Rabi modelの数理　2016/12/22

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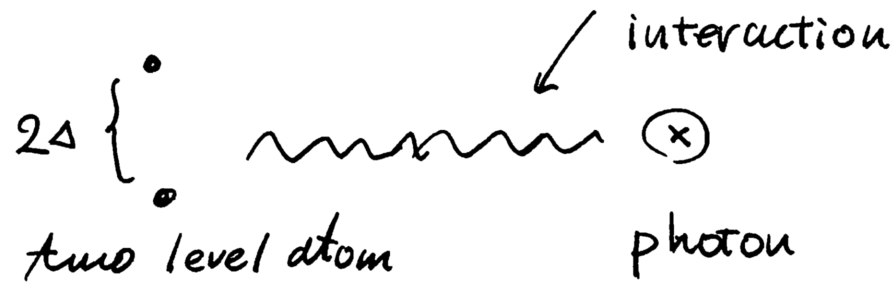
Rabi-model



Quantum Rabi Hamiltonian H is given by

$$H = \Delta \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes a^\dagger a + g \sigma_x \otimes (a + a^\dagger),$$

which acts on the Hilbert space

$$\mathcal{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R}), \quad \text{where}$$

- $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $\Delta \geqq 0$, $g \in \mathbb{R}$
- $a = \frac{1}{\sqrt{2}}(x + \partial)$, $a^\dagger = \frac{1}{\sqrt{2}}(x - \partial)$

⊙ a and $a^\dagger$ satisfy CCR : $[a, a^\dagger] = 1$, and
$\varphi_n = \underbrace{a^\dagger \cdots a^\dagger}_{n} \varphi_0$ with $\varphi_0 = (\pi)^{-1/4} \exp\left(-\frac{x^2}{2}\right)$
satisfies

$$a^\dagger a \varphi_n = n \varphi_n.$$

⊙ $\text{L.H.} \{ \prod^{n} a^\dagger \varphi_0 \mid n \in \mathbb{N}^\times \} = L^2(\mathbb{R})$

$$H \cong -\Delta\sigma_x + a^\dagger a + g\sigma_z(a+a^\dagger)$$

$$= \begin{pmatrix} a^\dagger a + \sqrt{2}gx & -\Delta \\ -\Delta & a^\dagger a - \sqrt{2}gx \end{pmatrix}$$

- $a^\dagger a = \frac{1}{2}(-\partial^2 + x^2 - 1)$ harmonic oscillator or Euler operator

$\mathrm{spec}(a^\dagger a) = \{n\}_{n=0}^{\infty}$ and

the multiplicity of each n is "1".

- We can see that

$$a^\dagger a^{+\sqrt{2}gx} = \frac{1}{2}(-\partial^2 + (x+\sqrt{2}g)^2 - 1) - g^2$$
$$\cong a^\dagger a - g^2$$

$\Delta = 0$ の場合

$$H \cong \begin{pmatrix} a^\dagger a & 0 \\ 0 & a^\dagger a \end{pmatrix} - g^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ and

$\mathrm{spec}(H_{\Delta=0}) = \{n - g^2\}_{n=0}^{\infty}$ and

the multiplicity of each $n-g^2$ is "2".

Bargman Representation

$\mathcal{B} \ni f$ if and only if

① $f : \mathbb{C} \to \mathbb{C}$ analytic
② $\pi^{-1} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \, dx dy < \infty$

$L^2(\mathbb{R}) \cong \mathcal{B}$ by the map

$$\underset{\substack{\cap \\ L^2(\mathbb{R})}}{h} \mapsto Th(z) = \sqrt{2} \int \overset{h}{\cancel{g}}(x) \, e^{2\pi x z - \pi x^2 - \frac{\pi}{2} z^2} dx \in \mathcal{B}$$

Under above identification we can see

- $a \sim \dfrac{d}{dz}$, $\quad a^{+} \sim z$
- $L.H. \{ z^n \mid n \in \overset{\mathbb{N}^\times}{\cancel{\mathbb{N}^+}} \} = \mathcal{B}$

Eigenvalue equation $H\varphi = E\varphi$ is reduced to

$$\left[\Delta \sigma_x + z \frac{d}{dz} + g \sigma_z \left(z + \frac{d}{dz} \right) \right] \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = E \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$\begin{cases} (z+g) \dfrac{d\varphi_1}{dz} + (gz - E)\varphi_1 + \Delta \varphi_2 = 0 \\[2ex] (z-g) \dfrac{d\varphi_2}{dz} - (gz + E)\varphi_2 + \Delta \varphi_1 = 0 \end{cases}$$

$\Delta = 0$ の場合

$$\left(\frac{d}{dz} - \frac{E+g^2}{z+g} + g\right)\varphi_1 = 0$$

$$\therefore \quad \varphi_1(z) = e^{-gz}(z+g)^{E+g^2}$$

$$\varphi_1 \in B \iff E+g^2 \in \mathbb{N}^\times$$

Similary it follows that

$$\varphi_2 \in B \iff E+g^2 \in \mathbb{N}^\times$$

$$\varphi_2(z) = e^{gz}(z-g)^{E+g^2}$$

※ Crossings between E_{2n} and E_{2n+1}

Let $\Delta = 0$. Then we have already seen that $E_{2n} = E_{2n+1} = n - g^2$ for $n \in \mathbb{N}^\times$, where E_m denotes m_{th} excited eigenvalue, i.e., $E_0 \leq E_1 \leq E_2 \leq \cdots$. We consider Rabi-Hamiltonian with $\Delta \neq 0$.

Let $P = \sigma_x \otimes (-1)^{a^\dagger a}$. Then P is self-adjoint and it follows that $P^2 = \mathbb{1}_{\mathcal{H}}$, which implies that

$$\mathrm{spec}(P) = \{-1, +1\}$$

and P is called "parity". Let $P\varphi = -\varphi$ (resp. $P\varphi = \varphi$), then φ is parity -1 (resp. $+1$). We give some examples. Recall that $\varphi_0(x) = \pi^{-1/4} \exp(-\frac{x^2}{2})$. Hence $\begin{pmatrix}1\\1\end{pmatrix} \otimes \varphi_0(x)$ is parity $+1$, and $\begin{pmatrix}1\\-1\end{pmatrix} \otimes \varphi_0(x)$ is parity -1. An ~~I~~ important fact is that

$$[H, P] = 0,$$

which yields that H can be decomposed ~~into~~ as

$$H = H_- \oplus H_+,$$

where ~~[illegible]~~ H_- denotes parity -1 part and H_+ parity $+1$ part.

In Bargumann representation, $H\varphi = E\varphi$ is reduced to the differential equation:

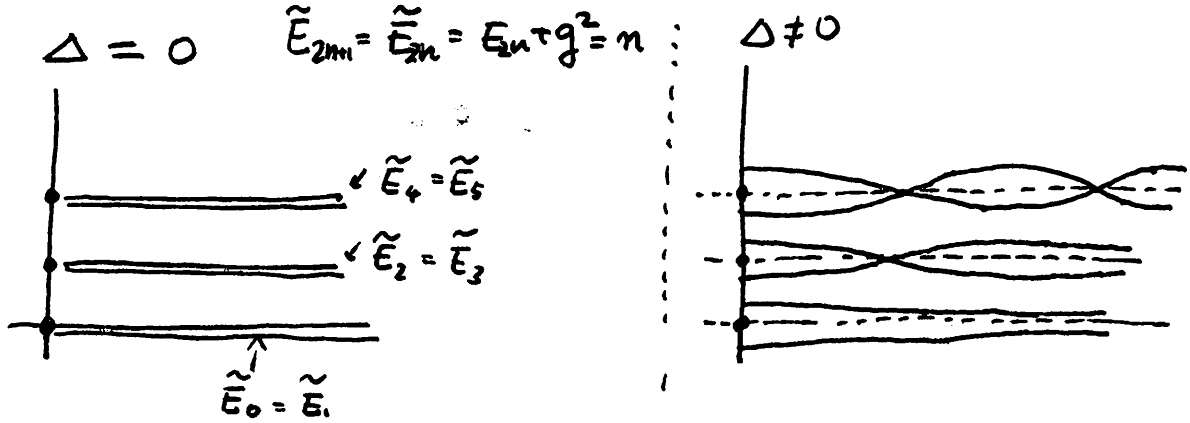
$$\begin{cases} (z+g)\dfrac{d\varphi_1}{dz} + (gz - E)\varphi_1 + \Delta\varphi_2 = 0 \\ (z-g)\dfrac{d\varphi_2}{dz} - (gz + E)\varphi_2 + \Delta\varphi_1 = 0 \end{cases}$$

$$(*) \quad \frac{d}{dz}\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix} + A(z)\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix} = 0,$$ where

$$A(z) = \begin{pmatrix} \dfrac{gz-E}{z+g} & \Delta \\ \Delta & -\dfrac{gz+E}{z-g} \end{pmatrix}$$

- Hence the dimension of the vector space of analytic solutions on $\mathbb{C}\setminus\{\pm g\}$ of $(*)$ is 2.

- By a numerical result eigen value curve $g \mapsto E_n(g)$ $n = 0, 1, 2, \cdots$ can be drawn as



Thm 1. $E_0(g)$ has no crossing.

☺ It is enough to show that $E_0(g)$ is simple eigenvalue for arbitrary $g \in \mathbb{R}$.
Let $(X_t)_{t \in \mathbb{R}_+}$ be an OU process such that

$$(f, e^{-th_0'} g) = \int_{\mathbb{R}} dx\, \varphi_0(x)^2\, \mathbb{E}^x\left[\overline{f(X_0)}\, g(X_t)\right]$$

where $h_0' = \frac{1}{\varphi_0} h_0 \varphi_0 = \frac{1}{2}(-\partial^2 + x\partial)$.

Rabi Hamiltonian can be transformed as

$$\frac{1}{\varphi_0} H \varphi_0 = h_0' + \Delta\sigma_x + \sqrt{2} g \sigma_z x = H'$$

$$(H'f)(x, \sigma) = (h_0' + \sqrt{2} g x) f(x, \sigma) + \Delta f(x, -\sigma)$$

where $f = \begin{pmatrix} f(x\ +1) \\ f(x\ -1) \end{pmatrix}$ and $\sigma = \pm 1$.

Hence we have Feynman-Kac type formula:

$$(f, e^{-tH'} g) = \sum_{\sigma=\pm 1} \int_{\mathbb{R}} dx\, \varphi_0(x)^2\, \mathbb{E}^{x\sigma}\left[\overline{f(X_0\sigma_0)}\, g(X_t\sigma_t)\, S\right],$$

where

$$S = e^{\sqrt{2} g \int_0^t X_s\, ds} \Delta^{N_t}$$

and $(N_t)_{t \in \mathbb{R}_+}$ denotes the Poisson process such that $\mathbb{E}[N_t = n] = \frac{t^n}{n!} e^{-t}$.

Let $f \geq 0$, $g \geq 0$ but $f \not\equiv 0$ and $g \not\equiv 0$. Then Feynman-Kac type theorem yields that

$$(f, e^{-tH'} g) \gneqq 0.$$

This implies that $e^{-tH'}$ is a positivity improving operator on $L^2(\mathbb{R} \times \{\mathbb{Z}_2\}, \varphi(x)^2 dx)$. By Perron-Frobenius theorem, E_0 is simple.

QED.

<u>Thm 2</u> (Kús, Wakayama-Yamazaki)
Let $0 < \Delta < 1$. Then the number of crossings between E_{2n} and E_{2n+1} is n.

Remak ① When $E_{2n} = E_{2n+1}$, $E_{2n} = E_{2n+1} = n - g^2$.
② When $E_{2n} = E_{2n+1}$, we can take as a base of eigenvectors. (positive parity and negative parity)

§ Spectral zeta functions

Hurwitz type zeta function is given by

$$\zeta(s;\tau) = \sum_{n=0}^{\infty} \frac{1}{(n+\tau)^s} \quad (s>1)$$

It is shown that $\zeta(s;\tau)$ can be extended to a meromorphic function on $\mathbb{C}$.

Let $\{E_j\}_{j=0}^{\infty}$ be eigenvalues of the Rabi Hamiltonian. It is easily seen that

$$E_j \geqq -g^2-\Delta, \quad \forall j.$$

We define the spectral zeta function by

$$\zeta_g(s;\tau) = \zeta_g(s) = \sum_{j=0}^{\infty} \frac{1}{(E_j+g^2+\tau)^s}$$

for $\tau > \Delta$. By the fact

$$^{\exists}c\, j \lesssim E_j \lesssim {}^{\exists}C j,$$

where c and C are independent of j.

Then $\zeta_g(s)$ converges at $s>1$.

<u>Thm 3</u> (S. Sugiyama) $\zeta_g(s)$ can be extened to the meromorphic function on $\mathbb{C}$.

By the Fourier transformation we can see

$$H \cong \begin{pmatrix} \frac{1}{2}(-\partial^2 + x^2 + 1) - g^2 & \Delta e^{i2\sqrt{2}gx} \\ \Delta e^{-i2\sqrt{2}gx} & \frac{1}{2}(-\partial^2 + x^2 + 1) - g^2 \end{pmatrix}$$

$$= \begin{pmatrix} h_0 & 0 \\ 0 & h_0 \end{pmatrix} - g^2 + T_g, \quad \text{where}$$

$$T_g = \begin{pmatrix} 0 & \Delta e^{i2\sqrt{2}gx} \\ \Delta e^{-i2\sqrt{2}gx} & 0 \end{pmatrix}$$

Let $f, h \in L^2(\mathbb{R})$. Hence we observe that

$$(f, e^{igx} h) = \int_{\mathbb{R}} \overline{\hat{f}(k)}\, \hat{g}(k+g)\, dk \to 0 \quad (g \to \infty)$$

This implies $e^{igx} \to 0$ weakly as $g \to \infty$.

Anothere observation is

$$\zeta_g(s) = \mathrm{Tr}\left(\begin{pmatrix} h_0 & 0 \\ 0 & h_0 \end{pmatrix} + T_g \right)^{-s}$$

Formaly we can conclude that

$$\lim_{g \to \infty} \zeta_g(s) = \mathrm{Tr}\left(\begin{pmatrix} h_0 & 0 \\ 0 & h_0 \end{pmatrix} \right)^{-s} = 2\, \mathrm{Tr}\, h_0^{-s}$$

$$= 2\, \zeta(s;\tau).$$

Thm 4. $\lim_{g \to \infty} \zeta_g(s) = 2\zeta(s:\tau)$ for $\tau > \Delta$.

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