Ground state of Rabi model

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Outline

Rabi model

Rabi Model(2-level atom vs 1-mode photon)

$$H = \Delta\sigma_z + \omega a^\dagger a + g\sigma_x(a + a^\dagger)$$

Notations

- $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- parameters $\Delta > 0$, $\omega > 0$
- Creation and annihilation: $a = (\frac{1}{\sqrt{\omega}}\frac{d}{dx} + \sqrt{\omega}x)/\sqrt{2}$, $a^\dagger = (-\frac{1}{\sqrt{\omega}}\frac{d}{dx} + \sqrt{\omega}x)/\sqrt{2}$.
- Two level atom $\Longrightarrow \Delta\sigma_z \Longrightarrow \mathrm{Spec}(\Delta\sigma_z) = \{-\Delta, \Delta\}$
- Harmonic oscillator $\Longrightarrow a^\dagger a = \frac{1}{2}(-\frac{d^2}{dx^2} + \omega^2 x^2 - \omega)$

NcHO and JC model

- Rotation Wave Approximation $\Longrightarrow H_{JC}$

$$H_{Rabi} = \sigma_z \Delta + \omega a^\dagger a + g\sigma_x(a + a^\dagger)$$

$$\Downarrow$$

$$H_{JC} = \sigma_z \Delta + \omega a^\dagger a + g(\sigma_- a + \sigma_+ a^\dagger)$$

- $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
- Non-commutative harmonic oscillator

$$Aa^\dagger a + \frac{1}{2}A + \frac{1}{2}J(aa - a^\dagger a^\dagger)$$

- $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

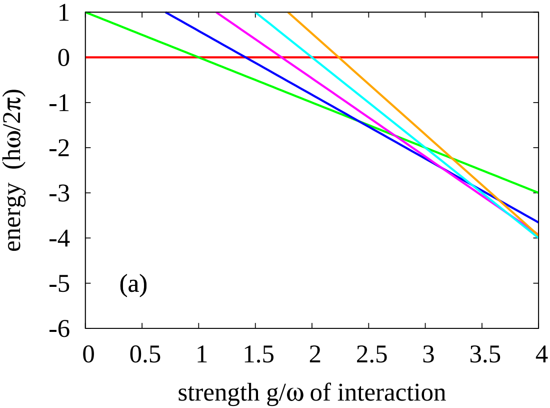


Figure: Spectral curves of JC Hamiltonian, $2\Delta = \omega$

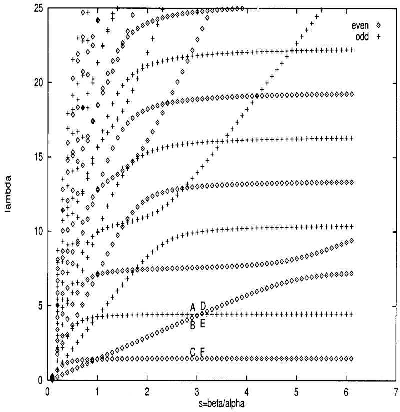


Figure 1. Approximate eigenvalue $\hat{\lambda}_N$.

Figure: Spectral curves of NcHO, Nagatou,Nakao and Wakayama, *Numerical Functional Analysis and Optimization* **23** 633-650, 2002

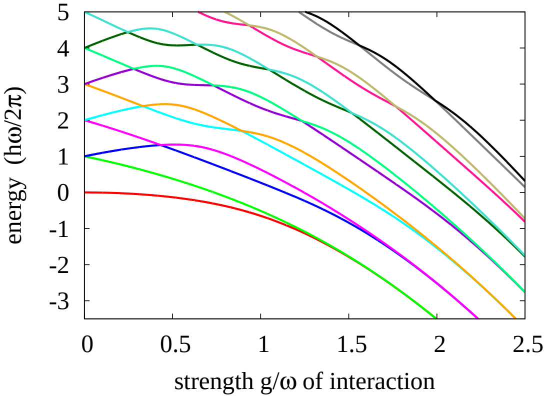


Figure: Spectral curves of Rabi Hamiltonian , $2\Delta = \omega$.

	NcHO	JC	Rabi
ground state	unique for $\alpha \neq \beta$?	degenerate?	unique

Figure: Uniqueness of ground state

- Rabi Hamiltonian $\Longrightarrow$ # crossing between $E_{2n} \sim E_{2n+1} = n$ for $n = 0, 1, 2, \ldots$.

$\sigma_z \to -\sigma_x, \sigma_x \to \sigma_z$

- $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is elements of $SU(2)$.
- The rotation group in $\mathbb{R}^3$ has an adjoint representation on $SU(2)$, i.e.,

$$e^{(i/2)\theta n\cdot\sigma}\sigma_\mu e^{-(i/2)\theta n\cdot\sigma} = (R\sigma)_\mu,$$

  where R denotes 3×3 matrix representing the rotation around $n \in \mathbb{R}^3$ with angle $\theta \in [0, 2\pi)$.
- For $n = (0,1,0)$ and $\theta = \pi/2$,

$$e^{(i/2)\theta n\cdot\sigma}\sigma_x e^{-(i/2)\theta n\cdot\sigma} = \sigma_z, \quad e^{(i/2)\theta n\cdot\sigma}\sigma_z e^{-(i/2)\theta n\cdot\sigma} = -\sigma_x.$$

Unitary transformation $U = e^{(i\pi/4)\sigma_y}$

$$\begin{aligned} UH_{\mathrm{Rabi}}U^{-1} &= -\sigma_x\Delta + \omega a^\dagger a + g\sigma_z(a+a^\dagger) \\ &= \begin{pmatrix} \omega a^\dagger a + g(a+a^\dagger) & -\Delta \\ -\Delta & \omega a^\dagger a - g(a+a^\dagger) \end{pmatrix} \end{aligned}$$

Ground state transformation

- $\varphi_{\rm g}(x) = (\omega/\pi)^{1/4} e^{-\omega x^2/2}$ is the ground state,i.e., $\omega a^\dagger a \varphi_{\rm g} = 0$.
- Define the unitary operator $U_g : L^2(\mathbb{R}) \to L^2(\mathbb{R}, \varphi_{\rm g}^2 dx)$ by $U_g f = \varphi_{\rm g}^{-1} f$.
- Probability measure $d\mu = \varphi_{\rm g}^2 dx$

Unitary transformation

$$U_g U H_{\rm Rabi} U^{-1} U_g^{-1} = -\sigma_x \Delta + \frac{1}{2}(-\frac{d^2}{dx^2} + \omega x \frac{d}{dx}) + g\sigma_z \sqrt{2\omega} x$$

$$on\ \mathbb{C}^2 \otimes L^2(\mathbb{R}, d\mu).$$

$\mathbb{C}^2 \to L^2(\mathbb{Z}_2)$

- $\mathbb{Z}_2 = \{-1, +1\}$
- Identification $\mathbb{C}^2 \otimes L^2(\mathbb{R}, d\mu) \cong \mathscr{H}$:
  - $\mathscr{H} = L^2(\mathbb{R} \times \mathbb{Z}_2) = \left\{ f(x,\sigma) \,\middle|\, \sum_{\sigma \in \mathbb{Z}_2} \int |f(x,\sigma)|^2 d\mu(x) < \infty \right\}$
  - $\mathbb{C}^2 \otimes L^2(\mathbb{R}, d\mu) \ni \begin{bmatrix} f_+(x) \\ f_-(x) \end{bmatrix} \mapsto f(x,\sigma) \in \mathscr{H}$.
- Under this identification

Final form of H_{Rabi}

$$Hf(x,\sigma) = \left\{ \frac{1}{2} \left(-\frac{d^2}{dx^2} + \omega x \frac{d}{dx} \right) + g\sqrt{2\omega}\sigma x \right\} f(x,\sigma) - \Delta f(x,-\sigma)$$

for $(x,\sigma) \in \mathbb{R} \times \mathbb{Z}_2$.

OU process and harmonic oscillator

- $(X_t)_{t\geq 0}$ the Ornstein-Uhrenbeck process on a probability space (C, Σ, P^x) st $P^x(X_0 = x) = 1$
- $\int d\mu(x)\mathbb{E}_{P^x}[X_t] = 0$, $\int d\mu(x)\mathbb{E}_{P^x}[X_t X_s] = \frac{e^{-|t-s|\omega}}{2\omega}$.
- $\mathbb{E}_Q[\cdots] = \int \cdots dQ$

The generator of X_t is given by $-h = -\frac{1}{2}(-\frac{d^2}{dx^2} + \omega x \frac{d}{dx})$ and

$$(f, e^{-th} g)_{\mathscr{H}} = \int d\mu(x)\mathbb{E}_{P^x}\left[\overline{f(X_0)} g(X_t)\right].$$

- The distribution $\rho_t(x,y)$ of X_t is given by

$$\rho_t(x,y) = \frac{\varphi_g(y)}{\varphi_g(x)} \frac{1}{\sqrt{\pi(1-e^{-2t})}} \exp\left(\frac{4xye^{-t} - (x^2+y^2)(1+e^{-2t})}{2(1-e^{-2t})}\right).$$

Poisson process and spin

In order to show the spin part by a path measure we introduce a Poisson process.

- $(N_t)_{t\geq 0}$ is a Poisson process on a probability space (C',Σ',ν) with unit intensity, i.e.,

$$\mathbb{E}_\nu\left[\mathbb{1}_{N_t=n}\right]=\frac{t^n}{n!}e^{-t},\quad n\geq 0.$$

- $\sigma_t=(-1)^{N_t}$ for $t\geq 0$.

$$(u,e^{-t\sigma_z}v)_{\mathbb{C}^2}=(u,e^{-t\sigma}v)_{L^2(\mathbb{Z}_2)}=e^t\sum_{\sigma\in\mathbb{Z}_2}\mathbb{E}_\nu[u(\sigma_0)v(\sigma_t)]$$

FK-formula

Spin boson model $\Longrightarrow$ H.and Lorinczi (JFA07)

$$\Delta\sigma_z + \int |k| a^\dagger(k)a(k)dk + g\sigma_x \int \lambda(k)(a^\dagger(k)+a(k))dk$$

Let $\sum_{\sigma\in\mathbb{Z}_2}\int d\mu(x)\mathbb{E}_{P^x}\mathbb{E}_\nu[\cdots] = \mathbb{E}[\cdots]$.

Theorem

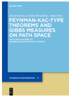
(Hirokawa and H.(2012))

$$(\Delta>0)\quad (f, e^{-tH}g)_{\mathscr{H}} = e^t\mathbb{E}\left[\overline{f(X_0,\sigma_0)}g(X_t,\sigma_t)e^{-g\sqrt{2\omega}\int_0^t \sigma_s X_s ds}\Delta^{N_t}\right],$$

$$(\Delta=0)\quad (f, e^{-tH}g)_{\mathscr{H}} = e^t\mathbb{E}\left[1\!\!1_{N_t=0}\overline{f(X_0,\sigma)}g(X_t,\sigma)e^{-g\sigma\sqrt{2\omega}\int_0^t X_s ds}\right]$$

Proof of Theorem:

- In the case of $\Delta > 0$



$$(f, e^{-tH} g)_{\mathscr{H}} = e^t \mathbb{E}\left[\overline{f(X_0, \sigma_0)} g(X_t, \sigma_t) e^{-g\sqrt{2\omega}\int_0^t \sigma_s X_s ds} e^{\int_{(0,t]} \log \Delta dN_s}\right].$$

  - $\displaystyle\int_{(0,t]} f(N_s) dN_t = \sum_{0<r\leq t, N_{r+}\neq N_{r-}} f(N_r), \quad e^{\int_0^t \log \Delta dN_s} = e^{\log \Delta^{N_t}} = \Delta^{N_t}.$

- In the case of $\Delta = 0$

$$(f, e^{-tH} g)_{\mathscr{H}} = e^t \mathbb{E}\left[\overline{f(X_0, \sigma_0)} g(X_t, \sigma_t) e^{-g\sqrt{2\omega}\int_0^t \sigma_s X_s ds} \mathbb{1}_{N_t=0}\right].$$

Avoided Crossing

Let $E = \inf \mathrm{Sp}(H)$.

Corollary

- $f \geq 0 \Longrightarrow e^{-tH} f > 0$
- $\dim \ker(H - E) = 1$

Proof

- $(f, e^{-tH} g)_{\mathscr{H}} = e^t \mathbb{E}\left[\overline{f(X_0, \sigma_0)} g(X_t, \sigma_t) e^{-g\sqrt{2\omega}\int_0^t \sigma_s X_s ds} \Delta^{N_t}\right] > 0$ for $\forall f, g \geq 0$
- $e^{-tH} f > 0$ for $f \geq 0$.

Proof

- $f \geq 0 \Longrightarrow \Omega_f = \{(x,\sigma) \in \mathbb{R} \times \mathbb{Z}_2 | f(x,\sigma) > 0\}$ has a positive measure.
- $(f, e^{-tH} g) \geq e^t \mathbb{E}\left[1\!\!1_{\Omega_f}(X_0,\sigma_0) 1\!\!1_{\Omega_g}(X_t,\sigma_t) e^{-g\sqrt{2\omega}\int_0^t \sigma_s X_s ds} \Delta^{N_t} \right]$.
- Let Ω be the set of paths starting from the inside of $(\Omega_f^+, +)$ and arriving at inside of $(\Omega_g^+, +)$.
-
$$\mathbb{E}[1\!\!1_\Omega] = \mathbb{E}\left[1\!\!1_{\Omega_f^+}(X_0) 1\!\!1_{\Omega_g^+}(X_t) 1\!\!1_{N_t=\text{even}} \right]$$
$$= \int_{\Omega_f^+} dx \int_{\Omega_g^+} dy \varphi_g(x)^2 p_t(x,y) \sum_{n=0}^\infty \frac{t^{2n}}{(2n)!} e^{-t} > 0.$$
- Hence Ω has a positive measure and
$$(f, e^{-tH} g) \geq e^t \mathbb{E}\left[1\!\!1_\Omega e^{-g\sqrt{2\omega}\int_0^t \sigma_s X_s ds} \Delta^{N_t} \right] > 0.$$
- $\dim\ker(H-E) = 1$ follows from Perron-Frobenius theorem.

Corollary

The ground state energy of H_{Rabi} has no crossing for all values of g and Δ.

- In this talk we have proven the fact the numerical computation predicts.
- While the JC model has many energy level crossings for the ground state energy in the ultra-strong coupling regime of circuit QED though it has no energy level crossing in the weak and strong coupling regimes $\Longrightarrow$ Sasaki's lecture tomorrow.

Thank you!