} \Psi_{\rm g} = \phi_{\rm p} \otimes \Omega,$$

where $\phi_{\rm p}$ is the ground state of $H_{\rm p}$.

Proof: It follows from the uniqueness of the ground state and Theorem 6.21. QED

6.4 Ground state energy

Let $f \in L^2(\mathbf{R}^d)$ be positive. Then, by Corollary 6.7,

$$(f \otimes \Omega, \Psi_{\rm g}) = (f \otimes \Omega, U\Psi_{\rm g}) \neq 0. \tag{6.17}$$

Theorem 6.23 ([110]) *Let $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \sqrt{\omega}\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$. We assume that there exists the ground state of H. Then*

$$E = E(\alpha^2) = -\lim_{t\to\infty} \frac{1}{t} \log \int_{\mathcal{W}} dP e^{-\int_0^t V(X_s)ds} f(X_0) f(X_t) e^{-(\alpha^2/2) q_0(\mathbf{K}_t(X))}. \tag{6.18}$$

In particular $E(\alpha^2)$ is a continuous, monotonously increasing, concave function in α^2.

Proof: From (6.17) it follows that

$$E(\alpha^2) := -\lim_{t\to\infty}\frac{1}{t}\log\left(f\otimes\Omega, e^{-tH}f\otimes\Omega\right).$$

By Theorem 4.1, (6.18) follows. By a Hölder inequality we see that $E(\alpha^2)$ is concave, which implies that $E(\alpha^2)$ is continuous in $\alpha^2 > 0$. Since H converges to $H_{\rm d}$ as $\alpha\to\infty$ uniformly in the sense of resolvent, $\lim_{\alpha^2\to 0} E(\alpha^2) = E(0)$. Hence $E(\alpha^2)$ is continuous in $\alpha^2 \geq 0$. Concave continuous function $E(\alpha^2)$ can be represented as

$$E(\alpha^2) = E(0) + \int_0^{\alpha^2}\phi(t)dt$$

with some increasing function $\phi(t)$. Moreover we have by a diamagnetic inequality, $\phi(t)\geq 0$. Thus $E(\alpha^2)$ is monotonously increasing.[30] QED

6.5 Degenerate ground states with singular potentials

In this subsection we give a simple example of external potentials for which H has degenerate ground states. For classical case see [65, 66, 69]. Assume that $\hat\lambda/\sqrt{\omega}, \hat\lambda, \sqrt{\omega}\hat\lambda, \omega\hat\lambda \in L^2(\mathbf{R}^d)$. Let D_j, $j = 1, ..., J$, be open sets such that

$$\bigcup_{j=1}^J \overline{D_j} = \mathbf{R}^d, \quad \bigcap_{j=1}^J D_j = \emptyset,$$

and the Lebesgue measure of the boundary $S := \partial\left(\bigcup_{j=1}^J D_j\right)$ is zero. Let V be such that $V_+ \in L^1_{\rm loc}(\mathbf{R}^d\setminus S)$, $D(\Delta)\cap D(V_+)$ is dense in $\mathbf{R}^d$, and V_- is infinitesimally small with respect to the Laplacian in the sense of form. We assume that

$$\int_0^t V_+(X_s)ds = 0, \tag{6.19}$$

if $X_0\in D_i$ and $X_t \in D_j$, $i\neq j$. Moreover we suppose that $H_{{\rm p}_j} := H_{\rm p}\lceil_{L^2(D_j)}$ is essentially self-adjoint on $C_0^\infty(D_j)$ and

$$-\Delta \leq aH_{{\rm p}_j} + b, \quad j = 1, ..., J,$$

on $L^2(D_j)$ with some constants a and b. Finally we make assumption:

$$E_{{\rm p}_j} := \inf\sigma(H_{{\rm p}_j}) \in \sigma_{\rm disc}(H_{{\rm p}_j}), \quad \sigma_{\rm ess}(H_{{\rm p}_j}) - E_{{\rm p}_j} > 0.$$

[30] For the Nelson model and a spin-boson model, we can get the similar expression of the ground state energy in terms of probability measures. For a spin-boson model M.Hirokawa [100] directly expands its pair potential term and get a bound of its ground state energy.

Lemma 6.24 *Let P_j be the projection of $L^2(\mathbf{R}^d)$ to $L^2(D_j)$. Then*

$$e^{-tH}P_j = P_j e^{-tH}, \quad t \geq 0.$$

Proof: Let $F, G \in C_0^\infty(D_j)\widehat{\otimes}L_0^2(Q)$. We extend functional integral representation in Theorem 4.1 to external potentials such as stated above. We see that, by (6.19),

$$\left(F, e^{-tH}P_j G\right)_{\mathcal{H}} = \int_{\mathcal{W}_j} dPe^{-\int_0^t V(X_s)ds}\left(J_0 F(X_0), e^{i\alpha\phi_0(\mathbf{K}_t(X))}J_t(P_j G)(X_t)\right)$$

$$\int_{\mathcal{W}_j} dPe^{-\int_0^t V(X_s)ds}\left(J_0\left(P_j F\right)(X_0), e^{i\alpha\phi_0(\mathbf{K}_t(X))}J_t G(X_t)\right) = \left(P_j F, e^{-tH}G\right)_{\mathcal{H}},$$

where $\mathcal{W}_j$ is the set of paths $q(\cdot)$ such that $q(s) \in D_j$ for $0 \leq s \leq t$. Thus lemma follows. QED

Lemma 6.25 *Let $|\alpha|$ be sufficiently small. Then H_j is reduced by $L^2(D_j) \otimes L^2(Q)$ and*

$$H_j := H\lceil_{L^2(D_j)\otimes L^2(Q)}$$

is essentially self-adjoint on $C_0^\infty(D_j)\widehat{\otimes}[L_0^2(Q) \cap D(H_\mathrm{f})]$. Moreover the ground state of H_j exits and it is unique.

Proof: By Lemma 6.24, H_j is reduced by $L^2(D_j) \otimes L^2(Q)$. Since $H_{\mathrm{p}j}$ is essentially self-adjoint on $C_0^\infty(D_j)$, the Kato-Rellich theorem yields the essential self-adjointness of H_j. In the similar way as the proofs of Theorems 6.6 and 6.21, one can prove the existence and uniqueness of the ground state of H_j. QED

Lemma 6.26 (A.Arai [15]) *We have $\sigma_{\mathrm{ess}}(H) = [E, \infty)$.*

Let $m(a)$ denote the multiplicity of a point spectrum a of H.

Theorem 6.27 ([115]) *Set $E_j = \inf \sigma H_j$, $j = 1, ..., J$. Then E_j is an eigenvalue of H and*

$$m(E_j) \geq \#\{E_k | E_k = E_j, k = 1, ..., J\}.$$

Moreover

$$\lim_{\alpha\to 0} E_j = \inf \sigma(H_{\mathrm{p}j}).$$

Proof: Let $\mathcal{H}_j := L^2(D_j) \otimes L^2(Q)$ and Ψ_j be the unique ground state of H_j. Since $H \cong \oplus_{j=1}^J H_j$ on $\mathcal{H} \cong \oplus_{j=1}^J \mathcal{H}_j$, vectors $\oplus_{j=1}^J \delta_{ij}\Psi_j$ are eigenvectors with eigenvalues E_j. Thus theorem follows. QED

Corollary 6.28 *Define* $E := \min_k E_k = \inf \sigma(H)$. *Let*

$$\overline{H} := H - E - \sum_{j=1}^{J} (E_j - E)\mathbf{1}_{D_j}.$$

Then $\overline{H}$ *has* J*-fold ground states.*

A typical example of $\{D_j\}$ and V is as follows: let $d = 3$, $J = 3$, and

$$D_1 := \{x \in \mathbf{R}^3 | x_1 > 0, x_2 > 0, x_3 > 0\},$$

$$D_2 := \{x \in \mathbf{R}^3 | x_1 < 0, x_2 < 0, x_3 < 0\},$$

$$D_3 := \mathbf{R}^3 \setminus \overline{D_1 \cup D_2}, \quad D := \cup_{j=1}^3 D_j.$$

Define

$$V_\nu(x) := \frac{\nu}{|x - \partial D|^3} + |x|^2 + m\mathbf{1}_{D_1} + n\mathbf{1}_{D_2},$$

where ν, m and n are positive constants. Taking sufficiently large ν, we see that $-\Delta/2 + V_\nu \lceil_{L^2(D_j)}$ is essentially self-adjoint on $C_0^\infty(D_j)$ ([136]) and satisfies the assumptions stated in the beginning of this subsection (see [115]). Let

$$H(\nu) := H_{\mathrm{p}} + H_{\mathrm{f}} + V_\nu.$$

From the functional integral representation it follows that

$$\lim_{\nu\to 0} \left(F, e^{-tH(\nu)}G\right) = \int_{\mathcal{W}_j} dP e^{-\int_0^t V_0(X_s)ds} (J_0 F(X_0), e^{i\alpha\phi_0(K(X))} J_t G(X_t))$$

$$\neq \int_{\mathcal{W}} dP e^{-\int_0^t V_0(X_s)ds} (J_0 F(X_0), e^{i\alpha\phi_0(K(X))} J_t G(X_t)) = \left(F, e^{-tH(0)}G\right).$$

Namely

$$s - \lim_{\nu\to 0} e^{-tH(\nu)} \neq e^{-tH(0)}.$$

This phenomena carries an interesting consequence that once turned on the effects of the singular potential cannot be completely turned off. See [144, 65, 66, 69, 187, the Klauder phenomena].

6.6 The Kato-Mugibayashi-H.Krohn type scattering theory

For instance we let $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \sqrt{\omega}\hat{\lambda}, \omega\hat{\lambda} \in L^2(\mathbf{R}^d)$,

$$V(x) := x^2, \quad |\alpha| \ll 1,$$

in this subsection. Let

$$a_t^{r\sharp}(f) := e^{itH} e^{-itH_0} a^{r\sharp}(f) e^{itH_0} e^{-itH}, \quad r = 1, ..., d-1.$$

We want to consider the strong limit of $a_t^{r\sharp}$ as $t \to \pm\infty$. We will focus on $s - \lim_{t\to\infty} a_t^{r\dagger}$ in what follows. The other statements are similar. From the definition of $a_t^{r\dagger}$ and fundamental limiting arguments [31], we have

$$a_t^{r\dagger}(f)\Psi = a_T^{r\dagger}(f)\Psi - i \int_T^t e^{isH} \alpha K_\mu^r(s, x, f) \left(\mathbf{p}_\mu - A_\mu(x)\right) e^{-isH} \Psi ds, \qquad (6.20)$$

where

$$K_\mu^r(s, x, f) := [A_\mu(\hat{\lambda}, x), a^{r\dagger}(e^{-is\omega} f)] = \left(\overline{\frac{\hat{\lambda} e_\mu^r e^{-ikx}}{\sqrt{2}}}, e^{-is\omega} f \right).$$

Let $\mathcal{E}$ be as follows: $f \in \mathcal{E}$ if

$$\lim_{t\to\infty} t^{\frac{d-1}{2}} \sup_{x\in\mathbf{R}^d} \left| \int_{\mathbf{R}^d} f(k) h(k) e^{ikx - it\omega(k)} dk \right| < \infty \quad \text{for all } h \in C_0^\infty(\mathbf{R}^d).$$

Lemma 6.29 *Let $\hat{\lambda}, \partial_\mu \hat{\lambda} \in \mathcal{E}$ and $f \in C_0^\infty(\mathbf{R}^d)$. Then $s - \lim_{t\to\infty} a_t^{r\sharp}(f)\Psi$ exits for $\Psi \in D(H)$.*

Proof: By virtue of (6.20) it is enough to prove that

$$\left\| K_\mu^r(t, x, f) \left(\mathbf{p}_\mu - \mathbf{A}_\mu(x)\right) e^{-itH} \Psi \right\|_{\mathcal{H}} \in L^1([T, \infty), dt). \qquad (6.21)$$

Using

$$e^{it\omega(k)} = \frac{\omega(k)}{k_\mu} \frac{1}{it} \frac{\partial}{\partial k_\mu} e^{it\omega(k)}, \quad k \in \mathbf{R}^d \setminus \{0\},$$

and integrating by parts, one sees that that

$$\text{L.H.S. } (6.21) \le C_1 t^{-(d+1)/2}$$

[31] Formally it follows that

$$a_t^{r\sharp}\Psi = a_T^{r\sharp}(f)\Psi + i \int_T^t e^{isH} [-\alpha \mathbf{p} A(x) + \alpha^2 A^2(x), a^{r\dagger}(e^{-is\omega} f)] e^{-isH} \Psi ds.$$

$$\times \left(\|x_\mu \mathbf{p}_\mu e^{-itH}\Psi\| + \|\mathbf{p}_\mu e^{-itH}\Psi\| + \|x_\mu A_\mu(x) e^{-itH}\Psi\| + \|A_\mu(x) e^{-itH}\Psi\| \right) \quad (6.22)$$

with some constant C_1. Since $V(x) = x^2$, we have

$$\|x_\mu \mathbf{p}_\mu e^{-itH}\Psi\| \leq C_2 (\|H\Psi\| + \|\Psi\|)$$

with some constant C_2. The other terms in (6.22) are estimated similarly and we have, with some constant C_3,

$$\text{L.H.S. } (6.21) \leq C_3 t^{-(d+1)/2} (\|H\Psi\| + \|\Psi\|) \in L^1([T, \infty), dt).$$

QED

We define, for $\Psi \in D(H)$,

$$s - \lim_{t \to \pm} a_t^{r\sharp}(f)\Psi := a_\pm^{r\sharp}(f)\Psi.$$

It is immediately seen that

$$\|a_\pm^{r\sharp}(f)\Psi\| \leq C_4 (\|f/\sqrt{\omega}\| + \|f\|)(\||H|^{1/2}\Psi\| + \|\Psi\|)$$

with some constant C_4. Hence we extend $a_\pm^{r\sharp}(f)$ to $f, f/\sqrt{\omega} \in L^2(\mathbf{R}^d)$. The closure of $a_\pm^{r\sharp}(f)$ is written as the same symbol. Then $D(a_\pm^{r\sharp}(f)) \supset D(|H|^{1/2})$. Moreover we have

$$[a_\pm^r(f), a_\pm^{s\dagger}(g)] = \delta_{rs}(\bar{f}, g), \quad [a_\pm^{r\sharp}(f), a^{s\sharp}(g)_\pm] = 0,$$

and

$$\begin{aligned} e^{itH} a_\pm^{r\dagger}(f) e^{-itH} &= a_\pm^{r\dagger}(e^{it\omega} f), \\ e^{itH} a_\pm^r(f) e^{-itH} &= a_\pm^r(e^{-it\omega} f) \end{aligned} \quad (6.23)$$

on $D(H)$. Let Ψ_g be the ground state of H. Then

$$a_\pm^r(f)\Psi_\mathrm{g} = 0, \quad \text{for all } f, f/\sqrt{\omega} \in L^2(\mathbf{R}^d).$$

We define an asymptotic Hilbert space $\mathcal{H}_{\pm\mathrm{asy}}$ by

$$\mathcal{H}_{\pm\mathrm{asy}} := \overline{\{a_\pm^{r_1\dagger}(f_1) \cdots a_\pm^{r_n\dagger}(f_n)\Psi_\mathrm{g}, \Psi_\mathrm{g} | f_j \in C_0^\infty(\mathbf{R}^d), r_j = 1, ..., d, j = 1, ..., n, n \in \mathbf{N}\}}.$$

Let $W_\pm : \mathcal{H}_{\pm\mathrm{asy}} \to \mathcal{F}_\mathrm{EM}$ be defined by

$$W_\pm a_\pm^{r_1\dagger}(f_1) \cdots a_\pm^{r_n\dagger}(f_n)\Psi_\mathrm{g} := a^{r_1\dagger}(f_1) \cdots a^{r_n\dagger}(f_n)\Omega_\mathrm{b},$$

$$W_\pm \Psi_\mathrm{g} := \Omega_\mathrm{b}.$$

Thus $W_\pm$ uniquely extends to a unitary operator of $\mathcal{H}_{\pm\mathrm{asy}}$ to $\mathcal{F}_\mathrm{EM}$.

Theorem 6.30 *We assume that the ground state of H exists. Then we have*

$$\sigma_{\rm ac}(H) = [E, \infty).$$

Proof: It is seen that e^{itH} is reduced by $\mathcal{H}_{\pm\rm asy}$. Then $H = \left(H\lceil_{\mathcal{H}_{\pm\rm asy}}\right) \oplus \left(H\lceil_{\mathcal{H}_{\pm\rm asy}^\perp}\right)$ under identification $\mathcal{H} \cong \mathcal{H}_{\pm\rm asy} \oplus \mathcal{H}_{\pm\rm asy}^\perp$. By the definition of W and (6.23), we have

$$W_\pm \left(e^{itH\lceil_{\mathcal{H}_{\pm\rm asy}}}\right) W_\pm^* = e^{it(H_{\rm f}+E)}.$$

Hence

$$H \cong (H_{\rm f} + E) \oplus H\lceil_{\mathcal{H}_{\pm\rm asy}^\perp}$$

under identification $\mathcal{H} \cong \mathcal{F}_{\rm EM} \oplus \mathcal{H}_{\pm\rm asy}^\perp$. Since $\sigma_{\rm ac}(H_{\rm f} + E) = [E, \infty)$, theorem follows. QED

Remark 6.31 *A.Arai [19] proved independently of the existence of the ground states of H that $\sigma_{\rm ess}(H) = [E, \infty)$ under some weaker conditions.*

7 Gibbs measures

In this section we assume that $V \in V_0$ and $\hat\lambda, \omega\hat\lambda \in L^2(\mathbf{R}^d)$. Related work of this section are V.Betz, F.Hiroshima, J.Lőrinczi, R.Minlos, H.Osada, H.Spohn [39, 38, 111, 116, 114, 158, 160, 168, 195].

7.1 The existence of an infinite time Gibbs measure

For positive $f \in L^2(\mathbf{R}^d)$, we define a finite-time Gibbs measure on the measure space $W_T := C([-T, \infty)) \times \mathbf{R}^d$ by

$$dW_{2T}^f := \frac{1}{Z_{2T}} f(q_{-T}) f(q_T) e^{\int_{-T}^T V(q_s) ds} e^{-(\alpha^2/2) q_0(\mathbf{K}_t(X))},$$

where $q_s := x + b(T + s)$, Z_{2T} is normalizing constant such as $\int dW_{2T}^f = 1$. Let $-T \le t_1 \ge \cdots t_m \le T$. Set

$$\mu_{A_1,\ldots,A_m}^{t_1,\ldots,t_m} := \int_{W_T} \mathbf{1}_{A_1}(q_{t_1}) \cdots \mathbf{1}_{A_m}(q_{t_m}) dW_{2T}^f.$$

From Theorem 4.1 it follows that

$$\mu_{A_1,\ldots,A_m}^{t_1,\ldots,t_m} = \frac{\left(f \otimes \Omega, e^{-(T+t_1)H} \mathbf{1}_{A_1} e^{-(t_2-t_1)H} \cdots e^{-(t_m-t_{m-1})H} \mathbf{1}_{A_m} e^{-(T-t_m)H} f \otimes \Omega\right)}{(f \otimes \Omega, e^{-2TH} f \otimes \Omega)}.$$

Thus $\mu^{t_1,\ldots,t_m}_{A_1,\ldots,A_m}$ is consistent. By Kolmogorov's construction, there exists a probability measure $(\Xi_T, \mathcal{B}(\Xi_T), \mu_T)$ such that

$$\mu^{t_1,\ldots,t_m}_{A_1,\ldots,A_m} = \int_{\Xi_T} \mathbf{1}_{A_1}(q_{t_1}) \cdots \mathbf{1}_{A_m}(q_{t_m}) \mu_T(dq),$$

where $\Xi_T := \left(\mathbf{R}^d\right)^{[-T,T]}$ and $\mathcal{B}(\cdot)$ denotes the smallest σ-field containing cylinder sets. Let Π_T be the projection of Ξ_∞ to Ξ_T. We define

$$\mu^{\rm ex}_T(A) := \mu_T(\Pi_T(A)), \quad A \in \mathcal{B}(\Xi_\infty).$$

We shall prove that

- there exists a continuous version of $(\Xi_\infty, \mathcal{B}(\Xi_\infty), \mu^{\rm ex}_T)$;
- there exists a subsequence T' such that $\mu^{\rm ex}_{T'}$ weakly converges to a measure μ on $(\Xi_\infty, \mathcal{B}(\Xi_\infty))$.

Note that there exists a constant C_n such that

$$\mathbf{E}|b(t) - b(s)|^{2n} = C_n |t-s|^n, \quad n \geq 0.$$

Lemma 7.1 *Let $\overline{H} = H - E$. Then we have*[32]

$$\left| \int_{\Xi_\infty} |q(t) - q(s)|^{2n} \mu^{\rm ex}_T(dq) \right| \leq |t-s|^n C_n e^{|t-s|(E - \inf V)} \left(\frac{\|f\|}{\|e^{-T\overline{H}} f \otimes \Omega\|} \right)^2 .$$

Proof: Let $q^a(s)$ and X^a_s are truncated paths defined by

$$q^a_\nu(s) := \begin{cases} q_\nu(s), & |q_\nu(s)| \leq a, \\ -a, & q_\nu(s) < -a, \\ a, & q_\nu(s) > a, \end{cases}$$

$$X^a_{\nu,s} := \begin{cases} X_{\nu,s}, & |X_{\nu,s}| \leq a, \\ -a, & X_{\nu,s} < -a, \\ a, & X_{\nu,s} > a. \end{cases}$$

Moreover we define

$$h^a_\nu(x) := \begin{cases} x_\nu, & |x_\nu| \leq a, \\ -a, & x_\nu < -a, \\ a, & x_\nu > a. \end{cases}$$

[32] Note that $E - \inf V \geq E - \inf \sigma(H_{\rm p}) \geq 0$.

We put

$$\psi := e^{-(T+t)\overline{H}}(f\otimes\Omega)/\left\|e^{-T\overline{H}}f\otimes\Omega\right\|, \quad \phi := e^{-(T-s)\overline{H}}(f\otimes\Omega)/\left\|e^{-T\overline{H}}f\otimes\Omega\right\|.$$

Then we have

$$\int_{\Xi_\infty}|q^a(s)-q^a(t)|^{2n}\mu_T^{\rm ex}(dq) = \sum_{k=0}^{2n} {}_{2n}C_k(-1)^k\int_{\Xi_\infty} q_\nu^a(s)^k q_\nu^a(t)^{2n-k}\mu_T^{\rm ex}(dq)$$

$$= \sum_{k=0}^{2n} {}_{2n}C_k(-1)^k \frac{\left(f\otimes\Omega, e^{-(T+t)\overline{H}}(h_\nu^a)^k e^{-(t-s)\overline{H}}(h_\nu^a)^{2n-k}e^{-(T-s)\overline{H}}f\otimes\Omega\right)}{\left(f\otimes\Omega, e^{-2T\overline{H}}f\otimes\Omega\right)}$$

$$= \sum_{k=0}^{2n} {}_{2n}C_k(-1)^k\left(\phi, (h_\nu^a)^k e^{-t(t-s)}(h_\nu^a)^{2n-k}\psi\right)$$

$$= \sum_{k=0}^{2n} {}_{2n}C_k(-1)^k\int_{\mathcal{W}} dP(X_{\nu,0}^a)^k(X_{\nu,t-s}^a)^{2n-k}e^{-\int_0^{t-s}V(X_{s'})ds'}\left(\phi(X_0), \mathbf{J}_{t-s}\psi(X_{t-s})\right)e^{|t-s|E}$$

$$\le \int_{\mathcal{W}} dP|\mathbf{b}(0)-\mathbf{b}(t-s)|^{2n}\,\|\phi(X_0)\|\,\|\psi(X_{t-s})\|\,e^{|t-s|(E-\inf V)}$$

$$\le C_n|t-s|^n\,\|\phi\|\left(\int_{\mathcal{W}} dP\,\|\psi(X_{t-s})\|^2\right)^{1/2}e^{|t-s|(E-\inf V)}$$

$$\le C_n|t-s|^n\,\|\phi\|\,\|\psi\|\,e^{|t-s|(E-\inf V)}.$$

Note that

$$\|\phi\|\le\|f\|/\left\|e^{-T\overline{H}}f\otimes\Omega\right\|, \quad \|\psi\|\le\|f\|/\left\|e^{-T\overline{H}}f\otimes\Omega\right\|.$$

Since $|q_\nu^a(t)-q_\nu^a(s)|\uparrow|q_\nu(t)-q_\nu(s)|$ as $a\uparrow\infty$, lemma follows by the Lebesgue monotone convergence theorem. QED

By this lemma there exists a continuous version of $(\Xi_\infty, \mathcal{B}(\Xi_\infty), \mu_T^{\rm ex})$, i.e., there exists $\Xi^{\rm cont}\in\mathcal{B}(\Xi_\infty)$ such that $\mu_T^{\rm ex}(\Xi^{\rm cont})=1$ and $\Xi^{\rm cont}\ni q(\cdot)$ is continuous. Define a probability measure $\overline{\mu}_T$ on $({\rm C}(\mathbf{R};\mathbf{R}^d), \mathcal{B}({\rm C}(\mathbf{R};\mathbf{R}^d)))$ by

$$\overline{\mu}_T(A) := \mu_T^{\rm ex}(A'),$$

where $A'\in\mathcal{B}(\Xi_\infty)$ such that $A'\cap{\rm C}(\mathbf{R};\mathbf{R}^d)=A$. It is immediate to see that $\overline{\mu}_T$ is well defined. Thus we had the following lemma:

Lemma 7.2 *We see that* $({\rm C}(\mathbf{R};\mathbf{R}^d), \overline{\mu}_T)$ *and* (W, dW_{2T}^f) *have the same finite dimensional distributions.*

Theorem 7.3 *We assume that there exists the ground state of H. Then there exists a subsequence T' such that $\overline{\mu}_{T'}$ weakly converges to a probability measure μ on $(\mathrm{C}(\mathbf{R};\mathbf{R}^d), \mathcal{B}(\mathrm{C}(\mathbf{R};\mathbf{R}^d)))$ as $T' \to \infty$.*

Proof: Let $\Pi := \{\overline{\mu}_{T'}\}_{T>0}$. From Lemma 7.1 it follows that

$$\int_{\mathrm{C}(\mathbf{R};\mathbf{R}^d)} |q(t)-q(s)|^{2n}\overline{\mu}_T(dq) \leq |t-s|^{2n} C_n e^{|t-s|(E-\inf V)} \left(\sup_{T>0} \frac{\|f\|}{\left\| e^{-T\overline{H}} f\otimes\Omega \right\|} \right)^2 .$$

Since

$$\lim_{T\to\infty} \left\| e^{-T\overline{H}} f\otimes\Omega \right\| = \|\Psi_{\mathrm{g}}\| \neq 0,$$

there exists a positive constant D_n independent of T such that

$$\int_{\mathrm{C}(\mathbf{R};\mathbf{R}^d)} |q(t)-q(s)|^{2n}\overline{\mu}_T(dq) \leq |t-s|^{2n} D_n.$$

Thus Π is tight ([138]). Hence Π is precompact by [172], i.e., there exists a subsequence T' such that $\overline{\mu}_{T'}$ weakly converges to a probability measure μ. QED

Remark 7.4 *In Theorem 7.3 we do* **not** *explicitly assume $|\alpha| \ll 1$.*

7.2 Expectation values and a boson-localization

In this subsection we assume that there exists the ground state of H. Let the expectation value of T with respect to the normalized ground state Ψ_{g} be defined by

$$\langle T\rangle := (\Psi_{\mathrm{g}}, T\Psi_{\mathrm{g}})_{\mathcal{H}}.$$

Corollary 7.5 *Let $h_j \in L^\infty(\mathbf{R}^d)$, $j=1,...,m$. Then*

$$\langle h_1 e^{-(t_2-t_1)\overline{H}} h_2 \cdots h_{m-1} e^{-(t_m-t_{m-1})\overline{H}} h_m \rangle = \int_{\mathrm{C}(\mathbf{R};\mathbf{R}^d)} h_1(q(t_1))\cdots h_m(q(t_m))\mu(dq). \tag{7.1}$$

Proof: We directly see that

$$\text{L.H.S.}(7.1) = \lim_{T\to\infty} \frac{(f\otimes\Omega, e^{-(T+t_1)H} h_1 e^{-(t_2-t_1)H} h_2\cdots h_m e^{-(T-t_m)H} f\otimes\Omega)}{(f\otimes\Omega, e^{-2TH} f\otimes\Omega)}$$

$$= \lim_{T\to\infty} \int_{\mathrm{C}(\mathbf{R};\mathbf{R}^d)} h_1(q(t_1))\cdots h_m(q(t_m))\overline{\mu}_T(dq) = \text{R.H.S.}(7.1)$$

Thus corollary follows. QED

Corollary 7.6 *We have*

$$\lim_{|t-s|\to\infty} \int_{\mathrm{C}(\mathbf{R};\mathbf{R}^d)} q(t)q(s)\mu(dq) = \langle x\rangle^2.$$

Proof: By a limiting argument we have

$$\int_{\mathrm{C}(\mathbf{R};\mathbf{R}^d)} q(t)q(s)\mu(dq) = \langle xe^{-|t-s|H}x\rangle.$$

Thus corollary follows. QED

Corollary 7.7 *Let V be as that of Theorem 6.11. Then, for sufficiently small $\delta > 0$,*

$$\int_{\mathrm{C}(\mathbf{R};\mathbf{R}^d)} e^{\delta|q(t)|^{m+1}}\mu(dq) = \langle e^{\delta|x|^{m+1}}\rangle < \infty. \tag{7.2}$$

Proof: By Corollary 7.5, we have

$$\langle e^{\delta|x|^{m+1}}\lceil_n\rangle = \int_{\mathrm{C}(\mathbf{R};\mathbf{R}^d)} e^{\delta|q(t)|^{m+1}}\lceil_n \mu(dq),$$

where $f(x)\lceil_n := f(x)$ if $f(x) \le n$, otherwise $f(x)\lceil_n = n$. Since $e^{\delta|\cdot|^{m+1}}\|\Psi_{\mathrm{g}}(\cdot)\| \in L^2(\mathbf{R}^d)$, the Lebesgue monotone convergence theorem yields (7.2). QED

Corollary 7.8 *Let V be as that of Theorem 6.12. Then*

$$\int_{\mathrm{C}(\mathbf{R};\mathbf{R}^d)} e^{\delta|q(t)|}\mu(dq) = \langle e^{\delta|x|}\rangle < \infty. \tag{7.3}$$

Proof: The proof is similar to that of Corollary 7.7. QED

By means of (4.8) we have

$$\left(\Psi_{\mathrm{g}}, e^{-\beta N}\Psi_{\mathrm{g}}\right) = \lim_{T\to\infty} \int_{\mathrm{C}(\mathbf{R};\mathbf{R}^d)} e^{(\alpha^2/2)F_T(q)}\mu_T(dq),$$

where

$$F_T(q) := 2q_1\left(\oplus_{\mu=1}^d \int_{-T}^0 \xi_0\lambda(\cdot - q_s)dq_\mu(s), \oplus_{\mu=1}^d \int_0^T \xi_\beta\lambda(\cdot - q_s)dq_\mu(s)\right).$$

Since $N = d\Gamma(\mathbf{1})$ (i.e., $h(k) = 1$), formally we can write down $F_T(q)$ as

$$F_T(q) = (1-e^{-\beta})\int_{-T}^0 dq_\mu(s)\int_0^T dq_\nu(s')\int_{\mathbf{R}^d} d_{\mu\nu}(k)e^{-|s-s'|\omega(k)}e^{ik(q_s-q_{s'})}|\hat{\lambda}(k)|^2 dk.$$

(See Remark 4.5). Our conjecture is as follows:

Conjecture 7.9 *There exist a function F_∞ on $\mathrm{C}(\mathbf{R};\mathbf{R}^d)$ and $a>0$ such that*

$$\int_{\mathrm{C}(\mathbf{R};\mathbf{R}^d)} e^{zF_\infty(q)}\mu(dq)$$

is analytic in $\Re z < a$ and

$$\left(\Psi_{\mathrm{g}}, e^{-\beta N}\Psi_{\mathrm{g}}\right) = \int_{\mathrm{C}(\mathbf{R};\mathbf{R}^d)} e^{(\alpha^2/2)(1-e^{-\beta})F_\infty(q)}\mu(dq)$$

for $\alpha\in\mathbf{R}$ and $\beta\in\mathbf{C}$ such that $\Re(\alpha^2/2)(1-e^{-\beta})<a$.

8 The dipole approximation

Let $\hat{\lambda}$ be sufficiently smooth and rotation invariant [33], and V also sufficiently smooth for simplicity. Let M be the mass of the electron in this section. The Pauli-Fierz Hamiltonian with the dipole approximation is defined by $A(\hat{\lambda},x)\to A(0):=A(\hat{\lambda},0)$, i.e.,

$$H_{\mathrm{dip}} := \frac{1}{2M}\left(\mathbf{p}\otimes\mathbf{1}-\alpha\mathbf{1}\otimes A(0)\right)^2 + V\otimes\mathbf{1}+\mathbf{1}\otimes H_{\mathrm{f}}.$$

The Pauli-Fierz Hamiltonian with the dipole approximation is solvable [7]-[16], namely, we can concretely construct a Bogoliubov transformation ([36]) T [8, 9, 10, 11] which diagonalize H_{dip}.

Let K be a Hilbert space. We say that a pair of bounded operators $\{A,B\}$ is of symplectic group $S_{\mathrm{ym}}(K)$ if the following operator equation holds on $K\oplus K$: [34]

$$\begin{pmatrix} A & B\\ \overline{B} & \overline{A}\end{pmatrix}^*\begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix}\begin{pmatrix} A & B\\ \overline{B} & \overline{A}\end{pmatrix} = \begin{pmatrix} A & B\\ \overline{B} & \overline{A}\end{pmatrix}\begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix}\begin{pmatrix} A & B\\ \overline{B} & \overline{A}\end{pmatrix}^* = \begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix},$$

where $\overline{T}f := \overline{T\overline{f}}$.

Proposition 8.1 (A.Arai [8, 9, 10, 11]) *There exists a pair of bounded operators $\{\mathbf{W}_+,\mathbf{W}_-\}\in S_{\mathrm{ym}}(\oplus^{d-1}L^2(\mathbf{R}^d))$ and a vector $L\in\oplus^{d-1}L^2(\mathbf{R}^d)$ such that $\mathbf{W}_+$ is a Hilbert-Schmidt operator on $\oplus^{d-1}L^2(\mathbf{R}^d)$, and*

$$B^{s\dagger}(f) := a^{r\dagger}(\mathbf{W}_{+rs}f) + a^r(\mathbf{W}_{-rs}f) - \alpha\mathbf{p}_\nu(L_\nu,f),$$

$$B^{s}(f) := a^{r\dagger}(\overline{\mathbf{W}}_{+rs}f) + a^r(\overline{\mathbf{W}}_{-rs}f) - \alpha\mathbf{p}_\nu(\overline{L}_\nu,f),$$

[33] $\hat{\lambda}(k)=\hat{\lambda}(|k|)$.
[34] See e.g.,[146, 147, 148].

satisfies

$$[B^r(f), B^{\dagger s}(g)] = \delta_{rs}(\bar{f}, g), \quad [B^{\sharp r}(f), B^{\sharp s}(g)] = 0,$$

and

$$e^{itH_{\rm dip}} B^{s\dagger}(f) e^{-itH_{\rm dip}} = B^{s\dagger}(e^{it\omega} f),$$

$$e^{itH_{\rm dip}} B^{s}(f) e^{-itH_{\rm dip}} = B^{s\dagger}(e^{-it\omega} f).$$

Thus by E.A.Berezin [37] we can concretely construct a Bogoliubov transformation T diagonalizing $H_{\rm dip}$. Also see S.N.M.Ruijsenaars [174, 173].

Theorem 8.2 ([8, 9, 10, 11, 117]) *For all $\alpha \in \mathbf{R}$. There exists a unitary operator T of $\mathcal{H}$ such that*

$$T H_{\rm dip} T^{-1} = -\frac{1}{2M_{\rm eff}} \Delta + H_{\rm b} + \alpha^2 g + V_{\rm eff},$$

where

$$M_{\rm eff} := M + \alpha^2 \|\hat{\lambda}/\sqrt{\omega}\|^2,$$

$$g := \frac{d-1}{2\pi} \int_{-\infty}^{\infty} \frac{t^2 \left\| \sqrt{\omega}\hat{\lambda}/(t^2+\omega^2) \right\|^2}{M + \alpha^2 (d-1)/d \left\| \sqrt{\omega}\hat{\lambda}/\sqrt{t^2+\omega^2} \right\|^2} dt,$$

and

$$V_{\rm eff}(x) := V(x - A(K))$$

with some $K \in \oplus^d L^2(\mathbf{R}^d)$.

Proof: See [117][35].

Remark 8.3 *Operators $\mathbf{W}_\pm$ can be extended to a negative mass $M < 0$. In this case $\{\mathbf{W}_+, \mathbf{W}_-\} \notin S_{\rm ym}(\oplus^{d-1} L^2(\mathbf{R}^d))$ ([102]).*

Corollary 8.4 *Let $V = 0$. Then* $\inf \sigma(H_{\rm dip}) = \alpha^2 g$.

Let $d = 3$ and $V \le 0$. Set

$$N(V) := a_3 \int_{\mathbf{R}^3} |V(x)|^{3/2} dx,$$

[35] By this transformation, several scaling limits of $H_{\rm dip}$ are investigated. In particular, taking a scaling limit, A.Arai obtained an effective potential which had been found by Welton [199]. This work was continued in F.Hiroshima [102, 104]. Another aspects of such scaling limits are investigated in [1, 53, 54, 59, 169].

where a_3 is a universal constant, and $a_3 \leq 0.116$ is established in [152, p.269],[151]. It is known as the Lieb-Thirring inequality that

$$N(V) \leq \#\{\text{negative eigenvalues of } -\Delta/2 + V\}.$$

In particular $H_{\rm p}$ for V with $N(V) < 1$ has no ground state and $\sigma(H_{\rm p}) = [0, \infty)$. Moreover $H_{\rm d} = H_{\rm p} + H_{\rm f}$ has no ground state.

Theorem 8.5 (F.Hiroshima and H.Spohn [117])
Let V be as above. Then there exist $\alpha_0 > 0$ and $\alpha_1 > 0$ such that $H_{\rm dip}$ for $\alpha_1 > |\alpha| > \alpha_0$ has a ground state and it is unique.

9 Concluding remarks

(A boson-localization)
For the Nelson model it is established in [38] that there exists $F_\infty(q)$ such that

$$|F_\infty(q)| \leq \|\hat{\lambda}/\omega\|^2 \text{ for all } q \in {\rm C}(\mathbf{R}; \mathbf{R}^d), \tag{9.1}$$

$$\left(\Psi_{\rm g}, e^{-\beta N}\Psi_{\rm g}\right) = \int_{{\rm C}(\mathbf{R};\mathbf{R}^d)} e^{-(\alpha^2/2)(1-e^{-\beta})F_\infty(q)}\mu(q). \tag{9.2}$$

Actually

$$F_\infty = \int_{-\infty}^0 dt \int_0^\infty ds \int_{\mathbf{R}^d} |\hat{\lambda}(k)|^2 e^{-|t-s|\omega(k)} e^{ik(X_s - X_t)} dk.$$

Thus we can see, by an analytic continuation argument, that for *all* $\beta \in \mathbf{C}$

$$\Psi_{\rm g} \in D(e^{\beta N})$$

and (9.2) holds for all $\beta \in \mathbf{C}$. Moreover we explicitly express both of the average momentum density $\langle a^\dagger(k)a(k)\rangle$ and the average spatial density $\langle a^\dagger(x)a(x)\rangle$ by the measure μ. Hence we have pointwise bounds of the densities. The key point of a proof of (9.2) is the uniform estimate (9.1) on paths. In the case of the Pauli-Fierz model, we, up to moment, do not have such uniform estimate and can not shed any light on this problem.

(Essential self-adjointness)
Essential self-adjointness of the Pauli-Fierz Hamiltonian H is proved only for one-particle Hamiltonian. For the Z-particle Hamiltonian (see footnote18), it has not been established. For the Z-particle case, an invariant domain exists. It is, however, not so small. See [112].

(The Zeeman effect)
Let $d=3$. The Hamiltonian with spin 1/2 is defined on Hilbert space $\mathbf{C}^2\otimes\mathcal{H}$ by

$$H_\sigma := \mathbf{1}\otimes H - (\alpha/2)\sigma\otimes B(\hat{\lambda}),$$

where

$$B(\hat{\lambda}) = \int_{\mathbf{R}^d}^{\oplus} B(\hat{\lambda},x)dx,$$

and

$$B(\hat{\lambda},x) := \mathrm{rot}A(\hat{\lambda},x) = \frac{i}{\sqrt{2}}\left\{a^{r\dagger}\left((k\times e^r)e^{-ikx}\tilde{\hat{\lambda}}\right) + a^r\left((-k\times e^r)e^{ikx}\hat{\lambda}\right)\right\}$$

and $\sigma := (\sigma_1,\sigma_2,\sigma_3)$ denotes the Pauli matrices. In this case PI-argument does not work. The uniqueness of the ground state of H_σ is not yet established [36].

In the classical case a *paramagnetic inequality* of a Pauli operator

$$(\mathbf{p}-A)^2 + V + \sigma\cdot B$$

is known under some conditions by L.Erdős [62]. Does there exist the paramagnetic inequality of H_σ?

(Semi-classical limits)
We can define a partial trace $\mathrm{Tr}_\Psi e^{-tH}$ for each $\Psi\in L^2(Q)$ in terms of functional integral representations. In [115], a semi-classical limit [50, 190] of the partial trace is shown:

$$\lim_{\hbar\to 0}\hbar^d \mathrm{Tr}_\Psi e^{-tH} = (2\pi)^{-d}\int_{\mathbf{R}^{2d}} e^{-t(p^2/2+V(x))}dpdx\|\Psi\|_{L^2(Q)}.$$

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