Ground states of scalar quantum field on pseudo Riemannian manifolds

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This is the joint work with

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- *Infrared divergence of a scalar quantum field model on a pseudo Riemannian manifold*, **IIS 15(2009) 399-421**
- *Infrared problem for the Nelson mode on a static space-times*, to appear in **CMP**
- *Absence of ground state for the Nelson model on a static-space-times*, **ArXiv 1012.2655**
- *Removal of UV cutoff for the Nelson model with variable coefficients*, **preprint 2011**
- *Existence and absence of ground states for a particle interacting through the quantized scalar field on a static spacetime*, **RIMS Kôkyûroku Bessatsu B21 (2010) 15-24**

Hilbert Space

$$\mathscr{H} = L^2(\mathbb{R}^{\nvDash}) \otimes \mathscr{F} \quad \mathscr{F} = \bigoplus_{n=0}^{\infty} L^2_{sym}(\mathbb{R}^{\nvDash\ltimes})$$

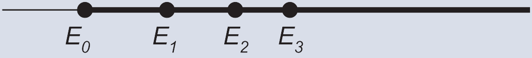
- (dispersion relation) $\omega = \omega(-i\nabla_x) = \sqrt{-\Delta_x + m^2}$
- $\mathrm{d}\Gamma(\omega)\Phi^{(n)}(x_1,\ldots,x_n) = (\sum_{j=1}^n \omega(-i\nabla_{x_j}))\Phi^{(n)}(x_1,\ldots,x_n)$
- $\phi(f) = \frac{1}{\sqrt{2}}(a^\dagger(\bar{f}) + a(f))$, $[a(f), a^\dagger(g)] = (\bar{f}, g)$
- $\phi_\rho(X) = \phi(\omega^{-1/2}\rho(\cdot - X))$
- UV cutoff $0 \le \rho \in \mathscr{S}$

Standard Nelson model

$$H = (-\frac{1}{2}\Delta_X + V(X)) \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega) + \phi_\rho(X)$$

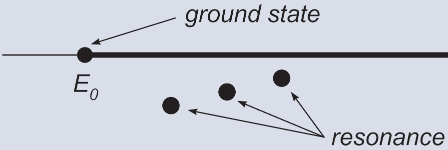
$\sigma(-\frac{1}{2}\Delta+V)=\{E_j\}$, $\sigma(\mathrm{d}\Gamma(\omega))=[0,\infty)$ $(m=0)$

- Embedded eigenvalues(no interaction)



$$\Downarrow (+\phi_\rho(X))$$

- Resonances and ground state



Infrared problem

$$I_{\mathrm{IR}} = \int \frac{|\hat{\rho}(k)|^2}{\omega(k)^3} dk$$

E.g., $\omega(k) = \sqrt{|k|^2 + m^2}$ and $\hat{\rho}(0) > 0$

- **($m > 0$)** $I_{\mathrm{IR}} < \infty$
- **($m = 0$)** $I_{\mathrm{IR}} = \infty$

Bach-Fröhlich-Sigal(98), Gérard(00), Spohn(99), Arai-Hirokawa-H.(99), Derezinski-Gérard(03), Hirokawa(07), Lőrinczi-Minlos-Spohn(03))....

- $I_{\mathrm{IR}} < \infty$ **(IR regular)** $\Longrightarrow$ $\exists$**ground state**
- $I_{\mathrm{IR}} = \infty$ **(IR singular)** $\Longrightarrow$ **no ground state**

Pull-through formula, number operator N,

$$(\Phi_{\mathrm{g}}, (\mathbb{1} \otimes N)\Phi_{\mathrm{g}})_{\mathscr{H}} \leq \frac{1}{2} I_{\mathrm{IR}} \|\Phi_{\mathrm{g}}\|^2$$

- **($m > 0$) $\exists$ ground state**
- **($m = 0$) no ground state**

How about variable mass $m(x)$?

Conjecture

- $m(x) \downarrow 0$ fast $\Longrightarrow$ no ground state
- $m(x) \downarrow 0$ slowly $\Longrightarrow$ $\exists$ ground state

Nelson model on static pseudo Riemannian manifold

- $e^{-itH}(\mathbb{1} \otimes \phi(f))e^{itH} = \int \phi(t,x)f(x)dx$
- $e^{-itH}(X \otimes \mathbb{1})e^{itH} = X_t$

Standard Nelson model (formally) satisfies that

$$(\partial_t^2 - \Delta_x + m^2)\phi(t,x) = \rho(x - X_t)$$
$$\partial_t^2 X_t = -\nabla V(X_t) - \int \phi(t,x)\nabla_X \rho(x - X_t)dx$$

Static (time independent) Lorenzian manifold

$$g = (g_{\mu\nu}) = \begin{pmatrix} \lambda & \\ & -\gamma \end{pmatrix}$$

$\lambda(x) > 0$, $\gamma(x)$ is a Riemannian metric on $\mathbb{R}^{\nvDash}$.

$$\Box_g = \sum |g|^{-1/2} \partial_\mu |g|^{1/2} g^{\mu\nu} \partial_\nu + \theta\eta$$

where $\theta \in \mathbb{R}$, η scalar curvature.

$$(\Box_g + m^2)\phi(t,x) = 0 \quad on\ L^2(\mathbb{R}^{\nvDash}, |g|^{1/2} dx)$$

Transform to the equation from $L^2(\mathbb{R}^d, |g|^{1/2}dx)$ to $L^2(\mathbb{R}^d)$:

$$(\partial_t^2 + h)\phi(t,x) = 0$$

$$h = -\sum_{ij} \frac{1}{c}\partial_i a^{ij}(x)\partial_j \frac{1}{c} + m(x)^2$$

Variable mass $m(x)$ appears even when $m = 0$.

Nelson model on static Lorenzian manifold

$$H = K \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega) + \phi_\rho(X)$$

where

- $K = -\sum \partial_i A^{ij}(X)\partial_j + V(X)$
- $\omega = h^{1/2}$
- $\phi(X) = \phi(\omega^{-1/2}\rho(\cdot - X))$

Existence of ground state

Assumptions

(F)
- $C_0 \mathbb{1} \leq [a^{ij}(x)] \leq C_1 \mathbb{1}$, $\partial^\alpha a^{ij}(x) \in O(\langle x \rangle^{-1})$, $|\alpha| \leq 1$
- $C_0 \leq c(x) \leq C_1$, $\partial^\alpha c(x) \in O(1)$, $|\alpha| \leq 2$
- $\partial^\alpha m(x) \in O(1)$, $|\alpha| \leq 1$

(P)
- $C_0 \mathbb{1} \leq [A^{ij}(X)] \leq C_1 \mathbb{1}$
- $V(X) \geq C_0 \langle X \rangle^{2\delta} - C_1$

Theorem (GHPS) Existence of ground state

Suppose $m(x) \geq a\langle x \rangle^{-1}$ for some $a > 0$, and $\delta > 3/2$. Then H has a ground state.

Proposition (Bruneau-Dereziński) general ω and K

Suppose that

(1) $\omega \geq 0$, Ker $\omega = 0$

(2) $\sup_X \|\omega^{-1/2}\rho(\cdot - X)\| < \infty$

(3) $(K + 1\!\!1)^{-1/2}$ is compact

(4) $\omega^{-1}\rho(\cdot - X)(K + 1\!\!1)^{-1/2}$ is compact

(5) $\omega^{-3/2}\rho(\cdot - X)(K + 1\!\!1)^{-1/2}$ **is compact** (general IR regularity).

Then $K \otimes 1\!\!1 + 1\!\!1 \otimes d\Gamma(\omega) + \phi_\rho(X)$ has a ground state.

Proof of Thm: Check (5).

$$m(x) \geq a\langle x\rangle^{-1} \Longrightarrow \omega^{-3/2}\langle x\rangle^{-3/2-\varepsilon} \textit{ is bounded}$$

$\Longrightarrow \omega^{-3/2}\langle x\rangle^{-3/2-\varepsilon}\langle x\rangle^{3/2+\varepsilon}\rho(x - X)\langle X\rangle^{-3/2-\varepsilon}\langle X\rangle^{3/2+\varepsilon}(K + 1\!\!1)^{-1/2}$
is compact, since $\langle X\rangle^{3/2+\varepsilon}(K + 1\!\!1)^{-1/2}$ is compact by
$V(X) > \langle X\rangle^{3+\varepsilon'}$.

Absence of ground state

Probabilistic approach

- e^{-tK} is positivity preserving.
- Let $\Phi_{\rm p} > 0$ be the ground state of K, $\Phi_{\rm p}(x) \leq C_0 e^{-C_1|x|^{\delta+1}}$
- (ground state transform) $U : L^2(\Phi_{\rm p}^2 dx) \to L^2(dx)$, $f \mapsto \Phi_{\rm p} f$
- $L = U(K - \inf \sigma(K))U^{-1}$

Feynman-Kac formula $\mathscr{X} = C(\mathbb{R}, \mathbb{R}^d)$

There exists a diffusion process $(X_t)_{-\infty<t<\infty}$ on a probability space $(\mathscr{X}, B(\mathscr{X}), \exists P^x)$ such that

$$(f, e^{-tL} g) = \int \mu_0(dx) \mathbb{E}_P^x[\overline{f(X_0)} g(X_t)]$$

where $\mu_0(dx) = \Phi_{\rm p}^2(x) dx$ is the probability measure on $\mathbb{R}^d$.

$\mathscr{F} \cong L^2(Q, d\nu)$, ν Gaussian measure such that

$$\int e^{\alpha\phi(f)} d\nu = e^{\alpha^2/4\|f\|^2}$$

$$H = L \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\omega) + \phi_\rho(X)$$

on $L^2(\mathbb{R}^{\nvDash}, d\mu_0) \otimes L^2(Q, d\nu) \cong L^2(\mathbb{R}^{\nvDash} \times Q, d\mu_0 \otimes d\nu)$

- e^{-TH} is positivity preserving.
- If H has a ground state $\Phi_{\mathrm{g}} \Longrightarrow \Phi_{\mathrm{g}} > 0$.
- $1\!\!1 = 1\!\!1_{L^2(\mathbb{R}^{\nvDash})} \otimes \Omega$. Then $\Phi_{\mathrm{g}}^T = e^{-TH} 1\!\!1 / \|e^{-TH} 1\!\!1\| \to \Phi_{\mathrm{g}}$ $(T \to \infty)$.

$$\gamma = \lim_{T\to\infty} (1\!\!1, \Phi_{\mathrm{g}}^T)^2 = \lim_{T\to\infty} \frac{(1\!\!1, e^{-TH} 1\!\!1)^2}{(1\!\!1, e^{-2TH} 1\!\!1)}$$

Lemma (Lőrinczi-Minlos-Spohn)

- $(\gamma > 0)$ H has a ground state
- $(\gamma = 0)$ H has no ground state

Theorem (GHPS) Absence of ground state

Suppose $m(x) \leq a\langle x\rangle^{-1-\varepsilon}$. Then H has no ground state.

$$(\mathbb{1}, e^{-TH}\mathbb{1}) = \int \mu_0(dx)\mathbb{E}_{\mathbb{P}}^{\frown}[^{\int_{\nvdash}^{\mathbb{T}} \approx \int_{\nvdash}^{\mathbb{T}} \sim \mathbb{W}(\mathbb{X}_{\approx}, \mathbb{X}_{\sim}, |\approx - \sim|)}]$$

$$W = W(X, Y, |t|) = \frac{1}{2}(\rho(\cdot - X), \omega^{-1}e^{-|t|\omega}\rho(\cdot - Y)).$$

Lemma (GHPS)

$$\gamma \leq \lim_{T\to\infty} \mathbb{E}_{\mu_{\mathbb{T}}}[^{-\nvDash \int_{-\mathbb{T}}^{\nvdash} \int_{\nvdash}^{\mathbb{T}} \mathbb{W}}]$$

where

$$\mu_T(\mathscr{O}) = \frac{1}{Z_T}\int \mu_0(dx)\mathbb{E}_{\mathbb{P}}^{\frown}[\mathscr{O}^{\int_{-\mathbb{T}}^{\mathbb{T}}\int_{-\mathbb{T}}^{\mathbb{T}} \mathbb{W}}]$$

Proof $\gamma = \lim_{T\to\infty} \frac{(1\!\!1, e^{-TH} 1\!\!1)^2}{(1\!\!1, e^{-2TH} 1\!\!1)}$

- Denominator:

$$(1\!\!1, e^{-2TH} 1\!\!1) = \int \mu_0(dx)\mathbb{E}_{\mathbb{P}}^{\frown}[^{\int_{\nvDash}^{\nvDash\mathbb{T}} \int_{\nvDash}^{\nvDash\mathbb{T}} \mathbb{W}}] = \int \mu_{\nvDash}(\frown)\mathbb{E}_{\mathbb{P}}^{\frown}[^{\int_{-\mathbb{T}}^{\mathbb{T}} \int_{-\mathbb{T}}^{\mathbb{T}} \mathbb{W}}]$$

- Numerator:

$$\begin{aligned}
(1\!\!1, e^{-TH} 1\!\!1)^2 &= \left(\int \mu_0(dx)\mathbb{E}_{\mathbb{P}}^{\frown}[^{\int_{\nvDash}^{\mathbb{T}} \int_{\nvDash}^{\mathbb{T}} \mathbb{W}}]\right)^2 \\
&\le \int \mu_0(dx)\mathbb{E}_{\mathbb{P}}^{\frown}[^{\int_{\nvDash}^{\mathbb{T}} \int_{\nvDash}^{\mathbb{T}} \mathbb{W}}]\mathbb{E}_{\mathbb{P}}^{\frown}[^{\int_{-\mathbb{T}}^{\nvDash} \approx \int_{-\mathbb{T}}^{\nvDash} \sim \mathbb{W}}] \\
&= \int \mu_0(dx)\mathbb{E}_{\mathbb{P}}^{\frown}[^{\int_{-\mathbb{T}}^{\nvDash} \int_{-\mathbb{T}}^{\nvDash} + \int_{\nvDash}^{\mathbb{T}} \int_{\nvDash}^{\mathbb{T}} \mathbb{W}}] \\
&= \int \mu_0(dx)\mathbb{E}_{\mathbb{P}}^{\frown}[^{\int_{-\mathbb{T}}^{\mathbb{T}} \int_{-\mathbb{T}}^{\mathbb{T}} - \nvDash \int_{-\mathbb{T}}^{\nvDash} \int_{\nvDash}^{\mathbb{T}} \mathbb{W}}]
\end{aligned}$$

Lemma (GHPS) Harnack type estimate

Suppose $m(x) \leq a\langle x\rangle^{-1-\varepsilon}$. Then

$$C_1 e^{C_2 t\Delta}(x,y) \leq e^{-t\omega^2}(x,y) \leq C_3 e^{C_4 t\Delta}(x,y)$$

Corollary

$$C_1 W_\infty(x,y,C_2|t|) \leq W(x,t,|t|) \leq C_3 W_\infty(x,y,C_4|t|)$$

where

$$W_\infty(X,Y,|t|) = \frac{1}{4\pi^2}\int \frac{\rho(x)\rho(y)}{|x-y+X-Y|^2+t^2}dxdy$$

Proof of Thm: $\mathbb{E}_{\mu_\mathbb{T}}[^{-\nvDash\int_{-\mathbb{T}}^{\nvdash}\int_{\nvdash}^{\mathbb{T}}\mathbb{W}}] = \mathbb{E}_{\mu_\mathbb{T}}[\mathbb{1}_{\mathbb{A}_\mathbb{T}}\cdots] + \mathbb{E}_{\mu_\mathbb{T}}[\mathbb{1}_{\mathbb{A}_\mathbb{T}}\cdots]$ where

- $A_T = \{(x,w) | \sup_{|s|\leq T}|X_s(w)| \leq T^\lambda, |X_0(w)| = |x| \leq T^\lambda\}$
- $\frac{1}{1+\delta} < \lambda < 1$

By

- $\int_{-T}^T dt \int_{-T}^T dsW \leq \int_{-T}^T dt \int_{-T}^T dsW_\infty \leq CT\|\hat\rho/|k|\|^2_{L^2(\mathbb{R}^{\nVdash})}$
- $P(\mathscr{O}) = \int \mu_0(dx)\mathbb{E}_{\mathbb{P}}^{\frown}[\mathscr{O}]$

we have

$$\mathbb{E}_{\mu_\mathbb{T}}[\mathbb{1}_{\mathbb{A}_\mathbb{T}}\cdots] \leq \mathbb{C}^{\mathbb{TC}}\left(\int \mathbb{1}_{\mathbb{A}_\mathbb{T}}\mathbb{P}\right)^{\nVdash/\nvDash}$$

By exponential decay $\Phi_\mathrm{p}(x) \leq C_0 e^{-C_1|x|^{\delta+1}}$ we have

Lemma (Kipnis-Varadhan)

$$\int \mathbb{1}_{A_T^c} dP \leq T^{-\lambda}(a+bT)^{1/2} e^{-T^{\lambda(\delta+1)}}$$

Since $\lambda(\delta+1) > 1$, $\mathbb{E}_{\mu_\mathbb{T}}[\mathbb{1}_{\mathbb{A}_\mathbb{T}}\cdots] \to \nvdash$ $(T \to \infty)$.

UV problem

Theorem (E. Nelson 1964) Removal of UV cutoff

Let

$$\hat{\rho}_\Lambda(k) = (2\pi)^{-3/2}\chi_\Lambda = \begin{cases} (2\pi)^{-3/2} & |k| \leq \Lambda \\ 0 & |k| > \Lambda \end{cases}$$

and

$$E_\Lambda = -\frac{1}{2}(2\pi)^{-3}\int \frac{|\chi_\Lambda(k)|^2}{|k|(|k|^2/2+|k|)}dk$$

Then

$$\lim_{\Lambda\to\infty} e^{-t(H_\Lambda - E_\Lambda)} = e^{-t\exists H_\infty}$$

$\rho_\Lambda(\cdot) = \Lambda^3 \rho(\Lambda \cdot)$. $\rho_\Lambda(x - X) \to \delta(x - X) \int \rho(y) dy$.
Symbols:

$$h_0(X,\xi) = \sum \xi_i a^{ij}(X) \xi_j \quad K(X,\xi) = \sum \xi_i A^{ij}(X) \xi_j$$
$$E(X) = -\frac{1}{2}(2\pi)^{-3} \int (h_0(X,\xi)+1)^{-1/2} \frac{K(X,\xi)}{(K(X,\xi)+1)^2} |\hat{\rho}(\xi/\Lambda)^2| d\xi$$

Theorem (GHPS) Removal of UV cutoff

There exists a self-adjoint operator H_{ren} bounded from below such that $e^{-t(H_\Lambda - E_\Lambda(X))} \to e^{-tH_{\mathrm{ren}}}$ $(\Lambda \to \infty)$.

Concluding Remarks

- Critical ratio

$$a\langle x\rangle^{-1} \le m(x) \quad H \text{ has a ground state}$$
$$m(x) \le a\langle x\rangle^{-1-\varepsilon} \quad H \text{ has no ground state.}$$

- Condition $V(x) \ge \langle x\rangle^{2\delta} - \varepsilon$ can be changed to "binding condition" by Griesemer-Lieb-Loss (Inv Math 01), which include Coulomb potentials.
- The standard Nelson model without UV cutoff also has a ground state (Hirokawa-H.-Spohn, Adv Math 05). **However it is unknown the uniqueness of the ground state.**