E11 St Petersburg
2017/7/10 16:30–

$\ell^2(\mathbb{Z}^d)$

$$L\varphi(x) = -\frac{1}{2d}\sum_{|x-y|}\left(\varphi(y)-\varphi(x)\right)$$

$$V\varphi(x) = \nu\,\delta_0(x)\,\varphi(x) \qquad 0\in\mathbb{Z}^d,\ \underline{\nu\geq 0}$$

$$H = L - V$$

$$F : \ell^2 \to L^2(\mathbb{T}^d) \qquad \mathbb{T}^d = (-\pi,\pi]^d$$

$$(F\varphi)(\theta) = \sum_{x\in\mathbb{Z}^d} e^{-i\theta x}\varphi(x)$$

$$(F^{-1}\varphi)(x) = (2\pi)^{-d}\int \varphi(\theta)e^{+i\theta x}\,d\theta$$

- $FLF^{-1} = 1 - \frac{1}{d}\sum^{d}\cos\theta_j = g(\theta)$
  $\sigma(L) = [0,2]$
- $FVF = \nu(\Omega,\cdot)\Omega \qquad \Omega = (2\pi)^{-\frac{d}{2}}\mathbb{1}$

$$H \to g - \nu(\Omega\ \cdot)\Omega$$

E
×

0 2

Edge behavior

① $^{\exists}E \quad \forall\nu$
② $\nu\to{}^{\exists}\nu_c \quad E\to 0$ resonance
③ $\nu\to{}^{\exists}\nu_c \quad E\to 0$ threshold

$$(L - V)\varphi = E\varphi$$

$$(g - E)\varphi = \nu(\Omega, \varphi)\Omega$$

$$\rightarrow \quad \varphi = \frac{1}{g-E}\mathbb{1} \qquad\qquad \mathbb{1} = \nu\left(\Omega, \frac{1}{g-E}\mathbb{1}\right)\Omega$$

$$\left.\begin{array}{l} \nu = \dfrac{(2\pi)^d}{\int \frac{1}{g-E}} \\[2ex] \|\varphi\|^2 = \int \dfrac{1}{|g-E|} \end{array}\right\} \qquad \begin{array}{lll} E \longmapsto \nu(E) & \downarrow \\ \nu \longmapsto E(\nu) & \downarrow \\ \nu(E=0) = \nu_c \end{array}$$

$$E < 0 \qquad \int \frac{1}{g-E} < \infty \qquad \int \frac{1}{|g-E|^2} < \infty$$

$$E = 0 \qquad \int \frac{1}{g} \qquad \int \frac{1}{g^2}$$

$$g(\theta) = 1 - \frac{1}{d}\sum \cos\theta_j \sim \frac{1}{2}\sum \theta_j^2$$

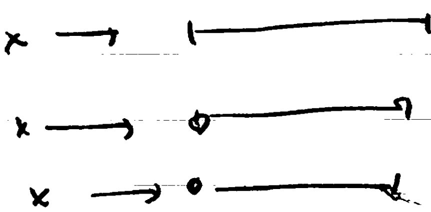
$$\theta = (0 \cdots 0)$$

$$\int \frac{1}{g} \sim \int \frac{1}{\sum \theta_j^2} \sim \int \frac{r^{d-1}}{r^2} < \infty \iff d \geq 3$$

$$\int \frac{1}{g^2} \sim \int \frac{r^{d-1}}{r^4} < \infty \iff d \geq 5$$

(Thm)

$$\begin{array}{llll} d=1,2 & \forall \nu & & {}^\exists E \in \sigma_p \quad (E<0) \\ d=3,4 & {}^\exists \nu_c > 0 \text{ s.t. } & \nu > \nu_c & {}^\exists E \in \sigma_p \\ & & \nu \leq \nu_c & {}^\nexists E \in \sigma_p \\ d \geq 5 & {}^\exists \nu_c > 0 \text{ s.t. } & \nu > \nu_c & {}^\exists E \in \sigma_p \\ & & \nu = \nu_c & 0 \in \sigma_p \\ & & \nu < \nu_c & {}^\nexists E \in \sigma_p \end{array}$$



$d=1,2$

$d=3,4$

$d \geq 5$

$\sqrt{L+m^2} - m + V$ $\quad m \geq 0$

$\Psi(u) = \sqrt{u+m^2} - m$ $\quad$ Berstein function

$\Psi(L) + V$ $\quad \Psi \in \underline{\underline{C^1(0,\infty)}}$ $\quad \Psi' > 0$

$\sigma(\Psi(L)) = \Psi(\sigma(L)) = [\Psi(0), \Psi(2)]$ $\quad \Psi(0) \quad \Psi(2)$

$$J(x) = \int \frac{1}{x - \Psi(g)} \qquad I(x) = \int \frac{1}{|x + \Psi(g)|^2}$$

$x < 0 \Rightarrow J(x) < \infty, \ I(x) < \infty$

Lemma $\quad$ E is e.v $\Leftrightarrow I(E) < \infty$ & $J(E) < \infty$
and $\quad V = (2\pi)^d / J(E)$

$E < 0$ ... trivial

$E = \Psi(2)$ or $\Psi(0)$.

$\underline{\text{Def}}$ $\quad \Psi$ has density index $(a,b) \Leftrightarrow$

$$\lim_{x \to 0+} \frac{\Psi(x) - \Psi(0)}{x^a} \neq 0 \qquad \lim_{x \to 0} \frac{\Psi(2+x) - \Psi(2)}{x^b} \neq 0$$

Ex. $\quad \Psi(u) \quad (1,1) \quad \log(1+u^{\alpha/2}) \ (\frac{\alpha}{2}, 1)$ $\quad | \quad \Psi$ Bernstein ft $(d/2, 1)$

$\Psi(u) = u^{\alpha} \ (\frac{\alpha}{2}, 1)$

$\Psi(u) = \sqrt{u+m^2} - m \quad (1,1) \quad m > 0$

lem $\quad \Psi(a,b)$, Then $\quad E = \Psi(0)$

$I(E) < \infty \Leftrightarrow d \geq 1 + 4a$ $\quad \left(E = \Psi(2) \ a \leftrightarrow b \right)$

$J(E) < \infty \Leftrightarrow d \geq 1 + 2a$

$$\Psi(0) - \Psi(g) \sim g^a \sim \left(\sum \theta_j^2\right)^a$$

$$J(E) \sim \int \frac{r^{d-1}}{r^{2a}} < \infty \Leftrightarrow d \geq 1 + 2a$$

$$I(E) \sim \int \frac{r^{d-1}}{r^{4a}} < \infty \Leftrightarrow d \geq 1 + 4a$$

	threshold	resonance	$V<0$
$d \leq 2a$	×	×	
~~[illegible]~~	×	o	$\leftarrow 2a < d \leq 4a$
~~[illegible]~~	o	×	$d > 4a$

$$H_m = \sqrt{1+m^2} - m + V \qquad m \geq 0$$

① $m = 0$ $\quad (\frac{1}{2}, 1)$

② $m \neq 0$ $\quad (1, 1)$

①

	T	R
$d=1$	×	×
$d=2$	×	O
$d \geq 3$	o	×

$V < 0$

	T	R
$d=1,2$	×	×
$d=3,4$	×	o
$d \geq 5$	o	×

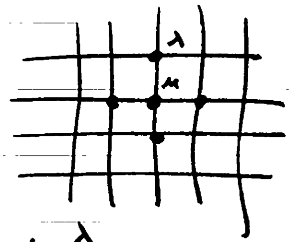
②

	T	R
$d=1,2$	×	×
$d=3,4$	×	O
$d \geq 5$	O	×

$(m \neq 0 \quad d=3) \rightarrow$ 

$(m=0 \quad d=3) \rightarrow$ 

$$V(x) = \mu\,\delta_0(x) + \lambda \sum_{|s|=1} \delta_s(x)$$



$$H = L - V$$

$$H \cong g + \mu(\Omega,\cdot\,\Omega) + \frac{\lambda}{2}\sum (c_j\,\cdot)c_j + \frac{\lambda}{2}\sum (s_j\,\cdot)s_j$$

$$c_j = (2\pi)^{-\frac{d}{2}}\cos\theta_j \qquad s_j = (2\pi)^{-\frac{d}{2}}\sin\theta_j$$

$$\Omega = (2\pi)^{-\frac{d}{2}}\,\mathbb{1}$$

$$L^2(\mathbb{T}^d) = L^2_e(\mathbb{T}^d) \oplus L^2_{od}(\mathbb{T}^d)$$

$$H = H_e \oplus H_{od}$$

$$\begin{cases} H_e = g + \mu(\Omega,\cdot\,)\Omega + \frac{\lambda}{2}\sum (c_j\,\cdot)c_j \\ H_{od} = g + \frac{\lambda}{2}\sum (s_j\,\cdot)s_j \end{cases}$$

$$H_e = g - V_{\mu\lambda} \qquad (g - V_{\mu\lambda})\varphi = E\varphi$$

$$\therefore \varphi = \underbrace{(g-E)^{-1}V_{\mu\lambda}}_{\text{B.S operator}}\varphi.$$

$$\left[\begin{array}{l}
\text{In general: } F\sum^{n}(h_j\,\cdot)h_j = A \\
\mathcal{L}\{h_j\} \ni \varphi \overset{C_2}{\mapsto} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} (h_1,\varphi) \\ \vdots \\ (h_n,\varphi) \end{pmatrix} \xrightarrow{C_1} F\sum z_j h_j \in \mathcal{L}(h_j) \\
A : \mathcal{L} \to \mathcal{L} \ \ \& \ \ A = C_1 C_2 \quad n\times n \ \underline{\text{matrix}} \\
\sigma(A) \ni 1 \Leftrightarrow \sigma(C_1C_2) \ni 1 \Leftrightarrow \sigma(C_2C_1) \ni 1
\end{array}\right]$$

$$\Omega = c_0$$

$$G = \underbrace{\begin{bmatrix} a & \lambda b & \cdots & \lambda b \\ 2\mu b & \lambda c & \lambda d \cdots & \lambda d \\ \vdots & \lambda d & \ddots & \\ 2\mu b & \lambda d & & \lambda c \end{bmatrix}}_{d+1} \Bigg\} d+1$$

$$\begin{aligned} a &= (\Omega,\ (g-z)^{-1}\Omega) \\ b &= (c_0,\ (g-z)^{-1}c_j) \\ c &= (c_j,\ (g-z)^{-1}c_j) \\ d &= (c_i,\ (g-z)^{-1}c_j)\,. \quad i \neq j \end{aligned}$$

Mumnov + Kuljianov

○ $\det\left(G(z) - I\right) \cdot \gamma\, H_{\lambda\mu}\, \delta_\lambda$

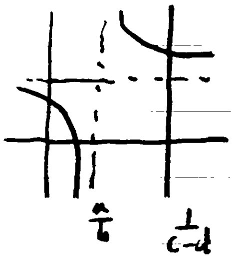
$\gamma \neq 0 \qquad H_{\lambda\mu} = \left(\lambda - \frac{a}{b}\right)\left(\mu - (d - z)\right) - d$

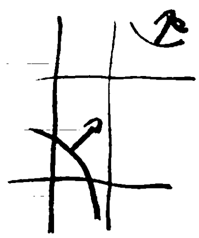
$\delta_\lambda = \left(\lambda(c-d) - 1\right)^{d-1} \ldots$

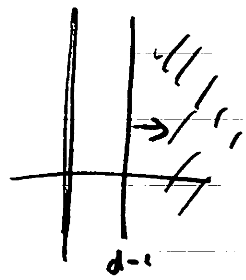
$\delta_\lambda = 0 \iff \lambda = \frac{1}{c-d}$

$z : 0 \to -\infty$

$H_{\lambda\mu} = 0 \iff$

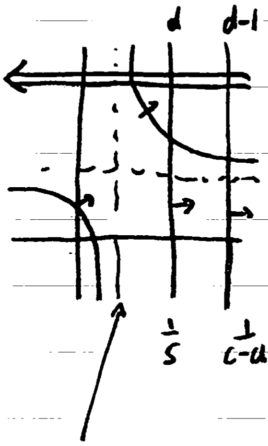






○ $H^0 - \quad \det\left(G(z) - 1\right) = (\lambda s - 1)^d$

$s = \left(s_j \quad (f-z)^{-1} s_j\right)$



$\frac{b_0}{a_0} = \lambda$

$d = 3$