

2015/4/14
①

§ 0 Harmonic Oscillator

$\infty \supset \text{Harmonic Oscillators} = \text{QFT}$

$$\mathcal{H} = \frac{1}{2} \langle \hat{p}^2 + \hat{x}^2 \rangle$$

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 - \frac{1}{2}$$

$$= \hat{a}^\dagger \hat{a}, \quad a = \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + x \right)$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + x \right)$$

$$\Omega = \pi^{1/4} e^{-\frac{1}{2}x^2} \quad \therefore \int |\Omega(x)|^2 dx = 1$$

$$\varphi_n = \underbrace{\hat{a}^\dagger \dots \hat{a}^\dagger}_{n \text{ times}} \Omega, \quad [\hat{a}, \hat{a}^\dagger] = 1$$

$$[\hat{a} \Omega = 0]$$

$$\hat{a}^\dagger \hat{a} \varphi_n = n \hat{a}^\dagger \dots \hat{a}^\dagger \Omega = n \varphi_n$$

$$\therefore H \varphi_n = n \varphi_n, \quad H \Omega = 0$$

$$\varphi_n \perp \varphi_m \Leftrightarrow n \neq m$$

$$\therefore \left(\underbrace{\hat{a}^\dagger \dots \hat{a}^\dagger \Omega}_n \quad \underbrace{\hat{a}^\dagger \dots \hat{a}^\dagger \Omega}_m \right) = 0 \quad n < m$$

$$\|\hat{a}^\dagger \dots \hat{a}^\dagger \Omega\|^2 = n! \quad \therefore \sqrt{\frac{1}{n!}} \hat{a}^\dagger \dots \hat{a}^\dagger \Omega = \varphi_n$$

↑ normalization

$$\bigoplus_{n=0}^{\infty} \{ \varphi_n \} \subset L^2(\mathbb{R}^1)$$

$$a^\dagger a = H, \quad \frac{1}{\sqrt{2}}(a^\dagger + a) = x \quad \text{field op}$$

$$\frac{i}{\sqrt{2}}(a^\dagger - a) = -i \frac{d}{dx}$$

conjugate momentum

$$\bigoplus_{n=0}^{\infty} \{ \varphi_n \} = L^2(\mathbb{R}) \quad \text{Fock space}$$

$a^\dagger a$ free Hamiltonian

$|0\rangle$ Fock vacuum

$$n = \phi = \frac{1}{\sqrt{2}}(a^\dagger + a) \quad \text{field op}$$

$$-i \frac{d}{dx} = \pi = \frac{i}{\sqrt{2}}(a^\dagger - a) \quad \text{conjugate momentum}$$

$$U_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}, \Omega^2 dx) \quad \xleftarrow{\text{prob space}}$$

$$f \mapsto \frac{1}{\hbar} f \quad \xrightarrow{\text{isometry}}$$

unitary 123

$$L^2 \xrightarrow{U_g} L^2(\mathbb{R}, \Omega^2 dx)$$

$$H \downarrow \quad \downarrow L$$

$$L^2 \xrightarrow{U_g} L^2(\mathbb{R})$$

$$U_g^\dagger L U_g = H \quad \therefore L = U_g H U_g^\dagger = \frac{1}{\hbar} H \Omega$$

$$\frac{1}{\Omega} H \Omega f = \left(-\frac{1}{2} \frac{d}{dx^2} + \alpha \frac{d}{dx} \right) f$$

$$1 \in L^2(\Omega, \Omega^2 dx) = \mathcal{L}$$

$$L 1 = 0, \quad \Omega^{-1} \varphi_n \text{ is } L \Omega \varphi_n = n \Omega^{-1} \varphi_n$$
$$\overset{n}{\underset{1}{\widehat{\varphi}_n}} \quad L \widehat{\varphi}_n = n \widehat{\varphi}_n$$

$$\mathcal{L} = \bigoplus_{n=0}^{\infty} \{ \widehat{\varphi}_n \} \quad \mathcal{G}_0 = \mathbb{C}$$

$$L 1 = 0, \quad L \widehat{\varphi}_n = n \widehat{\varphi}_n$$

§ 1 Fock space

\mathcal{H} Hilbert space / \mathbb{C}
 complete normed space $(\cdot, \cdot)_{\mathcal{H}}$
 $= (\cdot, \cdot)$

$$\mathcal{H} \hat{\otimes} \dots \hat{\otimes} \mathcal{H} = L^2 \{ f_1 \otimes \dots \otimes f_n \}$$

$$(f_1 \otimes \dots \otimes f_n, g_1 \otimes \dots \otimes g_n)_n = \prod (f_j, g_j)$$

$$\mathcal{H} \hat{\otimes} \dots \hat{\otimes} \mathcal{H}^{\overline{\| \cdot \|_n}} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$$

$$= \hat{\otimes} \mathcal{H}$$

$(e_j) \in \mathcal{H}$ a base となる

$\otimes (e_{j_1} \otimes \dots \otimes e_{j_n})$ が $\mathcal{H} \hat{\otimes} \dots \hat{\otimes} \mathcal{H}$ の base

$\hat{\otimes} \mathcal{H}$ は n -th Hilbert space である

Symmetrization (projection)

$S_n : \hat{\otimes}^n \mathcal{H} \rightarrow \hat{\otimes}^n \mathcal{H}$

$$S_n(f_1 \otimes \dots \otimes f_n) = \frac{1}{n!} \sum_{\sigma} f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(n)}$$

Lemma 1.1 $S_n^* = S_n$, $S_n^2 = S_n$

正確な証明

$$\begin{aligned} \mathcal{H} \hat{\otimes} \cdots \hat{\otimes} \mathcal{H} &\subset \mathcal{H} \otimes \cdots \otimes \mathcal{H} \\ \downarrow S_n &\quad \downarrow \tilde{S}_n \end{aligned}$$

$$\mathcal{H} \otimes \cdots \otimes \mathcal{H} \subset \mathcal{H} \otimes \cdots \otimes \mathcal{H}$$

$$\|S_n F\|^2 = (S_n F, S_n F) = (F, S_n F) \leq \|F\| \cdot \|S_n F\|$$

$$\therefore \|S_n F\| \leq \|F\| \quad \forall F \in \hat{\otimes}^n \mathcal{H}$$

$$\therefore \forall F \in \bigoplus_{m=1}^n \mathcal{H} \quad \exists F_m \rightarrow F \quad (m \rightarrow \infty)$$

$\mathcal{H} \hat{\otimes} \cdots \hat{\otimes} \mathcal{H}$

$$\|S_n F_m - S_n F_{m'}\| \leq \|F_m - F_{m'}\| \rightarrow 0$$

Cauchy 3'ly

$$\lim_m S_n F_m = G$$

$$\tilde{S}_n F := G \quad \textcircled{1} \quad \tilde{S}_n F = S_n F \quad \forall F \in \hat{\otimes}^n \mathcal{H}$$

$$\textcircled{2} \quad \|\tilde{S}_n F\| \leq \|F\|.$$

左辺 \Rightarrow $\tilde{S}_n^* = \tilde{S}_n$ projection
 $\tilde{S}_n^2 = \tilde{S}_n$

$S_n(\hat{\otimes} \mathcal{H}) = \stackrel{n}{\hat{\otimes}} \mathcal{H}$ n-fold symmetric tensor product.

$$\left\{ \begin{array}{l} \bigoplus_{n=0}^{\infty} \hat{\otimes}^n \mathcal{H} = \mathcal{F}(\mathcal{H}) = \mathcal{F} \\ \\ \end{array} \right.$$

$$(\bar{\Phi}, \bar{\Psi}) = \sum_{n=0}^{\infty} (\bar{\Phi}^{(n)}, \bar{\Psi}^{(n)}) \hat{\otimes}^n \mathcal{H}$$

Symmetric (Boson) Fock space

$$S_n \rightarrow A_n = \frac{1}{n!} \hat{\otimes}^n \mathcal{H}$$

$$A_n(f_1 \otimes \dots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\deg \sigma} f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(n)}$$

Anti-symmetric Fock space or
Fermion Fock space

$$\stackrel{0}{\hat{\otimes}} \mathcal{H} = \mathbb{C}$$

$$\Omega = \{ 1, 0, 0, \dots \} \quad \text{Fock vacuum}$$

Lemma 2.2

$$[a(f), a^*(g)] = (\bar{f}, g)$$

$$[a^*(f), a^*(g)] = 0$$

on \mathbb{F}_0

$$\textcircled{2} \quad \left(a(f) a^*(g) \bar{\Sigma} \right)^{(k)} \quad \bar{\Sigma} = \sum_k (f_i \alpha \dots \alpha f_k)$$

$$= a(f) S_{k+1} (g \otimes f_1 \otimes \dots \otimes f_k) \sqrt{k+1}$$

$$= (\bar{f}, g) S_k (f_1 \alpha \dots \alpha f_k) \sqrt{k+1} / \sqrt{k+1}$$

$$+ \sum_{j=1}^k S_{k-j} (g \otimes f_1 \dots \hat{f}_j \dots f_k) \frac{\sqrt{k+1}}{\sqrt{k+1}} (\bar{f}, f_j)$$

$$(a^*(g) a(f) \bar{\Sigma})^{(k)}$$

$$= a^*(g) \sum_{j=1}^k (\bar{f}, f_j) S_{k-j} (f_1 \alpha \dots \hat{f}_j \dots f_k) \frac{1}{\sqrt{k}}$$

$$= \frac{\sqrt{k}}{\sqrt{k}} \sum_{j=1}^k (\bar{f}, f_j) S_k (g \otimes f_1 \dots \hat{f}_j \dots f_k)$$

$$\begin{aligned} \sigma(1) &= j \\ 1 &= \bar{\sigma}(j) \end{aligned}$$

$$\sqrt{k} \left(S_k(f_1 \otimes \dots \otimes f_k), \bar{f} \otimes g_1 \otimes \dots \otimes g_{k-1} \right)$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} (f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(k)}, \bar{f} \otimes g_1 \otimes \dots \otimes g_{k-1})$$

$$= \frac{1}{\sqrt{k}} \frac{1}{(k-1)!} \sum_{\sigma \in S_{k-1}} (f_{\sigma(1)}, \bar{f}) (f_{\sigma(2)}, g_1) \dots (f_{\sigma(k)}, g_{k-1})$$

$$= \prod_{j=1}^k \prod_{\substack{1 \leq i \leq k \\ i \neq j}} (f_i, \bar{f}) \quad \begin{matrix} 1, 2, \dots, k \\ \downarrow \sigma \downarrow \quad \downarrow \\ j \quad 1 - j = k \end{matrix}$$

$$= \frac{1}{\sqrt{k}} \sum_{j=1}^k (f_j, \bar{f}) \underbrace{\frac{1}{(k-1)!} \sum_{\sigma \in S_{k-1}}}_{\text{Fix } f_j} (f_{\sigma(2)}, g_1) \dots (f_{\sigma(k)}, g_{k-1})$$

$$= \frac{1}{\sqrt{k}} \sum_j (f_j, \bar{f}) \left(S_{k-1}(f_1 \otimes \hat{f}_j \otimes f_{k-1}), g_1 \otimes \dots \otimes g_{k-1} \right)$$

$$= \left(\frac{1}{\sqrt{k}} \sum_j (\bar{f}, f_j) S_{k-1}(f_1 \otimes \dots \otimes \hat{f}_j \otimes \dots \otimes f_{k-1}), g_1 \otimes \dots \otimes g_{k-1} \right)$$

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§ 3 2nd quantization

- $T : \mathcal{H} \rightarrow \mathcal{H}$ s.t. $\|Tf\| \leq \frac{1}{2} \|f\| \quad \forall f \in \mathcal{H}$
 since T is b'ld operator \Leftarrow .
 $\inf_{\|f\| \neq 0} \|Tf\| / \|f\| = \|T\| \quad \Leftarrow T$ is bound \Leftarrow .
- ∴ $\|Tf\| \leq \|T\| \cdot \|f\| \quad \forall f \in \mathcal{H}$.

$T_j : \mathcal{H}_j \rightarrow \mathcal{H}_j$, $T_1 \otimes T_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$

$$T_1 \otimes T_2 \left(\sum_{\alpha, \beta} C_{\alpha \beta} \varphi_{1\alpha} \otimes \varphi_{2\beta} \right) = \sum_{\alpha, \beta} C_{\alpha \beta} T_1 \varphi_{1\alpha} \otimes T_2 \varphi_{2\beta}$$

Lemma 3.1 $T_1 \otimes T_2$ is b'ld op. base & dots

$$\therefore \|T_1 \otimes 1_{\mathcal{H}_2} \varphi\|^2 = \sum_{\alpha} \sum_{\alpha'} \sum_{\beta} C_{\alpha \beta} \bar{C}_{\alpha' \beta} (T_1 \varphi_{1\alpha}, T_1 \varphi_{1\alpha'}) (\varphi_{2\beta}, \varphi_{2\beta})$$

$$= \sum_{\beta} \sum_{\alpha} \sum_{\alpha'} C_{\alpha \beta} \bar{C}_{\alpha' \beta} (T_1 \varphi_{1\alpha}, T_1 \varphi_{1\alpha'})$$

$$= \sum_{\beta} \left\| \sum_{\alpha} T_1 \varphi_{1\alpha} \right\|^2 \leq \|T_1\|^2 \sum_{\beta} \left\| \sum_{\alpha} \varphi_{1\alpha} \right\|^2$$

$$= \|T_1\|^2 \sum_{\alpha} |C_{\alpha \beta}|^2 = \|T_1\|^2 \|\varphi\|^2$$

$$\therefore \|T_1 \otimes 1_{\mathcal{H}_2}\| \leq \|T_1\| \quad (278)$$

$$\begin{aligned} \therefore \|T_1 \otimes T_2\| &= \|T_1 \otimes 1_{\mathcal{H}_2} \cdot 1_{\mathcal{H}_2} \otimes T_2\| \\ &\leq \|T_1\| \cdot \|T_2\| \end{aligned}$$

$$\begin{aligned}
 \text{证} &= \|T_j \varphi_j\| \stackrel{?}{\geq} \|T_j\| - \varepsilon \quad \|\varphi_j\|=1 \\
 \| (T_1 \otimes T_2) (\varphi_1 \otimes \varphi_2) \| &= \|T_1 \varphi_1\| \cdot \|T_2 \varphi_2\| \\
 &\geq (\|T_1\| - \varepsilon)(\|T_2\| - \varepsilon) = \|T_1\| \cdot \|T_2\| \\
 &\quad - \varepsilon (\|T_1\| + \|T_2\|) + \varepsilon^2
 \end{aligned}$$

$$\therefore \|T_1 \otimes T_2\| \geq \|T_1\| \cdot \|T_2\|$$

Def 3.2 $T: \mathcal{H} \rightarrow \mathcal{H}$ contraction

$$\Gamma(T) : \mathcal{F}(\mathcal{H}) \rightarrow \mathcal{F}(\mathcal{H})$$

$$\Gamma(T) = \bigoplus_{n=0}^{\infty} \left(T \underbrace{\otimes \dots \otimes T}_n \right)$$

证 Lem 3.3 $\Gamma(T)$ is contraction

$$\begin{aligned}
 \|\Gamma(T)\psi\|^2 &= \sum_{n=0}^{\infty} \left\| (T \otimes \dots \otimes T) \underbrace{\psi}_{\psi^{(n)}} \right\|^2 \\
 &\leq \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2 = \|\psi\|^2
 \end{aligned}$$

$$\therefore \|\Gamma(T)\| \leq 1.$$

$C(\mathcal{H}) = \{ T: \mathcal{H} \rightarrow \mathcal{H}; \text{ unitary} \}$

$\Gamma : C(\mathcal{H}) \rightarrow C(F(\mathcal{H}))$

$$\begin{array}{ccc} \downarrow & & \uparrow \\ T & \mapsto & \Gamma(T) \end{array}$$

Lemma 3.4 $\Gamma(T)\Gamma(S) = \Gamma(TS)$
 $\Gamma(T)^* = \Gamma(T^*)$

\therefore easy

Example h s.a. in \mathcal{H}

$\left\{ e^{-ith} \right\}_{t \in \mathbb{R}}$ one parameter unitary group

- Stone's thm - \mathcal{H} Hilbert space

$(S_t)_{t \in \mathbb{R}}$ one-parameter strongly cont unitary groups

$\Leftrightarrow \exists h \text{ s.t. } S_t = e^{-ith}$

Def $\Gamma(e^{-ith}) = S_t$

Lemma 3.5 } S_t { $t \in \mathbb{R}$ } is strongly cont
1-parameter unitary groups.

$$\textcircled{(1)} \quad S_0 = 1, \quad S_t S_s = S_{t+s}, \quad \text{ok}$$

cont. + $\textcircled{(2)}$ & check $\textcircled{(3)}$
dense $\tau \in \mathbb{C}^*$ $\subset \mathcal{T}'$.

$$\begin{aligned} (S_t \Phi, \bar{\Phi}) &= (\tilde{e}^{ith} f_1 \otimes \dots \otimes \tilde{e}^{ith} f_n, g_1 \otimes \dots \otimes g_n) \\ &= \prod_{j=1}^n (\tilde{e}^{ith} f_j, g_j) \rightarrow \prod_j (f_j, g_j) \\ &= (f_1 \otimes \dots \otimes f_n, g_1 \otimes \dots \otimes g_n) \end{aligned}$$

$\therefore S_t \rightarrow 1$ weakly

$$\therefore \|S_t \bar{\Phi} - \bar{\Phi}\|^2 = \|\bar{\Phi}\|^2 - 2(S_t \bar{\Phi}, \bar{\Phi})$$

$$S_t = e^{-it} \sum dP(h) \quad dP(h) \text{ s.a.}$$

Lemma 3.6 $\bar{\Phi} = f_1 \otimes \dots \otimes f_n \quad f_j \in \mathcal{D}(h)$

$$dP(h) \quad f_1 \otimes \dots \otimes f_n = \sum_{j=1}^n f_1 \otimes \dots \otimes h f_j \otimes \dots \otimes f_n$$

$$\text{i.e.,} \quad dP(h) = \bigoplus_{n=0}^{\infty} \sum_{j=1}^n 1 \otimes \dots \otimes h \dots \otimes 1$$

$$\text{Def: } \hat{e}^{-itdP(h)} f_1 \otimes \dots \otimes f_n = \hat{e}^{-ith} f_1 \otimes \dots \otimes \hat{e}^{-ith} f_n$$

\Rightarrow When $t=0$, it's the original function.

大事 $dP = \text{id}$ $dP(n) : \mathcal{F}^{(n)} \rightarrow \mathcal{F}^{(n)}$

$$\text{Ex. } \mathcal{H} = L^2(\mathbb{R}^d), \quad h = \sqrt{k_x^2 + k_y^2}$$

$$dP(n) f_1 \otimes \dots \otimes f_n = \sum_j f_1 \otimes \dots \otimes h f_j \otimes f_n$$

$$f_1 \otimes \dots \otimes f_n = f(k_1, \dots, k_n) \quad \text{by definition}$$

$$\sum_j f_1 \otimes \dots \otimes h f_j \otimes f_n = \sum_j \sqrt{|k_j|^2 + V^2} f(k_1, \dots, k_n)$$

§ 4 Annihilation & creation

$\exists = \exists(\bar{f}e)$ \exists ~~is~~ complex conjugate -

$$J: \mathcal{H} \rightarrow \mathcal{H} \quad J^2 = 1$$

~~is anti-unitary~~

Def 3.1 $f \in \mathcal{H}$

$$\left\{ \begin{array}{l} \left(a^*(f) \bar{\Phi} \right)^{(n+1)} = \sqrt{n+1} S_{n+1} (f \otimes \bar{\Phi}^{(n)}), \quad n \geq 0 \\ \left(a^*(f) \bar{\Phi} \right)^{(0)} = 0 \end{array} \right.$$

creation

$$D(a^*(f)) = \left\{ \bar{\Phi} \in \mathcal{F} ; \sum_{n=0}^{\infty} (n+1) \| S_{n+1} (f \otimes \bar{\Phi}^{(n)}) \| ^2 < \infty \right\}$$

∴ $a(f) = \left(a^*(\bar{f}) \right)^*$ ~~annihilator~~

∴ a unbounded op or adjoint \Rightarrow 1.2

$A : \mathbb{H}^n \rightarrow \mathbb{H}^n$ linear map

A : $n \times n$ matrix

$$(u, A v) = (A^* u, v) \text{ where}$$

$A^* = \overline{^t A}$ adjoint .

$$T : \mathcal{H} \rightarrow \mathcal{H} \quad D(T) \subset \mathcal{H}$$

Def 3.2 adjoint $(f, Tg) = (\exists_h g)$ $\forall g \in D(T)$

$f \in D(T^*)$, $h = T^* f$ \Leftarrow 定義了。

$g \in D(T)$ defines \exists_h well-defined

-般 $\exists_h \in D(T^*) \subset \mathcal{H}$

$$\left(\prod_{j=1}^n \alpha(f_j) \Omega \right)^{(m)} = \begin{cases} \sqrt{n!} S_n(f_1 \otimes \dots \otimes f_n) & m=n \\ 0 & m \neq n \end{cases}$$

Lemma 3.3

$$J_0(\omega) = \text{L.H.} \left\{ \prod_{j=1}^n \alpha(f_j) \Omega_j \Omega; f_j \in \mathcal{H}, j=1..n \right\}_{n \geq 1}$$

is dense

Lemma 3.4 $J_0(\omega)$ 上で CCP

$$[\alpha^*(f), \alpha^*(g)] = (\bar{f}, g) \mathbb{I} \quad \forall \epsilon \neq \bar{f}, \bar{g}$$

∴ $\bar{\Phi} = S_n(f_1 \otimes \dots \otimes f_n) = S_n(f_1, \dots, f_n) \in \text{CCP}$

更に $\begin{cases} \alpha(f) S_n(f_1, \dots, f_n) = \frac{1}{\sqrt{n}} \sum_j (\bar{f}, f_j) S_n(f_1, \dots, \hat{f}_j, \dots, f_n) \\ \alpha^*(f) S_n(f_1, \dots, f_n) = \sqrt{n+1} S_{n+1}(f, f_1, \dots, f_n) \end{cases}$

が成り立つ。 これは $\frac{1}{\sqrt{n}} \sum_j (\bar{f}, f_j) S_n(f_1, \dots, \hat{f}_j, \dots, f_n)$

$$\alpha(f) \alpha^*(g) S_n(f_1, \dots, f_n) = \alpha(f) \sqrt{n+1} S_{n+1}(g, f_1, \dots, f_n)$$

$$= \frac{\sqrt{n+1}}{\sqrt{n+1}} \left((\bar{f}, g) + \sum_{j=1}^n (\bar{f}, f_j) S_n(f, f_1, \dots, \hat{f}_j, \dots, f_n) \right)$$

$$\alpha^*(g) \alpha(f) S_n(f_1, \dots, f_n) = \alpha^*(g) \frac{1}{\sqrt{n}} \sum_j (\bar{f}, f_j) S_n(g, f_1, \dots, \hat{f}_j, \dots, f_n)$$

$$= \frac{1}{\sqrt{n}} \sum_j (\bar{f}, f_j) \sqrt{n} S_n(g, f_1, \dots, \hat{f}_j, \dots, f_n) \quad \text{なぜなら}$$

$$a^*(g) S_n(f_1 \dots f_n) = \sqrt{n+1} g \otimes S_n(f_1 \otimes \dots \otimes f_n)$$

$$= \sqrt{n+1} S_{n+1}(g \otimes f_1 \dots \otimes f_n) \quad \text{OK}$$

$$(a^*(f)\Phi, S_n(f_1 \dots f_n)) = (\Phi, h)$$

$$= (\sqrt{n} S_n(\bar{f} \otimes \Phi), S_n(f_1 \dots f_n))$$

$$= \sqrt{n} \left(\bar{f} \otimes \Phi, \frac{1}{n!} \sum_{\sigma} f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(n)} \right)$$

$$= \frac{1}{\sqrt{n}} \frac{1}{(n-1)!} \sum_{\sigma} (\bar{f}, f_{\sigma(1)}) (\Phi, f_{\sigma(2)} \otimes \dots \otimes f_{\sigma(n)})$$

$$= \frac{1}{\sqrt{n}} \sum_{j=1}^n (\bar{f}, f_j) \frac{1}{(n-1)!} \sum_{\sigma} (\Phi, f_{\sigma(2)} \otimes \dots \otimes f_{\sigma(n)})$$

$$= (\Phi, \frac{1}{\sqrt{n}} \sum_{j=1}^n (\bar{f}, f_j) S(f_1 \dots \hat{f}_j \dots f_n)),$$

Lemma 3.5 $\Phi \in \mathcal{F}_0(H)$

$$\|a(f)\Phi\| \leq \|f\| \|N^{1/2}\Phi\|$$

$$\|a^*(f)\Phi\| \leq \|f\| (\|N^{1/2}\Phi\| + \|\bar{\Phi}\|)$$

$$\because \left\| a(f)\bar{\Phi} \right\|^2 = \sum_{n=0}^{\infty} \left\| S_n(f \otimes \frac{\Phi}{n+1}) \right\|_{(n+1)}^2$$

$$\leq \sum_{n=0}^{\infty} \left\| f \otimes \bar{\Phi}^{(n)} \right\|_{(n+1)}^2$$

$$= \|f\| \left\| \bar{\Phi}^{(n)} \right\|_{(n+1)}^2$$

V

Thm 3.6 $\frac{1}{\sqrt{2}} \left(\hat{a}^*(\bar{f}) + a(f) \right)$ is Fréchet ess. s. q.

(\Leftarrow) Nelson's analytic vector then $\mathfrak{f} = \bar{f}$.

$$\left\| \underbrace{\hat{a}^*(\bar{f}_1) \dots \hat{a}^*(\bar{f}_k)}_R \bar{\Phi} \right\| \leq \sqrt{n+k} \| f \| \prod_{k=1}^R \|\bar{\Phi}\|$$

$$\leq \sqrt{n+k} \dots \sqrt{n+1} \| f \|_R^R \|\bar{\Phi}\|$$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \|\bar{\Phi}_{\bar{\Phi}}^k\| \leq \sum \frac{t^k}{k!} \frac{-k}{2} 2^k \sqrt{n+k} \dots \sqrt{n+1} \| f \|_R^R \times \|\bar{\Phi}\| < \infty$$

\Rightarrow Nelson's unanalytic vector then

H symmetric op. on D $H \subset H^*$

$$\forall \varphi \in D \quad \sum \frac{t^n}{n!} \| H^n \varphi \| < \infty$$

$\Rightarrow H \Gamma_D$ is ess. s. q.

- $\phi(f) \mid_{\mathcal{F}_0}$ is symmetric

- $\phi(f) \Gamma_{\mathcal{F}_0}$ is ess. s. q. i.e., 自己共役 所以 $\Gamma_{\mathcal{F}_0} = -\Gamma_{\mathcal{F}_0}$

2nd quantization $\Gamma(T)$, $d\Gamma(T)$ の
CCR.

$$\Gamma(T) \sum_{j=1}^n a^*(f_j) \Omega = \sum_{j=1}^n a^*(Tf_j) \Omega \quad (1=23)$$

$$\sqrt{n} S_n(f_1 \otimes \dots \otimes f_n) \rightarrow 0$$

$$\Gamma(T) a^*(f) \bar{\Phi} = a^*(Tf) \Gamma(T) \bar{\Phi}$$

$$\therefore [\Gamma(T), a^*(f)] \bar{\Phi} = a^*((T-f)) \Gamma(T) \bar{\Phi}$$

$$Tf := \Gamma(e^{ith}) \text{ とおぼえ}$$



$$\Gamma(e^{ith}) a^*(f) \bar{\Phi} = a^*(e^{ith} f) \Gamma(e^{ith}) \bar{\Phi}$$

$$\Gamma(e^{ith}) a^*(f) \Gamma(e^{-ith}) \bar{\Phi} = a^*(e^{ith} f) \bar{\Phi}$$

$$t = 0 \text{ の } \frac{1}{2} \text{ 分で } \frac{1}{2} \text{ オーク$$

$$e^{it d\Gamma(h)} a^*(f) e^{-it d\Gamma(h)} = a^*(e^{ith} f)$$

$$i [d\Gamma(h), a^*(f)] = i a^*(hf)$$

$$\therefore [d\Gamma(h), a^*(f)] = a^*(hf)$$

$$[d\Gamma(h), a^*(f)] = -a^*(hf)$$

$$\begin{aligned} a(\bar{hf}) &= -[d\Gamma(h), a(\bar{f})] \\ &= \boxed{[d\Gamma(h), a(f)]} \\ &= a(\bar{hf}) \end{aligned}$$

(3) N number operator $\alpha \in \mathbb{Z}$

$$[N, \alpha^+(t)] = \alpha^+(t)$$

$$[N, \alpha(t)] = -\alpha(t) \quad i=2, 3 = \pm \pi/3.$$

$$(3) e^{itdP(h)} \alpha^+(t) e^{-itdP(h)} = \alpha^+(\frac{ith}{e} f)$$

$$e^{itdP(h)} \alpha(f) e^{-itdP(h)} = \alpha(\overline{e^{\frac{ith}{e}} f}) \\ = \alpha(\overline{e^{\frac{ith}{e}} f})$$

$$\pi(f) = \frac{i}{\sqrt{2}} \left[c^*(\bar{f}) - c(f) \right] = \alpha \in \mathbb{Z}$$

$$[\phi(f), \pi(g)] = i \operatorname{Re}(f, g)$$

$$[\psi(t), \psi(g)] = i \operatorname{Im}(t, g)$$

$$[\pi(t), \pi(g)] = i \operatorname{Im}(t, g)$$

Example エカルカリオ $\mathcal{H} = L^2(\mathbb{R}^d)$

$$\mathcal{F}^{(n)} \cong L^2_{\text{sym}}(\mathbb{R}^d) \quad T_1, T_2.$$

$g(k_1, \dots, k_n)$ は k_j は ≥ 0 の ~~対称~~ 対称.

$$\begin{cases} G(f) S_{nl}(f_1 \dots f_n) = \frac{1}{\sqrt{n}} \sum_j (\bar{f} f_j) \sum_{k=1}^n (f_1 \dots \overset{\wedge}{f_j} \dots f_n) \\ G^*(f) \underline{S_{nl}(f_1 \dots f_n)} = \sqrt{n+1} \underline{S_{n+1}(f f_1 \dots f_n)} \end{cases}$$

$$\begin{aligned} (G(f)g)(k_1 \dots k_{n-1}) &= \frac{1}{\sqrt{n}} \sum_j \int f(k_j) g(k_1 \dots \overset{\wedge}{k_j} \dots k_n) dk_j \\ &= \frac{1}{\sqrt{n}} \sum_j \int f(k_j) g(k_1 k_2 \dots k_{n-1}) dk_j \\ &\rightarrow \sqrt{n} \int f(k_j) g(k_1 k_2 \dots k_{n-1}) dk_j \end{aligned}$$

$$(G^*(f)g)(k_1 \dots k_{n-1}) = \cancel{\sqrt{n+1}} \cancel{\frac{1}{\sqrt{n+1}}} \cancel{\sum_j} \cancel{f} \cancel{g} \cancel{S_{n+1}(f_1 \dots f_n)}$$

$$\delta(1) = j \quad \begin{array}{ccccccccc} 1 & 2 & \dots & n & n+1 \\ \downarrow & & & & & & & & \\ b & & & & & & & & \\ \downarrow & & & & & & & & \end{array}$$

$$\begin{aligned}
& (G^+(f)g)(k_1 \dots k_n k_{n+1}) = \frac{\sqrt{n+1}}{(n+1)!} \sum_{\sigma} f(k_{\sigma(1)}) g(k_{\sigma(2)} \dots k_{\sigma(n)}) \\
&= \frac{\sqrt{n+1}}{(n+1)!} \sum_d f(k_d) \frac{1}{n!} (n+1)! g(k_1 \dots \hat{k}_d \dots k_{n+1}) \\
&= \frac{1}{\sqrt{n+1}} \sum_d^{n+1} f(k_d) g(k_1 \dots \hat{k}_d \dots k_{n+1})
\end{aligned}$$

$$\begin{aligned}
& \| (a(t)\bar{\Phi})^{(n)} \|^2 = n \int dk_1 \dots dk_n \left| \int_{\mathbb{R}^d} f(k_1) \bar{\Phi}^{(n)}(k_1 \dots k_n) dk_1 \right|^2 \\
& \leq n \|f\|^2 \| \bar{\Phi}^{(n)} \|^2 = \|f\|^2 \left(\bar{\Phi}^{(n)}, N \bar{\Phi}^{(n)} \right) \\
& \quad = \|f\| \|N^{1/2} \bar{\Phi}^{(n)}\|^2
\end{aligned}$$

$$||a(t)\bar{\Phi}||^2 = \sum_{n=1}^{\infty} \left\| (a(t)\bar{\Phi})^{(n)} \right\|^2 \leq \|f\|^2 \|N^{1/2} \bar{\Phi}\|^2$$

$$\begin{aligned}
& (a(t)\bar{\Phi}, a(t)\bar{\Phi}) = (\bar{\Phi}, a(t)a^*(t)\bar{\Phi}) = \|f\| \|\bar{\Phi}\|^2 \\
& \leq \|f\|^2 (\bar{\Phi}, (N+1)\bar{\Phi}),
\end{aligned}$$

§ 5 Bounds

⑤ 5/19

Example $\mathcal{F} = L^2(\mathbb{H}^d)$

$$\mathcal{F} = \mathcal{F}(\mathcal{F}) \doteq \bigoplus_{n=0}^{\infty} \overset{\sim}{\otimes} L^2 = \bigoplus_{n=0}^{\infty} L^2_{\text{sym}}(\mathbb{H}^{dn})$$

$$(a(f)\bar{\Phi})^{(n)}(k_1 \dots k_n) = \sqrt{n} \int f(k) \bar{\Phi}^{(n)}(k k_1 \dots k_{n-1}) dk$$

$$(dP(\omega)\bar{\Phi})^{(n)} = \left(\sum_{j=1}^n \omega(k_j) \right) \bar{\Phi}^{(n)}(k_1 \dots k_n) \quad \leftarrow$$

$$dP(\omega) S_n(f_1 \dots f_n) = \sum_j S_n(f_1 \dots \omega f_j \dots f_n)$$

$$\|a(f)\bar{\Phi}\|^2 = \sum_{n=0}^{\infty} \| (a(f)\bar{\Phi})^{(n)} \|^2$$

$$= \sum_{n=0}^{\infty} (n+1) \int dk_1 \dots dk_n \left| \int f(k) \bar{\Phi}^{(n+1)}(k k_1 \dots k_n) dk \right|^2$$

$$\leq \sum_{n=0}^{\infty} (n+1) \int dk_1 \dots dk_n \left| \int f(k)^2 dk \right|^{1/2} \left| \int \bar{\Phi}^{(n+1)}(k k_1 \dots k_n) dk \right|^{1/2}$$

$$= \|f\|^2 \sum_{n=0}^{\infty} (n+1) \int |\bar{\Phi}^{(n+1)}(k_1 \dots k_{n+1})|^2 dk_1 \dots dk_{n+1}$$

$$\therefore \|a(f)\bar{\Phi}\|^2 \leq \|f\|^2 \|N^{1/2} \bar{\Phi}\|^2$$

不一样:

$$\|a(f)\bar{\Phi}\|^2 = \sum_{n=0}^{\infty} \int dk_1 \dots dk_n \int \frac{f(k)}{\sqrt{n}} \sqrt{n} \bar{\Phi}^{(n+1)}(k k_1 \dots k_n) dk$$

$$(n+1) \int \sqrt{n} \bar{\Phi}(k k_1 \dots k_n) |f(k)|^2 dk dk_1 \dots dk_n$$

$$= \int \left(\sum_{j=0}^n w(k_j) \right) |\bar{\Phi}(k k_1 \dots k_n)|^2 dk dk_1 \dots dk_n$$

$$\therefore \|\alpha(f)\bar{\Phi}\| \leq \|f/\sqrt{\omega}\| \|\, dP(\omega)^{1/2} \bar{\Phi}\| \quad \text{证毕.}$$

CCR 在哪儿

$$\|\alpha^*(f)\bar{\Phi}\| \leq \|f\| (\|N^{1/2}\bar{\Phi}\| + \|\bar{\Phi}\|)$$

$$\|\alpha^*(f)\bar{\Phi}\| \leq \|f/\sqrt{\omega}\| \underbrace{\|\, dP(\omega)^{1/2} \bar{\Phi}\|}_{\|f\|} + \underbrace{\|\bar{\Phi}\|}_{\|f\|}$$

- 用反言命 13 页定理 13m = 2, 12, 13.

例 解析的 vector 例

$$\sum_{n=0}^{\infty} \frac{\|H^n f\| t^n}{n!} < \infty \quad \exists t \quad f \in D(H)$$

$D \ni f$ 为 H 的解析的 vector H 是 D 上 ess. s.a.

由 $t^n A^n f$ 为 A 的 vector.

由 $A^n f$ 为 A 的 vector.

$\|A^n f\| = \|f\|$

$$\bar{\Phi} \in \mathcal{F}^{(k)} \text{ & } 3$$

$$\|\phi(f)^n \bar{\Phi}\| \leq \frac{1}{\#}^2 \|f\| \sqrt{k+m-1} \quad \text{--- (1)}$$

$$\|\phi(f) \bar{\Phi}\| \leq \frac{1}{\sqrt{2}} \|f\| \|N^k \bar{\Phi}\| + \|f\| (\|N^k \bar{Q}\| + \|\bar{Q}\|)$$

$$= \frac{1}{\sqrt{2}} \left[\|f\| \sqrt{n} \|\bar{\Phi}\| + \|f\| (\sqrt{n} \|\bar{Q}\| + \|\bar{Q}\|) \right]$$

$$\frac{1}{\sqrt{2}} \|f\| \left(\underbrace{\frac{2\sqrt{n}+1}{\sqrt{2\sqrt{n+1}}}}_n \right) \|\bar{\Phi}\|$$

$$\begin{aligned} & f(n+1) = (4n+4\sqrt{n}+1) \\ & 4n+8 - 4\sqrt{n}-1 \\ & = 4n - 4\sqrt{n} + 7 > 0 \\ & 16 - 1/2 < 0 \end{aligned}$$

$$\therefore (1) \leq 2^n \|f\|^n \sqrt{k+m-1} \sqrt{k+m-2} \cdots \sqrt{k}$$

$$\sum_{n=0}^{\infty} \frac{\|\phi(f)^n \bar{\Phi}\| t^n}{n!}, \quad \sum_{n=0}^{\infty} \frac{2^n \|f\|^n \sqrt{k+m-1} \cdots \sqrt{k} t^n}{n!} \|\bar{\Phi}\| < \infty$$

$\therefore \mathcal{F}_b$ is $\phi(f)$ a analytic vector 12/3.

$\therefore \phi(f) \mathcal{F}_b$ is ess. s.a.

$$H_{\text{VH}} = dP(\omega) + g\phi(f) \quad g \in \mathbb{R}$$

von Hove model で g 結合定数

Kato Rellich の 定理

je Hilbert space

A, B s.a. B sym s.t:

$$\textcircled{1} \quad D(A) \subset D(B)$$

$$\textcircled{2} \quad \|B\varphi\| \leq a\|A\varphi\| + b\|\varphi\| \quad \forall \varphi \in D(A) \\ 0 \leq a \leq 1$$

$\Rightarrow A+B$ is s.g. on $D(A)$.

Then $f/\sqrt{\omega}, f \in L^2$ と \exists .

~~は~~ H_{VH} is $D(dP(\omega))$ 上 s.g.

$$\textcircled{-1} \quad \|\phi(f)\bar{\Phi}\| \leq \|dP(\omega)^{1/2}\bar{\Phi}\| \|f/\sqrt{\omega}\| + \|f\| \|\bar{\Phi}\|$$

//

§ 6 van Hove model

$H_f + gH(f)$, $f, f/\sqrt{w} \in L^2 \cap \mathcal{Z}$ $H_{12} \xrightarrow{fg \in H_2}$
 $\mathcal{L}^2 \otimes \mathcal{L}^2$ self-adjoint $T_2 \otimes T_2$.

T : self-adjoint operator.

$\sigma(T) \subset H_2$ spectrum

T : closed operator $\Rightarrow T^* \supseteq T$

$$\sigma(T) = \{\lambda \in \mathbb{C}; \text{Ran}(\lambda - T) = \mathcal{H}$$

\uparrow resolvent set $\lambda - T$ injective, $(\lambda - T)^{-1}$ bdd

$$\sigma(T) = \mathbb{C} \setminus \rho(T)$$

Example $A: \mathbb{H}^n \rightarrow \mathbb{H}^n$ linear op.

$$\sigma(A) = \{e.v.\}$$

Example $T\varphi = a\varphi \quad (\varphi \neq 0) \Rightarrow a \in \sigma(T)$

• a is called eigenvalue

• $\sigma_p(T) \ni \lambda \Leftrightarrow \lambda - T$ not injective

• $\sigma_c(T) \ni \lambda \Leftrightarrow (\lambda - T)^{-1}$ not densely defined but bdd

• $\sigma_r(T) \ni \lambda \Leftrightarrow (\lambda - T)^{-1}$ not densely defined

$$\mathbb{C} = \rho(T) \cup \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

Closed symmetric op T

$$G(T) \subset \mathbb{R} \Leftrightarrow T \text{ s.a.}$$

$\inf G(T) > -\infty \Rightarrow T \text{ is bdd from below}$

- $\inf G(T) = E_{\text{ground state energy}}$

- $T\varphi = E\varphi \quad \varphi : \text{ground state}$

$$-\Delta \cong k_1^2 \quad \delta(-\omega) = \underbrace{[0, \infty)}_{\text{ground state}}$$

$$- \quad \delta(UA\bar{U}) = G(A) \quad u : \text{unitary op}$$

Then Spectrum of 2n quantization

$$\delta(dP(n)) = \overline{\{ \lambda_1 + \dots + \lambda_n ; \lambda_j \in G(\mathfrak{u}) \}} \cup \{0\}$$

$$\delta_p(dP(n)) = \{ \lambda_1 + \dots + \lambda_n ; \lambda_j \in \delta_p(\mathfrak{u}) \} \cup \{0\}$$

$$-\frac{1}{2}\sum_{i=1}^n \delta(A \otimes \mathbb{1} + \mathbb{1} \otimes B) = \overline{\{ \lambda + \mu ; \lambda \in \delta(A), \mu \in \delta(B) \}}$$

$$\delta(A \oplus B) = \delta(A) \cup \delta(B)$$

$$1 \oplus 1 \otimes 1 \quad A\varphi = E\varphi \quad B\varphi = E'\varphi$$

$$(A \otimes \mathbb{1} + \mathbb{1} \otimes B)\varphi \otimes \varphi = (E + E')\varphi \otimes \varphi$$

$$[H_0, \pi(t/\omega^2)] = i\phi(f)$$

$$[\phi(f), \pi(t/\omega^2)] = i(f, f/\omega^2) \quad (= \pm 1)$$

$$\begin{aligned} H_0 \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \pi(t/\omega^2) &= H_0 \left[1 + (-i) \pi(t/\omega^2) + \sum_{n=2}^{\infty} \dots \right] \\ &= 1 H_0 + (-i) i \phi(f) + (-i) \pi(t/\omega^2) H_0 \\ &\quad + \underbrace{\sum_{n \geq 2} \frac{(-i)^n}{n!} \left[H_0 \pi(t/\omega^2)^n \right]}_{n-(n-1)-1} \\ H_0 \pi(t/\omega^2)^n &= \sum_{j=0}^{n-1} \pi(t/\omega^2)^j i \phi(f) \pi(t/\omega^2)^{n-j-1} \\ &\quad + \pi(t/\omega^2)^n H_0 \\ &= \sum_{j=0}^{n-1} \pi(t/\omega^2)^j i (n-j-1) i (f, f/\omega^2) \pi(t/\omega^2)^{n-j-2} \\ &\quad + \sum_{j=0}^{n-1} \pi(t/\omega^2)^{n-1} i \phi(f) \quad (= 2 \times 3) \\ &\quad \times \cancel{H_0} + \pi(t/\omega^2)^{n+1} \end{aligned}$$

2x3x2

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \pi(t/\omega^2) H_0 + \left[(-i) + \sum_{n \geq 2} \frac{n}{n!} (-i)^n \pi(t/\omega^2)^{n-1} \right] \\ + \sum_{n \geq 2} \frac{(-i)^n}{n!} (-1) \underbrace{\sum_{j=0}^{n-1} (n-j-1) \pi(t/\omega^2)^{n-2} (f, f/\omega^2)}_{(n-1)+\dots+0} \times i \phi(f) \\ \frac{n(n-1)}{2} \end{aligned}$$

$$e^{-t\pi(t/\omega^2)} H_0 + (-i) e^{-t\pi(t/\omega^2)} i\phi(t)$$

$$+ \frac{1}{2} \sum_{n \geq 2} \frac{(-i)^{n-2}}{(n-2)!} \frac{(-i)^2(-1)}{(+1)} \pi(t/\omega)^{n-2} \|t/\omega\|^2$$

$$\therefore e^{-t\pi(t/\omega^2)} \left(H_0 + \phi(t) + \frac{1}{2} \|t/\omega\|^2 \right)$$

$$\begin{aligned} & \therefore e^{i\pi(t/\omega^2)} H_0 e^{-i\pi(t/\omega^2)} \\ &= H_0 + \phi(t) + \frac{1}{2} \|t/\omega\|^2 \end{aligned}$$

Lemma

$$H_0 + g \phi(t) \cong H_0 - \frac{1}{2} g^2 \|t/\omega\|^2$$

$(f/\omega, f/\omega^3 \in L^2)$

$$\text{Thm 6 } (H_0 + g \phi(t)) = \left\{ -\frac{1}{2} g^2 \|t/\omega\|^2 \right\} \cup \left[-\frac{1}{2} g \sqrt{1 + g^2/\omega^2}, \infty \right)$$

$f/\omega, f/\omega^3 \in L^2$

Van Hove model \approx image

$$-\frac{1}{2}\Delta + \frac{1}{2}x^2 + gx = H_g$$

$$= -\frac{1}{2}\Delta + \frac{1}{2}(x+g)^2 - \frac{1}{2}g^2$$

$$\approx -\frac{1}{2}\Delta + \frac{1}{2}x^2 - \frac{1}{2}g^2$$

$$e^{igp} H_0 e^{-igp} = H_g \quad l=2, 3$$

$$\therefore H_g \approx H_0 - \frac{1}{2}g^2$$

↑ ground state energy T_{min}
 T_{cusp}

{ ζ_1 } number of N $N = dP(1)$

- $G(N) = \{n\} \cup \{0\}$
- $G_p(N) = \{n\} \cup \{0\}$

{ ζ_1 } $dP(w)$ $w(\zeta) = \sqrt{|\alpha|^2 + v^2}$
 $G(w(\zeta)) = [v, \infty)$ $G_p(w) = \emptyset$

- $G(dP(w)) = \{0\} \cup [v, \infty)$
- $G_p(dP(w)) = \{0\}$ ↗

van Hove model $G(H_f + g\phi(f))$?

Lemma 6.2

$$e^{i\pi(f/w^2)} dP(w) e^{-i\pi(f/w^2)} \\ = dP + \phi(f) + \frac{1}{2} \|f(w)\|^2$$

∴ Roughly $\approx \frac{i}{h} \int B P d\{$

$$\pi(f/w^2) = \frac{i}{h} \left[a^*(\tilde{f} w^{3/2}) - a(\hat{f} w^{3/2}) \right]$$

$$[H_0, \pi(f)] = i \phi(w^2 f)$$

$$[H_0, \pi(f/\omega^2)] = i\phi(f)$$

$$dP = H_0 \quad H_0 \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \pi(f/\omega)^n$$

$$\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \underbrace{H_0 \pi(f/\omega^2)^n}_{\pi^j [H_0 \pi] \pi^{n-j-1}} + \underline{\pi^n H_0}$$

$$\sum_{n=1}^{\infty} \pi^j i \phi(f) \pi^{n-j-1} + \underline{\pi^n H_0}$$

$$\sum_{n=1}^{\infty} \underline{i \pi^j \pi^{n-j-2} : (f, f/\omega)} + \underline{\pi^{n-1} i \phi(f)}$$

$$= H_0 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} i \pi^{n-1} \phi(f)$$

$$+ \sum_{n=2}^{\infty} \frac{(-i)^n}{n!} i \pi^{n-2} (-i)$$

$$= H_0 + (-i)i \sum_{n=1}^{\infty} \frac{(-i)^{n-1} n}{(n-1)!} \pi^{n-1} \phi(f)$$

$$+ \sum_{n=2}^{\infty} \frac{(-i)^n}{n!} (-i)n \pi^{n-2}$$

2015/6/2

§ #6 moment

$$(\Omega, e^{\alpha \phi(f)} \Omega) \text{ is } \mathbb{E}^{\frac{\alpha^2}{2} \|f\|^2} \quad \underline{\alpha \in \mathbb{Q}}$$

~~Then the~~
Theorem 5.1 $(\Omega, e^{\alpha \phi(f)} \Omega) = e^{\frac{\alpha^2}{2} \|f\|^2}$

準備 Wick 積

$$: a^\#(f_1) \dots a^\#(f_n) : \\ \leftarrow a^+ \quad a \rightarrow$$

Lemma 5.2 $\langle \phi(f_1) \dots \phi(f_n) \rangle$:

$$= \phi(f) : \phi(f_1) \dots \phi(f_n) : -\frac{1}{2} \sum_{j=1}^n (f_j f_j) : \phi(f_1) \dots \hat{\phi}(f_j) \dots \phi(f_n) :$$

由 2.3 \Leftrightarrow 同上

Lemma 5.3 $:\phi(f)^n: = \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!} \phi(f)^{n-2k} (-\frac{1}{2} \|f\|^2)^k \left(\frac{1}{2}\right)^k$

$\Leftrightarrow n=1 \text{ or } n \geq 2 \text{ is ok}$

WIP

$$\sum_{k=1}^m \frac{(2m+2)!}{k! (2m-2k+2)!} A^k \phi^{2m+2-2k}$$

① $k=0 \rightarrow \phi^{2m+2}$

② $k=m+1 \rightarrow \frac{(2m+1)!}{m!} A^{m+1} 2$

$$\cancel{\frac{k=k}{k=(m+1)}} \frac{(2m+2)!}{(2m+2-2k)!} = \frac{2(m+1) \cdot (2m+1)!}{(2m+1-m)! m!}$$

$$= \sum_{k=0}^{m+1} \frac{(2m+2)!}{k! (2m+2-2k)!} \phi^{2m+2-2k} A^k$$

273!

$$\sum_{n=0}^{\infty} \frac{\phi(f)^n}{n!} \alpha^n = : e^{\phi(f)}: \quad \text{273} = 273.$$

$$\Rightarrow \sum_{n=0}^N \frac{\alpha^n}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k! (n-2k)!} A^k \phi^{n-2k}.$$

$$= \sum_{k=0}^{\lfloor N/2 \rfloor} \sum_{n=2k}^N \frac{A^k}{k!} \alpha^k \frac{\alpha^{n-2k}}{(n-2k)!} \phi^{n-2k}$$

$$= \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{A^k}{k!} \alpha^k \sum_{n=2k}^N \frac{\alpha^{n-2k}}{(n-2k)!} \phi^{n-2k}$$

$$e^{-\frac{1}{4}\|f\|^2} e^{\phi}$$

$$2m+1 = N$$

$$\begin{aligned} \phi(t) &:= \phi(t) : \phi(t) : -\frac{1}{2} \|f\|^2 N : \phi(t)^{N-1} : \\ &= \phi \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{N!}{k! (N-2k)!} \phi \frac{(-\frac{1}{2} \|f\|^2)^k (\frac{1}{2})^k}{(N-1)!} \\ &\quad - \frac{1}{2} \|f\|^2 N \sum_{k=0}^{\lfloor N-1/2 \rfloor} \frac{(N-1)!}{k! (N-1-2k)!} \phi \frac{(-\frac{1}{2} \|f\|^2)^k (\frac{1}{2})^k}{(N-1-2k)!} \end{aligned}$$

$$i) N = 2m+1 \text{ or } 2$$

$$\phi \sum_{k=0}^m \frac{(2m+1)!}{k! (2m+1-2k)!} \phi^{2m+1-2k} A^k$$

$$-\frac{1}{2} \|f\|^2 \cancel{\sum_{k=0}^m} \frac{(2m)!}{k! (2m-2k)!} \phi^{2m-2k} A^k$$

$$= \sum_{k=0}^m \frac{(2m+1)!}{k! (2m+1-2k)!} \phi \frac{(\cancel{2m+2-2k})(\cancel{2m-2k})}{A^k}$$

$$+ \sum_{k=0}^m \frac{(2m+1)!}{k! (2m-2k)!} \cancel{\frac{2m+2-2k}{A^{k+1}}} \cdot 2 \phi^{2m+2-2k-2}$$

$$\sum_{k=1}^{m+1} \frac{(2m+1)!}{(k-1)! (2m-2(k-1))!} \phi^{2m+2-2k} A^k \cdot 2$$

$$\textcircled{O} \quad 1 \leq k \leq m \text{ or } 2 \quad \frac{2(2m+1)! A^k}{(k-1)! (2m-2k+2)!} + \frac{(2m+1)! A^k}{k! (2m-2k+1)!}$$

$$= \frac{(2m+1)! A^k}{(k-1)! (2m-2k+1)!} \left[\frac{2}{2m-2k+2} + \frac{1}{k} \right]$$

$$= \frac{(2m+2)! A^k}{k! (2m-2k+2)!} \quad 2k+2m-2k+2 = 2m+2$$

\therefore of Thm 5.1

$$(\Omega, e^{\alpha\phi} \omega) = (\Omega, : e^{\alpha\phi} : e^{-\alpha^2 \left(-\frac{1}{4} \|f\|^2\right)} \omega)$$
$$= e^{\frac{\alpha^2}{4} \|f\|^2} (\Omega, : e^{\alpha\phi} : \omega) = e^{\frac{\alpha^2}{4} \|f\|^2},$$

∴

$$(\Omega, e^{i\phi(t)} \omega) = e^{-\frac{1}{4} \|f\|^2} \quad t=2, 3.$$

§ 7 \mathbb{Q} -space and Schrödinger representation

Def 7.1 $(\mathbb{Q}, \Sigma, \mu)$ prob space

Σ : real vector space

$\{\phi(f) : f \in \Sigma\}$ is a family of Gaussian random variable indexed by $\Sigma \Leftrightarrow$

① $\phi(f)$ is Gamma on $(\mathbb{Q}, \Sigma, \mu)$ w.t.

$$\int_{\mathbb{Q}} \phi(f) d\mu = 0 \quad \int \phi(t_1) \phi(t_2) d\mu = \frac{1}{2} (t_1 t_2)_\Sigma$$

i.e. $\int e^{i\phi(f)} d\mu = e^{-\frac{1}{4}\|f\|^2}$

② $f \mapsto \phi(f)$ IR-linear

③ Σ is the minimal σ -field generated by $\{\phi(t_i) : i \in I\}$

Minlos Thm 7.2 — $\Sigma = \sigma(\phi(t_i))$

Lemma 7.13 $\mathcal{S}_{\mathbb{Q}} = \{F(\phi(t_1) \dots \phi(t_n)) : F \in \mathcal{S}(L^2)\}$ 6/9

① $\mathcal{S}_{\mathbb{Q}}$ is dense $\Leftrightarrow \Sigma = \sigma(\phi(f) : f \in \Sigma)$
in $L^2(\mathbb{Q})$

② ① \Rightarrow ② $F : \Sigma$ -measurable

$\mathcal{S}_{\mathbb{Q}}$ is Σ -measurable

$\sigma(\mathcal{S}_{\mathbb{Q}}) \subset \Sigma$ $\text{4=3 fm dense in } L^2 \text{ or } \mathcal{S}$

$\exists f_n \in \mathcal{S}_{\mathbb{Q}}$ s.t. $f_n \rightarrow \phi(f)$ in $L^2 \Rightarrow f_n \rightarrow \phi(f)$ a.e.
 $\therefore \phi(f) \in \sigma(\mathcal{S}_{\mathbb{Q}})$

$$G(\phi(t); t \in \mathcal{E}) = B \subset \Sigma \text{ if } \sigma_K$$

$\Sigma \ni A \Leftrightarrow \exists \varepsilon \subset \mathbb{1}_A \in L^2 : \exists f_n \in \mathcal{F}_Q \text{ s.t.}$
 $f_n \rightarrow \mathbb{1}_A \text{ a.e., i.e., } A \in G(\mathcal{F}_Q)$

$$\therefore \Sigma \subset G(\mathcal{F}_Q)$$

$\mathcal{F}_Q \ni F(\phi(t_1), \dots, \phi(t_n))$ ist B -meas.

$$\therefore G(\mathcal{F}_Q) \subset B$$

$$\therefore \Sigma \subset G(\mathcal{F}_Q) \subset B \quad \therefore \Sigma = B.$$

② \Rightarrow ① ist operator algebra \Rightarrow ~~axiom~~ v.

Wich 積

$$:\phi(f) := \phi(g)$$

$$:\rho(f) \prod_j^n \phi(f_j) := \phi(f) \prod_j^n \phi(f_j) - \frac{1}{2} \sum_j (f_j) \prod_{j \neq j}^n \phi(f_j) :=$$

$$= \alpha \in \mathbb{Z} \quad \left(:\prod_j^n \phi(f_j) := \prod_j^n \rho(g_j) := \right) =$$

$$\delta_{nm} \sum_{\pi \in P_n} 2^{-n} (f_1, g_{\pi(1)}) \cdots (f_m, g_{\pi(m)})$$

$$L^2(Q) = \bigoplus_{n=0}^{\infty} L_n^2(Q)$$

$$\mathcal{E}_Q = \{ \{f, g\}; f, g \in \mathcal{E} \}$$

$$\lambda \{f, g\} = \{\lambda f, \lambda g\}$$

$$\lambda \{f, g\} = \{-\overline{g}, \lambda f\}$$

$$\{f, g\} + \{f', g'\} = \{f+f', g+g'\}$$

$$(\{f, g\}, \{f', g'\})$$

$$\overline{\{f, g\}} = \{f, -g\}$$

$$= (f f') + (g g') + i((f g') - (g f'))$$

$(\mathcal{E}_Q, (\cdot, \cdot))$ Hilbert space / Q.

$$\Theta_w : (\mathcal{E}_Q) \rightarrow L^2(Q)$$

$$1) \Theta_w \Omega = 1 \quad 2) \Theta_w f_n = L_n^2(Q)$$

$$3) \Theta_w \Phi(f) \Theta_w^{-1} = \Phi(f) \quad f \in \mathcal{E} \quad \text{def: } f \in \mathcal{E}_Q \text{ ist } \mathbb{R}\text{-linear}$$

$$\Phi(f) = \frac{1}{\sqrt{2}} (a^*(f) + a(f)) \quad f \text{ ist } \mathbb{R}\text{-linear}$$

$$(a(\bar{f}))^* = a^*(f)$$

$$:\Phi(f):\rightarrow :\phi(f): \quad \text{この対応を } \tilde{\Phi} \text{ とする.}$$

この問題

$$\textcircled{1} \quad : \prod_{f=1}^m \Phi(f) : \Omega \rightarrow : \prod_{f=1}^m \varphi(f) : \quad f_j \in \mathcal{E}$$

$\Omega \rightarrow 1$ eu = maps in \mathbb{Z}
2nd column

2nd quantization $T: \mathcal{E} \rightarrow \mathcal{E}$ contraction
 $T: \mathcal{E}_c \rightarrow \mathcal{E}_c$ contraction

$$\begin{cases} \theta_w^\Gamma(T) \tilde{\theta}_w^\Gamma : \prod_f \varphi(f) : = : \prod_{f=1}^m \varphi(Tf) : \\ \theta_w^\Gamma(T) \tilde{\theta}_w^\Gamma 1 = 1 \end{cases}$$

$$\theta_w^\Gamma T(\Gamma) \theta_w^{-1} = T(\Gamma) \quad \text{and} \quad$$

同様に $d\Gamma(h) \neq \det T$.

Prop Let T be contraction on \mathcal{E} .
 Then $\Gamma(T)$ is pp.

$$\begin{aligned} \textcircled{2} \quad \Gamma(T) e^{\alpha \varphi(T)} &= \Gamma(T) : e^{\frac{\alpha}{4} \alpha^2 \|T\|^2} : \\ &= : e^{\alpha \varphi(Tf)} : e^{\frac{1}{4} \alpha^2 \|f\|^2} \\ &= e^{\alpha \varphi(Tf)} e^{\frac{1}{4} \alpha^2 (f, (I - T^* T)f)} \end{aligned}$$

$$\begin{aligned} \Gamma(T) F(\phi_0, \dots, \phi_m) &= (2\pi)^{-\frac{m}{2}} \int dk \hat{F}(k) \frac{e^{i \sum_k \phi(Tf)}}{e^{\frac{1}{4} \sum_k (f, (I - T^* T)f)}} \\ &\quad (f_i, (I - T^* T)f_i) \text{ positive semi definite} \end{aligned}$$

$$= (2\pi)^{\frac{m}{2}} \left(F * D_T \right)^{(P(Tf_1), \dots, P(Tf_m))} \geq 0 \quad (F \geq 0)$$

$$D_T(n) = -\frac{1}{4} (x, Ax)$$

$$A_{ij} = (f_i, (1 - T^* T) f_j)$$

$$\Theta : \Pi \bar{\Phi}(f_j) : \Omega = : \Pi \phi(f_j) : \quad f_j \in \mathcal{E}$$

$$\Theta : \mathbb{B}(\mathcal{E}_0) \rightarrow L^2(Q) \quad \text{unitary equivalence}$$

$$P(T) : \Pi \bar{\Phi}(f_j) : \Omega = : \Pi \bar{\Phi}(Tf_j) : \Omega$$

$$\Theta P(T) \Theta^{-1} : L^2(Q) \text{ op.}$$

$\Phi = T : \mathcal{E} \rightarrow \mathcal{E}$ は

$$\Theta P(T) \Theta^{-1} : \Pi \bar{\Phi}(f_j) : = : \Pi \bar{\Phi}(Tf_j) : //$$

具体例 13) $F : L^2 \rightarrow L^2$ Fouriertrans.

$$P(F) \bar{\Phi}(t) P(F^{-1}) = \bar{\Phi}(\hat{t}) \quad 1=23.$$

$$\stackrel{\text{より}}{\Theta} \bar{\Phi}(t) \Theta^{-1} = \phi(t) \quad f \in L^2_{\mathbb{R}}$$

$$\therefore \bar{\Phi}(\hat{t}) \cong \phi(t) \quad f \in L^2_{\mathbb{R}}$$

$$\frac{d}{dt} \mathbb{E}[\Phi] = \bar{\Phi} = 1 \quad a.s.$$

$$\begin{aligned} (\mathbf{1}, e^{-tH} \mathbf{1}) &= e^{\frac{1}{4} \left\| \int_0^t \hat{S}_s h \, ds \right\|^2} \\ &= e^{\frac{1}{4} \int_0^t \int_s^t \int -k-s u |h(u)|^2 \, dk \, ds} \end{aligned}$$

$$② \quad e^{-tH_p} \text{ is } \mathcal{F}_{t-} \text{-adapted}$$

$\rho = (([0, \infty) : \mathbb{H}^d), \mathcal{W}$ wiener meas

$$(\beta_t)_{t \geq 0} \sim \mathcal{N}(0, I_d)$$

$= a.s.$

$$(f, e^{-tH_p} g) = \int_{\mathbb{R}^d} dx \mathbb{E}_w^x \left[f(\tilde{B}_0^x) g\left(\frac{\beta_x}{t}\right) \right]$$

$$\text{Lemma 8.4} \quad (f, e^{-tH_p} g) = \int_{\mathbb{R}^d} dx \mathbb{E}_w^x \left[f(\tilde{B}_0^x) g\left(\frac{\beta_x}{t}\right) e^{-\int_0^t V(s) \, ds} \right]$$

③ Trotter product $\approx \mathbb{E}[e^{-tH_p}] = \mathbb{E}[e^{-tH_0} e^{-tV}]$

$$\begin{aligned} &\lim_n (f, (e^{-\frac{t}{n} H_0} e^{-\frac{t}{n} V})^n g) \\ &= \lim_n \int_{\mathbb{R}^d} dx \mathbb{E}^x \left[f(\tilde{B}_0^x) g\left(\frac{\beta_x}{t}\right) - e^{-\sum_{j=1}^n \frac{t}{n} V(B_{\frac{j-1}{n}})} \right], \end{aligned}$$

Nelson Model

$$H = H_p \otimes 1 + 1 \otimes H_f + g \phi(\tilde{\Phi}(-x)) \quad \text{or}$$

経路積分表示で表す

Lemma 8.5

$$g=0 \text{ のとき } L^2(\mathbb{R}^d) \otimes L^2(Q) \cong L^2(\mathbb{R}^d; L^2(Q))$$

$$(F, \bar{e}^{-TH_0} G) = \int dx E_W^x \left[\left(J_0 F(B_0), \bar{e}^{\int_0^t V} J_t G(B_t) \right) \right]$$

$$\because \bar{e}^{-TH_0} = \bar{e}^{-TH_p} \otimes \bar{e}^{-TH_f}$$

$$(F, \bar{e}^{-TH_p} \otimes \bar{e}^{-TH_f} G) = (J_0 F, (\bar{e}^{-TH_p} \otimes 1) J_t G)$$

$$(\bar{e}^{-TH_p} \otimes 1)(1 \otimes \bar{e}^{-TH_f})$$

$$F = f_1 \otimes \bar{\Phi}_1, \quad G = f_2 \otimes \bar{\Phi}_2 \quad \text{とする。}$$

となる

$$(f_1, \bar{e}^{-TH_p} f_2) \cdot (\bar{\Phi}_1, \bar{e}^{-TH_f} \bar{\Phi}_2)$$

$$= \int E^x \left[f_1(B_0) f_2(B_t) \bar{e}^{\int_0^t V} \right] \cdot (\bar{\Phi}_1, J_t \bar{\Phi}_2)$$

$$= \int E^x \left[(J_0 \bar{\Phi}_1 f_1(B_0), J_t \bar{\Phi}_2 f_2(B_t)) \bar{e}^{\int_0^t V} \right]$$

$$(1 \otimes J_t F)(x) = (J_t F)(x)$$

$$\begin{aligned}
& \int dx \mathbb{E}_w^0 \left[(J_0 \tilde{F}(x), J_t \tilde{G}^m(\beta_x)) \bar{e}^{-\int_V} \right] \\
& - \int dm \mathbb{E}_w^0 \left[(J_0 F(x), J_t G^m(\beta_{t+x})) \bar{e}^{-\int_V} \right] \\
& = \int dm \mathbb{E}_w^0 \left[(J_0 (F^m(\beta_x) - F(x)), J_t G^m(\beta_{t+x})) \bar{e}^{-\int_V} \right] \\
& \leq \int dm \mathbb{E}_w^0 \left[\| F^m(\beta_x) - F(x) \| . \| G^m(\beta_{t+x}) \| \bar{e}^{-\int_V} \right] \\
& \leq \int dm \| F^m - F \| \| G^m \| \mathbb{E}_w^0 \left[\| G^m(\beta_{t+x}) \| \bar{e}^{-\int_V} \right] \\
& \leq \| F^m - F \| \| G^m \|
\end{aligned}$$

Thm 8.6 $(F, \bar{e}^{T H_g} G) = \int dx \mathbb{E}_w^x \left[(J_0 F(\beta_0), e^{-\int_0^T \phi_E(i_s \tilde{\Phi}) ds} \bar{e}^{-\int_V} J_t G(\beta_t)) \right]$

\checkmark Lemma 8.7

$$\begin{aligned}
& (F, \bar{e}^{-(t_1-t_0)H_0} e^\phi \bar{e}^{-(t_2-t_1)H_0} \dots \bar{e}^{-(t_n-t_{n-1})H_0} G) \\
& = \int dm \mathbb{E}_w^x \left[(J_0 F, \prod_{g=1}^m e^{\phi \left(\frac{j}{g} \tilde{\Phi}(-\beta_g) \right)} J_t G) \right] \bar{e}^{-\int_V}
\end{aligned}$$

$$\begin{aligned}
& \left(F_1, \bar{e}^{-tH_0} e^{\phi(\tilde{\varphi}(t-x))} \bar{e}^{-sH_0} G \right) \\
&= \lim_n \lim_m \left(F_1, \left(\bar{e}^{-\frac{t}{n}H_P} \bar{e}^{-\frac{t}{n}H_F} \right)^n e^{\phi(\tilde{\varphi}(-x))} \left(\bar{e}^{-\frac{s}{m}H_P} \bar{e}^{-\frac{s}{m}H_F} \right)^m G \right) \\
&= \lim_{n,m} \left(J_0 F_1, \underbrace{\left(J_t \bar{e}^{\frac{t}{n}H_P} J_t^* \right) \dots \left(J_t \bar{e}^{\frac{t}{n}H_P} J_t^* \right)}_{J_t} \underbrace{e^{\phi(\tilde{\varphi}(-x))}}_{J_{t+s} J_{t+s}^*} G \right) \\
&\quad \cdots J_{t+s} G
\end{aligned}$$

$$\begin{aligned}
&= \left(J_0 F_1, \bar{e}^{-tH_P} e^{\phi_E(j_t \tilde{\varphi}(-x))} \bar{e}^{-sH_P} J_{t+s} G \right) \\
&= \lim_{n,m} \left(J_0 F_1, \left(\bar{e}^{-\frac{t}{n}H_0} \bar{e}^{-\frac{t}{n}V} \right)^n e^{\phi_E(\dots)} \left(\bar{e}^{-\frac{s}{m}H_0} \bar{e}^{-\frac{s}{m}V} \right)^m G \right) \\
&= \int d\alpha \mathbb{E}^* \left[(J_0 F(B_j), e^{-\bar{z} \frac{t}{n} V(B \frac{j}{n})} \hat{e}^{\phi_E(\tilde{\varphi}(t-B_j))} J_{t+s} G) \right] \\
&\rightarrow \int d\alpha \mathbb{E}^* \left[(J_0 F(B_j), \bar{e}^{\int_0^t V} \bar{e}^{\int_t^{t+s} V} e^{\phi_E(j_t \tilde{\varphi}(-B_j))} J_{t+s} G(B \frac{j}{t+s})) \right]
\end{aligned}$$

$\vdash \exists \text{ 極限存在 } \text{ Lemma E 3.}$

∴ Thm 8.6 の

Lemma 8.7 & Trotter product $\hookrightarrow \mathbb{E}^t$

$$\begin{aligned}
 & \lim_n \left(F_i \left(e^{-\frac{t}{n} H_0} e^{-\frac{t}{n} g \phi(\hat{\varphi}(1 \cdots n))} \right)^n G \right) \\
 &= \lim_n \int d\omega \mathbb{E}^x \left[\left(J_0 F(B_s), e^{-\sum_{s=1}^n \frac{t}{n} g \phi_E(j \frac{s}{n}, \hat{\varphi}(1 \cdots B_{\frac{s}{n}}))} J_t G(B_{\frac{t}{n}}) \right) \right] \\
 &= \int d\omega \mathbb{E}^x \left[\left(J_0 F(B_s), e^{-\int_0^t g \phi_E(j_s, \hat{\varphi}(1 \cdots B_s)) ds} J_t G(B_t) \right) \right] \\
 &\quad \times - \int_0^t v ds
 \end{aligned}$$

Cor 8.8 $F = G = f \otimes 1 \otimes \bar{1}$

$$(f \otimes 1, \bar{e}^T H f \otimes 1) = \int dk \mathbb{E}_W^x \left[f(B_0) f(B_T) e^{-\int_0^T v ds} + \frac{g^2}{2} \int_0^T \int_0^T W \right]$$

$$:= \mathbb{E}^x W = \int \frac{|\hat{\varphi}|}{2\pi} e^{-(t-s)W} e^{-ik \cdot (B_t - B_s)} dk.$$

§5 真空期待値と基底状態能

$$\bar{e}^{TH} f @ 1 = \varphi_g^T \quad \gamma(T) = (\varphi_1, \varphi_g^T)^T \\ = (f @ 1, \bar{e}^{TH} f @ 1)^T / (f @ 1, \bar{e}^{-2TH} f @ 1)$$

Lemma 9.1 $\lim_{T \rightarrow \infty} \gamma(T) = a = \begin{cases} > 0 & \exists \text{ g.s.} \\ = 0 & \nexists \text{ g.s.} \end{cases}$

$$\therefore \gamma(T) = \left(\int_0^\infty \bar{e}^{T\lambda} dE \right)^2 / \int_0^\infty \bar{e}^{-2T\lambda} dE$$

$$\textcircled{1} \quad \exists \varphi_g \in \mathcal{S} \text{ s.t. } E(\{\varphi\}) > 0$$

$$\therefore \int_0^\infty \bar{e}^{-T\lambda} dE \rightarrow E(\{\varphi\})$$

$$\therefore E(\{\varphi\})^2 / E(\{\varphi\}) = E(\{\varphi\}) > 0$$

$$\therefore \text{if } \exists \varphi \text{ s.t. } E(\{\varphi\}) > 0 \quad \therefore a = 0 \Rightarrow \nexists \text{ g.s.}$$

$$\textcircled{2} \quad \begin{aligned} \sqrt{\gamma(T)} &= \sqrt{\frac{\int_0^\delta + \int_\delta^\infty}{\left(\int_0^\infty \bar{e}^{-2T\lambda} dE \right)^{1/2}}} \leq \frac{\left(\int_0^\delta \bar{e}^{-2T\lambda} dE \right)^{1/2} + \bar{e}^{-\delta T}}{\left(\int_0^\infty \bar{e}^{-2T\lambda} dE \right)^{1/2}} \rightarrow E([0, \delta])^{1/2} \\ &\therefore a \leq E([0, \delta])^{1/2} \quad \therefore a \leq E(\{\varphi\}) \quad , \end{aligned}$$

Perron-Frobenius の 定理

(M, μ) σ -finite meas sp.

K s. a. on $L^2(M, d\mu)$

① $\inf_{g \in \mathbb{C}} \text{spec}(K) = \lambda$ e.v.

② \bar{e}^K is positivity improving
 $\Leftrightarrow (f, \bar{e}^K g) > 0 \quad \forall f, g \geq 0$.

Then g is ≥ 0 if g is ≥ 0 .

$\therefore \lambda = \inf \text{spec}(K) \quad \| \bar{e}^K \| = \bar{e}^\lambda$

\bar{e}^K : real-valued ft & real-valued ft
 $f = f_+ - f_-$.

f is also an eigenvector

$f = f_+ + f_-$

$$(g, \bar{e}^K f) = (g, \bar{e}^K f_+ - \bar{e}^K f_-)$$

$$\leq (g, \bar{e}^K f_+ + \bar{e}^K f_-) = (g, \bar{e}^K |f|) \quad g \geq 0$$

$$\therefore \bar{e}^\lambda \|f\|^2 = (\bar{e}^K f, f) \leq (\bar{e}^K |f|, |f|)$$

$$\leq \|\bar{e}^K\| \|f\|^2 = \bar{e}^\lambda \|f\|^2$$

$$\therefore (\bar{e}^K f, f) = (\bar{e}^K |f|, |f|)$$

$$\therefore (\bar{e}^K f_+, f_-) = -(\bar{e}^K f_-, f_+)$$

Theorem 9.2 (H. Spohn) $\text{In} \infty \approx \pi \cdot \pi$

$$\sum_p -E_p > \int \frac{|\hat{\psi}|^2}{2\omega^2} \frac{(\epsilon_1)^2}{2\omega + (\epsilon_1)^2} d\epsilon$$

$\alpha \approx 3$ ground states 131253 .

$$\varphi_g^T = \bar{e}^{TH} (f \otimes 1) / \| \bar{e}^{TH} f \otimes 1 \|$$

Lemmer 9.3

$$\left(\psi_g^T, e^{i\beta \phi(f)} \psi_g^T \right) = \frac{1}{Z_f} \int dx E_W^x \left[f(\beta_{-T}) f(\beta_T) e^{-\int_{-T}^T V} e^{-\phi_E^x \left(\begin{matrix} T \\ -T \end{matrix} \right)} e^{i\beta \phi_E^x(f, t)} \right]$$

- (注) $BM(B_e)$ は $t \in \mathbb{R}$ のとき $T^{\frac{1}{2}}(T^{\frac{1}{2}}B_e)$

$$\int d\lambda \times E_w^x \left[\prod_{n=1}^m f_{\bar{\lambda}}(B_{t_n}) \right] = \int d\lambda \times E_w^x \left[\prod_{n=1}^m f_{\bar{\lambda}}(B_{t_n^{+1}}) \right]$$

Lemur 9.3 2"

$$\text{分子} \quad e^{\frac{1}{2} \int_{-T}^T \int_{-T}^T W(\beta_t - \beta_s) + s}$$

$$(\Omega, e^{-\phi_E(f)}) \quad e^{i\beta \phi_E(j_0 f)} \quad Z\phi_E(f)$$

$$\rightarrow (\Omega, e^{\phi_E(-\int_{-T}^T + i\beta j_0 f)} \Omega) \quad \rightarrow \mathbb{Z}^2$$

$$= e^{\frac{1}{2} (-\int_{-T}^T + i\beta j_0 f, -\int_{-T}^T + i\beta j_0 f)} \leftarrow \text{内積計算} \\ (\cdot, \cdot) = (\cdot, \cdot)$$

$$= e^{\frac{1}{4} \left(\| \int_{-T}^T \| f \| \right)^2 - \beta^2 \| j_0 f \|^2 - 2i\beta \left(\int_{-T}^T, j_0 f \right)}$$

$$\downarrow \quad \downarrow \quad \downarrow \\ \int_{-T}^T ds \int_{-T}^T dt W \quad \beta^2 \| f \|^2 - 2i\beta \int_{-T}^T dt \left(\hat{\Phi}_{\omega}^{-ik\beta t} e^{i\beta t} \hat{f} \right)$$

$$\text{分子} \quad \mathbb{E}_{\mu_T} \left[e^{-\frac{\beta^2}{4} \| f \|^2} e^{-2i\beta \int_{-T}^T dt (\dots)} \right]$$

$$\therefore (\hat{\Phi}_g^T, e^{i\beta \phi_g(f)} \hat{\Phi}_g^T) = e^{-\frac{\beta^2}{4} \| f \|^2} \mathbb{E}_{\mu_T} \left[e^{i\beta K(f)} \right]$$

where

$$K(f) = -\frac{1}{2} \int_{-\infty}^{\infty} \left(\bar{e}^{ir\omega} \bar{e}^{-ik\beta_s} \hat{\Phi}(\sqrt{\omega}, \hat{f}) dr \right)$$

$$(2\pi)^{-\frac{1}{2}} \int e^{-\frac{|k|^2}{2}} e^{ikx} dk = e^{-\frac{|x|^2}{2}} \quad \text{by Hint.}$$

$$\begin{aligned} (\varphi_g^T, e^{-(\beta^2/2)\Phi(f)^2} \varphi_g^T) &= \int (\varphi_g, e^{-ik\phi(\beta)} \varphi_g) e^{-\frac{|k|^2}{2}(2\pi)^{-\frac{1}{2}}} dk \\ &= \mathbb{E}_{\mu_T} \left[\left(e^{-ikF_f \beta} e^{-\frac{|k|^2}{2}(2\pi)^{-\frac{1}{2}}} dk \right) e^{-\kappa^2 \beta^2 \|f\|^2 / 4} \right] \\ &= \int \mathbb{E}_{\mu_T} \left[e^{-ikF_f \beta} e^{-\frac{|k|^2}{2}} e^{-\frac{\beta^2 \|\beta\|^2}{4} |k|^2} \right] (2\pi)^{-\frac{1}{2}} dk \end{aligned}$$

$$-\frac{1}{2} \underbrace{\left(1 + \frac{\beta^2 \|\beta\|^2}{2} \right)}_{\{ } |k|^2 \quad \text{on Fourier transf.}$$

$$\sqrt{3}k = u \quad \text{eq 329} \quad dk = \frac{1}{\sqrt{3}} du$$

$$(2\pi)^{-\frac{1}{2}} \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{2}} u^2 e^{-iF_f \beta \cdot \frac{1}{\sqrt{3}} u} du = \frac{1}{\sqrt{3}} \bar{e}^{-\frac{1}{2} \frac{(F_f \beta)^2}{3}}$$

$$\begin{aligned} \text{7.21) } (\varphi_g^T, e^{-(\beta^2/2)\Phi(f)^2} \varphi_g^T) \\ &= \frac{1}{\sqrt{3}} \mathbb{E}_{\mu_T} \left[\bar{e}^{-\left(\frac{(F_f \beta)^2}{3} \right)} \right] \quad \text{eq 323} \end{aligned}$$

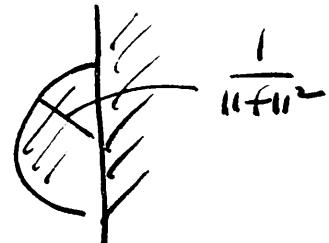
Lemma 9.4

$$(\varphi_g^T, e^{-\frac{(\beta l_2)^2}{2} \phi(f)^2} \varphi_g^T) = \frac{1}{\sqrt{3}} \mathbb{E}_{\mu_T} \left[e^{-\frac{(F_f \beta)_{l_2}^2}{3}} \right]$$

$\forall \beta = \beta l_2 \rightarrow \beta \text{ 为常数}$

$$(\varphi_g^T, e^{\beta \phi(f)^2} \varphi_g^T) = \sqrt{\frac{1}{1 + \beta \|f\|^2}} \mathbb{E}_{\mu_T} \left[e^{-\frac{F_f^2 \beta}{1 + \beta \|f\|^2}} \right]$$

解不等式成立



$$(\varphi_g^T, e^{\beta \phi(f)^2} \varphi_g^T) = \sqrt{\frac{1}{1 - \beta \|f\|^2}} \mathbb{E}_{\mu_T} \left[e^{+\frac{F_f^2 \beta}{1 - \beta \|f\|^2}} \right]$$

Cor 9.5

$$\begin{cases} \mathbb{E}_{\mu_T} \rightarrow \mathbb{E}_{\mu_m} & \text{(-矛盾)} \\ |F_f| \leq I_{L^2} \|f\| < \infty \end{cases}$$

$$\lim_{\beta \rightarrow \frac{1}{\|f\|^2}} \|e^{(\beta l_2) \phi(f)^2} \varphi_g^T\| \uparrow \infty.$$