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1. Abstract Hilbert spaces

$(\mathbb{R}, |\cdot|)$

V. linear space over $\mathbb{C}$

$\|\cdot\| : V \to \mathbb{R}$ is said to be norm if and only if

(i) $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|\alpha x\| = |\alpha| \|x\|$ $\forall \alpha \in \mathbb{C}$

(iii) $\|x+y\| \leq \|x\| + \|y\|$.

$(V, \|\cdot\|)$ normed space

$x_n \to x \Leftrightarrow \lim \|x_n - x\| = 0$

topology

$\mathcal{H}$. linear space

$(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$

(i) $(x, x) \geq 0$, $(x, x) = 0 \Leftrightarrow x = 0$

(ii) $(x, \alpha y) = \alpha(x, y)$

(iii) $(x, y+z) = (x, y) + (x, z)$

(iv) $\overline{(x, y)} = (y, x)$ $(\mathcal{H}, (\cdot, \cdot))$ inner product space

Claim $\sqrt{(x, x)} := \|x\|$ is the norm.

$(\mathcal{H}, (\cdot, \cdot))$ $\|\cdot\|$ complete $\Leftrightarrow$ Hilbert space

- Complete $\{x_n\}$ Cauchy seq $\Leftrightarrow$ convergence seq.

$x \perp y \Leftrightarrow (x, y) = 0$.

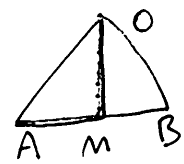
Example $y \neq 0$ $\underbrace{x - \left(\frac{y}{\|y\|}, x\right)\frac{y}{\|y\|}}_{X} + \underbrace{\left(\frac{y}{\|y\|}, x\right)\frac{y}{\|y\|}}_{Y}$

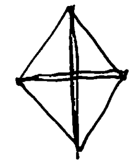
$(X, Y) = 0$

$\therefore \|X + Y\|^2 = \|X\|^2 + \|Y\|^2 \geq \|Y\|^2$

$\therefore \|x\|\|y\| \geq |(x, y)|$ This is called Schwarz ineq.

$OA^2 + OB^2 = (AM^2 + OM^2)\times 2$

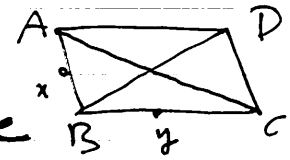




Parallelogram law

$(\mathcal{H}, (\cdot,\cdot)) \Rightarrow \|x-y\|^2 + \|x+y\|^2 = 2(\|x\|^2 + \|y\|^2)$ — (*)

& $(x,y) = \frac{1}{4}\sum_{n=1}^{4} \frac{1}{i^n}\|x+i^n y\|^2$ — (**)



Prop 1.1 $(V, \|\cdot\|)$ normed space

Suppose ~~d~~ that (*) is satisfied.

Then (**) is a norm on V.

(Orthogonal)

$(x,y) = 0 \Leftrightarrow x \perp y$

$M \subset \mathcal{H}$ subspace

$M^{\perp} = \{x \mid (x,y) = 0 \ \forall y \in M\}$

orthogonal complement of M.

Prop 1.2 $M^{\perp}$ is a closed subspace.

Prop 1.3 (the projection thm) $M \subset \mathcal{H}$ closed subsp.

Then $\mathcal{H} = M \oplus M^{\perp}$

(Base)

$\mathcal{H}$ is separable $\Leftrightarrow \exists \mathcal{D} \subset \mathcal{H}$ ① dense ② $\#\mathcal{D} = \mathbb{N}$

prop 1.4 $\mathcal{H}$ separable, D ONS $\Rightarrow \# D = \mathbb{N}$

where D ONS $\Leftrightarrow \forall \varphi, \psi \in D$ $(\varphi, \psi) = 0$

$\|\varphi\| = \|\psi\| = 1.$

Let D ons i.e. $D = \{\phi_n\}_{n\in\mathbb{N}}$

Prop 1.5 $\forall x \in \mathcal{H} \Rightarrow \|x\|^2 \geq \sum |(\phi_n, x)|^2$

M closed subspace $\subset \mathcal{H}$

$\mathcal{H} = M \oplus M^{\perp}$ $\qquad x = x_1 + x_2$

Map $x \mapsto x_1$ is denoted by P_M.

Let D ONS. $M = \langle D \rangle$ closed subsp.

Prop 1.6 $P_M x = \sum_{n=1}^{\infty} (\phi_n, x) \phi_n$ and

$(P_M x, P_M y) = \sum (x\, \phi_n)(\phi_n\, y)$

Prop 1.7 $\mathcal{H}$ separable D ONS $\langle D \rangle = M$

① - ⑤ are equivalent

① $\mathcal{H} = M$ ② $x = \sum (\phi_n\, x) \phi_n$

③ $\|x\|^2 = \sum |(\phi_n x)|^2$ ④ $(x y) = \sum (x \phi_n)(\phi_n y)$

⑤ $(x, \phi_k) = 0 \ \forall k \Rightarrow x = 0$.

Prop 1.8 $\mathcal{H}$ separable. $\mathcal{H}$ has CONS.

D is CONS ① ONS ② $M = \mathcal{H}$

$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\langle D \rangle}$

Prop 1.2

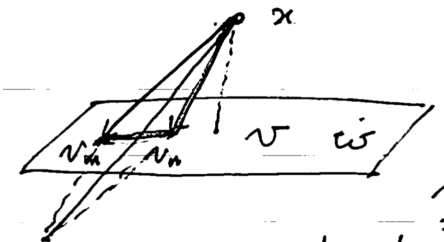
$M^\perp = \{ y \mid (x, y) = 0 \ \forall x \in M \}$

subspace → ok

$y_n \in M^\perp \mapsto y$ as $n \to \infty$.

$(x, y_n) = 0 \quad \therefore (x, y) = 0 \ \forall x \in M \quad \therefore y \in M^\perp$

Prop 1.3



$d = \inf_{v \in M} \| x - v \|$

By the def of "inf", we can see that $\exists v_m \in M$ s.t. $\| x - v_m \| \to d$ (as $m \to \infty$) ∵) $d + \varepsilon \geq \| x - \exists v_\varepsilon \| \geq d$

$\{ v_m \}$ is Cauchy

(∵) $$\| v_m - v_n \|^2 + \left\| 2 \left\{ \left(\frac{v_m + v_n}{2} \right) - x \right\} \right\|^2$$

$$= 2 \| v_m - x \|^2 + 2 \| v_n - x \|^2$$

$$\therefore \| v_m - v_n \|^2 = 2 \underset{\downarrow\, 2d^2}{\| v_m - x \|^2} + 2 \underset{\downarrow\, 2d^2}{\| v_n - x \|^2} - \left\| 2 \left\{ \left(\frac{v_m + v_n}{2} \right) - x \right\} \right\|^2$$

$$\leq 4d^2 - 4d^2 + \varepsilon \quad // \quad \forall \varepsilon$$

$\left\| \frac{v_m + v_n}{2} - x \right\|^2 \geq d^2$ に注意する. ∴ Cauchy sequence.

$\therefore \exists y = \lim_n v_n \in M$ (closed)

$w \in M$, $\forall \lambda \in \mathbb{R}$

$$d^2 \leq \| x - (y + \lambda w) \|^2 = \| (x - y) - \lambda w \|^2 = \underbrace{\| x - y \|^2}_{= d^2} - 2 \mathrm{Re}\, \lambda (x - y, w) + \lambda^2 \| w \|^2$$

$$\therefore 0 \leq -2 \mathrm{Re}\, \lambda (x - y, w) + \lambda^2 \| w \|^2 \quad \forall \lambda.$$

$$\therefore 4 | \mathrm{Re}\, (x - y, w) |^2 \leq 0. \quad \therefore$$

同様に $\mathrm{Im}\, (x - y, w) = 0$.

$\therefore x - y \perp M$

$\therefore\ x = (x-y) + y \in M^{\perp} \oplus M$

(uniqueness)

$$x = z + w \in M + M^{\perp}$$
$$= z' + w'$$

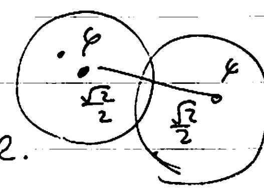
Then $(z-z') + (w-w') = 0$

$\therefore\ z - z' = w - w' \quad \therefore\ z-z' \in M \cap M^{\perp}$, $w - w' \in M \cap M^{\perp}$

$\therefore\ z = z',\ w = w'.$ //

Prop 1.4

$D \ni \varphi, \psi \Rightarrow \|\varphi - \psi\|^2 = 2$



$^{\exists} M \subset \mathcal{H}$ dense $\# M$ countable.

$\exists\ u_\varphi \in M$ s.t. $\|\varphi - u_\varphi\| \lneq \frac{1}{\sqrt{2}}$

$\therefore\ \sqrt{2} = \|\psi - \varphi\| \le \|\varphi - u_\varphi\| + \|u_\varphi - u_\psi\| + \|\psi - u_\psi\|$

$\lneq \sqrt{2} + \|u_\varphi - u_\psi\|$

$\therefore\ \|u_\varphi - u_\psi\| \gneq 0$

Then we can define the map.

$$p : D \to M$$
$$\varphi \mapsto u_\varphi.$$

$p(D) = M' \quad \therefore\ p : D \to M'$ bijective

1:1 onto.

$\therefore\ \# D \le \# M' \le \# M$ //

Thm 1.5 ☺ $D = \{\phi_n\}_{n=1}^{\infty}$ ONS

$$0 \le \left\| x - \sum_{n=1}^{N} (\phi_n, x)\phi_n \right\|^2 = \|x\|^2 - \sum^{N} |(\phi_n, x)|^2$$

$$\therefore \sum^{\infty} |(\phi_n, x)|^2 \le \|x\|^2 .$$

Prop 1.6 ☺ $\sum_{n=1}^{\infty} (\phi_n, x)\phi_n = y \in M$.

Hence we have $(x-y, \phi_m) = 0 \quad \forall m$

$\therefore$ LHD $\ni^{\forall} z \quad (x-y, z) = 0$.

$\langle M \rangle z \to {}^{\exists} z_n \in$ LHD s.t. $z_n \to z \quad \therefore (x-y, z) = 0$

$\therefore x - y \in M^{\perp}$

$\therefore x = x-y+y \in M^{\perp} \oplus M$

$\therefore P_M x = y$

$(P_M x, P_M y) = \sum (x, \phi_n)(\phi_n, y)$. of $\{\phi_n\}$

Prop 1.7 ☺ $\mathcal{H}$ separable D ONS

$\langle D \rangle = M$

① ↔ ② $\mathcal{H} = M \quad \therefore x = P_M x = \sum (\phi_n, x) x$.

① → ④ → ③ → ⑤ → ①

easy (1.6) | easy | easy | $P_M x = \sum (\phi_n, x)\phi_n$

$(x - P_M x, \phi_n) = 0 \quad \therefore x - P_M x = 0$

$\therefore x = P_M x$

$\therefore M = \mathcal{H}$.

Prop 1.8 $\mathcal{H}$ separable $\to$ $\exists$ CONS

$\exists D = \{\phi_n\} \subset \mathcal{H}$ We exclude vectors according to the following rules.
dense

$\varphi_k \in \langle \{\varphi_1, \dots, \varphi_{k-1}\} \rangle \Rightarrow \varphi_k$ is excluded

$\varphi_k \notin \langle \qquad \rangle \Rightarrow \varphi_k$ is not excluded

Remaing vectors are described by $\{\varphi_k'\}$
but $LH\{\varphi_n\} = LH\{\varphi_n'\}$

$LH\{\varphi_n'\}$ is dense. & $\{\varphi_1' \dots \varphi_n'\}$ linea indep.

Schmidt's orthogality method.

$\phi_1 = \varphi_1' / \|\varphi_1\|$

$\phi_2 = \varphi_2 - \phi_1 / \|\varphi_2 - \phi_1\|$

$\phi_3 = \varphi_3 - P_2\varphi_3 / \|\varphi_3 - P_2\varphi_3\|$

$\vdots$

$\phi_n = \varphi_n - P_{n-1}\varphi_n / \|\varphi_n - P_{n-1}\varphi_n\|$

$P_n = \langle \phi_1, \dots, \phi_n \rangle$

$\underline{\underline{n \le m}}$

Example $(\phi_n, \phi_m) = (\varphi_n - P_{n-1}\varphi_n, \varphi_m - P_{m-1}\varphi_m)$

$= (\varphi_n, \varphi_m) - (\varphi_n, P_{m-1}\varphi_m) - (\varphi_n, P_{n-1}\varphi_m)$

$+ (P_{n-1}\varphi_n, P_{m-1}\varphi_m)$

$= (\varphi_n \varphi_m) - (\varphi_n \varphi_m) - (\varphi_n \; P_{n-1}\varphi_m) + (\varphi_n \; P_{n-1}\varphi_m)$

$= 0$

Examples of Hilbert spaces.

① $(\mathbb{R}^n, (\,,\,))$, $(\mathbb{C}^n, (\,,\,))$

note that $\mathbb{C}$, $\mathbb{R}$ are complete

② $\ell^2 = \{ (a_n) \mid \sum_{n=1}^{\infty} |a_n|^2 < \infty \quad a_n \in \mathbb{C} \}$

$((a_n), (b_n))_{\ell^2} = \sum_{n=1}^{\infty} \bar{a}_n b_n$

③ $\mathcal{L}^2(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{R} \mid \int |f(x)|^2 dx < \infty \}$

$\mathcal{L}^2(\mathbb{R}^d)/\sim \; = L^2(\mathbb{R}^d)$

$([f], [g]) = \int \bar{f} g$

Note that ℓ^p, $\mathcal{L}^p$ $\quad 1 \le p < \infty$ Banach space

∵ of ② $f_n = (a_j^{(n)}) \in \ell^2$ Cauchy

$|f_n - f_m| = \sum_j |a_j^n - a_j^m|^2 < \infty$ $\quad$ Cauchy → bdd

$\therefore \exists a_j$ s.t. $a_j^n \to a_j$ $\quad \therefore \sum^N |a_j^n|^2 \le M \quad \therefore \sum^{\infty} |a_j| \le M \quad \forall N$

$\therefore (a_j) \in \ell^2$

$\therefore \sum^N |a_j^n - a_j^m|^2 \le \sum^{\infty} |a_j^n - a_j^m|^2 < \varepsilon \quad \forall m$

$\therefore \sum^N |a_j^n - a_j|^2 \le \varepsilon \quad \forall N$

$\therefore \sum^{\infty} |a_j^n - a_j|^2 < \infty$

∵ of ③ omit.

separability of L^2 and ℓ^2

① $e_j = \cancel{(a_j)} \, (\delta_{nj}) = \{0 \cdots \underset{\substack{\uparrow \\ j}}{1} \cdots\}$

$\{e_j\} = D$ is CONS

② $H_n(x) e^{-x^2/2}$ $\quad H_n(x)$ Hermite polynomial

$$\left(\frac{d^2}{dx^2} - 2x\frac{d}{dx}\right) H_n(x) = -2n\, H_n(x)$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$\left(H_n(x) e^{-x^2/2} . \ H_m(x) e^{-x^2/2}\right) = \delta_{mn}\, 2^n \sqrt{\pi}\, n!$$

$$2^{-\frac{n}{2}} \pi^{-\frac{1}{4}} \frac{1}{\sqrt{n!}} H_n(x)\, e^{-\frac{x^2}{2}} = \varphi_n(x) \in L^2$$

$\{\varphi_n\} = D$ CONS.

I.e. $\forall f \in L^2$

$$f = \sum_{n=0}^{\infty} (\varphi_n, f)\, \varphi_n$$

$(\varphi_n, f) = 0 \quad \forall n$

$$\int H_n(x)\, e^{-x^2} f(x)\, dx = 0$$

$$\int e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \cdot f(x)\, dx = 0$$

$$= \int \frac{d^n}{dx^n} e^{-x^2} \cdot e^{x^2} f(x)\, dx = \int e^{-x^2} \cdot \boxed{\frac{d^n}{dx^n} e^{x^2} f(x)} = 0$$

2 Bounded operators

$\mathcal{H}$, K Hilbert spaces.
$T: \mathcal{H} \to K$ linear operator
$D(T) \subset \mathcal{H}$ domain of T

T is continuous $\Leftrightarrow$ $\overset{\in D(T)}{x_n} \to x \to Tx_n \to Tx$.
T is bdd $\Leftrightarrow$ $\exists c$ st $\|Tx\| \le c\|x\|$ $\forall x \in D(T)$

<u>Prop 2.1</u> T is cont $\Leftrightarrow$ T is bdd.

$$\|Tx\| \le c\|x\| \quad \forall x \in D(T)$$

$$\sup_{\substack{x \neq 0 \\ x \in D(T)}} \frac{\|T(x)\|}{\|x\|} = \|T\| \quad \text{Hence } \|Tx\| \le \|T\| \, \|x\|$$

$\|T\|$ is also called the norm of T

<u>Prop 2.2</u> T is bdd, and $D(T) \subset \mathcal{H}$ dense

Then $\exists_1 \overline{T}: \mathcal{H} \to K$ st
① $\overline{T}$ is bdd
② $D(\overline{T}) = \mathcal{H}$
③ $T \subset \overline{T}$
④ $\|T\| = \|\overline{T}\|$

In what follows we assume $D(T) = \mathcal{H}$ for bdd op T.

$B(\mathcal{H}, K) = \{T: \mathcal{H} \to K\}$, $B(\mathcal{H}, \mathcal{H}) = B(\mathcal{H})$

<u>Prop 2.3</u> $(B(\mathcal{H}, K), \|\cdot\|)$ complete.

suppose that $\|Tf_n\| \geq n\|f_n\|$

Prop 2.1

(∵) b'dd → cont easy

cont → b'dd. Suppose that T is not b'dd.

$g_n = f_n / \|Tf_n\|$. Hence $\|Tg_n\| = 1$

but $\|g_n\| = \|f_n\| / \|Tf_n\| \leq \frac{1}{n} \to 0$.

It is a contradiction.

Prop 2.3 (⇐) $\{T_n\} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$

$\|T_n - T_m\| \to 0 \quad (n, m \to \infty)$

$\therefore \|T_n x - T_m x\| \to 0 \quad \therefore \exists \lim_n T_n x = y.$

Define the map. $T : x \mapsto y$

Then $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ ∵) $\|T_n x\| \leq \|T_n\| \, \|x\|$

$\downarrow$

$\|Tx\| \leq c\|x\|$

Prop 2.2 $T : \mathcal{H} \to \mathcal{K}$ b'dd $D(T) \subset \mathcal{H}$ dense

$\forall x \in \mathcal{H} \ \exists x_n \in D(T)$ s.t. $x_n \to x$

$\|Tx_n - Tx_m\| \leq \|T\| \, \|x_n - x_m\| \to 0.$

$\therefore \exists \lim_n Tx_n = y$ Define $\bar{T} : \mathcal{H} \to \mathcal{K}$ by

$\bar{T}x = y$ Remark that $z_n \to x$

$\|Tz_n - y\| \leq \|Tz_n - Tx_n\| + \|Tx_n - y\|$

$\leq \|T\| \, \underbrace{\|z_n - x_n\|}_{\downarrow 0} + \underbrace{\|Tx_n - y\|}_{\downarrow 0}.$

$\therefore Tz_n \to y$

Note that $\|\bar{T}x\| = \lim_n \|\bar{T}x_n\| \leq \lim \|\bar{T}\| \, \|x_n\| = \|\bar{T}\| \, \|x\|.$

$\therefore \|\bar{T}\| \leq \|T\|$

$\|\bar{T}\| = \sup_{\substack{x \neq 0 \\ \|x\| = 1}} \|\bar{T}x\| \geq \sup_{\substack{x \neq 0 \\ x \in D(T) \\ \|x\| = 1}} \|Tx\| = \|T\| \quad \therefore \|\bar{T}\| = \|T\|$

(uniqueness) $S \in B(\mathcal{H}, K)$ st

① $Sx = Tx \quad \forall x \in D(T)$

② $\|S\| = \|T\|$

$\Big) \Rightarrow S = \bar{T}$

$\because$ $Sx_n = Tx_n$

$\downarrow \qquad \downarrow$

$Sx \qquad \bar{T}x$

$\therefore \bar{T} = S$

Examples of b'dd operators

① $Tf = (\varphi, f)\varphi$ b'dd. one rank op

$Tf = \sum_{j=1}^{n} (\varphi_j, f)\varphi_j$ finite rank op.

② $\mathcal{H} = L^2(\mathbb{R}^d)$, f b'dd function

$Tg = fg$ b'dd

Young inequality

③ $\|f*g\|_{L^r} \le \|f\|_p \|g\|_q$ $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$

$r = 2 = p$ $q = 1$

$\therefore$ $Tf = f*g$ $\therefore$ $\|Tf\|_{L^2} \le \|g\|_{L^1} \|f\|_{L^2}$

integral operator

④ $\mathcal{H} = \ell^2$ gym

$T_m(a_n) = \{a_{m}, a_{m+1}, \cdots\}$ shift op

$\|T_m(a_n)\|^2 \le \|(a_n)\|^2$ b'dd.

We can see that

$\|T_m(a_n)\|^2 = \sum_{n=m}^{\infty} |a_n|^2 \to 0$ $(m \to \infty)$

$\therefore$ $T_n \to 0$ strongly

if $a_n = (0 \cdots a_m \cdots)$

$\|T_m\| = 1$ $\because)$ $\|T_m(a_n)\| = \|(a_n)\|$ $\therefore$ $\|T_m\| \ge 1$

$\therefore$ $T_m \not\to 0$ uniformly

$B(\mathcal{H}, \mathbb{C}) = \mathcal{H}^*$ the dual of $\mathcal{H}$.

$F \in \mathcal{H}^*$, $\quad |F(f)| \leq \exists C \|f\| \quad \forall f \in \mathcal{H}$

Ex. let $\varphi \in \mathcal{H}$ $\quad F(f) = (\varphi, f)$

$|F(f)| \leq \|\varphi\| \cdot \|f\| \qquad \therefore \|F\| \leq \|\varphi\|$

$|F(\varphi)| = \|\varphi\|^2 = \|\varphi\| \cdot \|\varphi\| \qquad \therefore \|F\| \geq \|\varphi\|$

Thus $\quad \|F\| = \|\varphi\| \qquad$ ok

Thm (Riesz Representation thm)
Let $F \in \mathcal{H}^*$. Then $\exists! \varphi$ s.t. $F(f) = (\varphi, f)$
and $\|F\| = \|\varphi\|$.

~~$\dim \operatorname{Ran} F = \dim \mathbb{C} = 1$~~ very small!

~~$(\operatorname{Ker} F)^\perp = (\operatorname{Ran} F)$~~

~~$\operatorname{Ker} F = \{\varphi \in \mathcal{H} \mid F(\varphi) = 0\}$~~

~~$\operatorname{Ker} F = \{\varphi \in \mathcal{H} \mid (\varphi, \varphi) = 0\}$~~

Linear algebra (finite dim case)

$A : \mathbb{R}^n \to \mathbb{R}^m$

$\dim \operatorname{Ran} A + \dim \operatorname{Ker} A = n$

(dimension theorem)

$\dim \operatorname{Ran} F = \dim \mathbb{C} = 1$

$1 + \dim \operatorname{Ker} A = \dim \mathcal{H} = \infty$

$\therefore \dim \operatorname{Ker} A = \infty$

Case 1 $\quad \mathrm{Ker}\, F = \mathcal{H}$ $\quad$ i.e. $F(f) = 0 \ \forall f$

$F(f) = (0, f)$. $\quad$ ok

Case 2 $\quad \mathrm{Ker}\, F \neq \mathcal{H}$

$(\mathrm{Ker}\, F)^{\perp} \ni \varphi$

$\mathrm{Ker}\, F$ is closed $\because$ $f_n \in \mathrm{Ker}\, F$

$F(f_n) = 0 \to F(f) = 0.$

$\mathcal{H} = \mathrm{Ker}\, F \oplus (\mathrm{Ker}\, F)^{\perp}$

$\varphi_F = c \cdot \varphi$ $\quad$ we define c later

(1) $f \in \mathrm{Ker}\, F$ $\quad F(f) = (f, c\varphi) = 0$

(2) $\cancel{f \in \mathrm{Ker}\, F^{\perp}}$ $\quad F(\alpha\varphi) = \alpha F(\varphi) = (f, c\varphi)$

$f = \alpha \varphi$

(3) $f \in \mathcal{H}$ $\quad f = \underbrace{(f - c\varphi)}_{\in\, \mathrm{Ker}\, F} \oplus \underbrace{c\varphi}_{\in\, \mathrm{Ker}\, F^{\perp}}$

$\varphi \in (\mathrm{Ker}\, F)^{\perp}$ $\quad \varphi_F = \varphi \cdot \dfrac{\overline{F(\varphi)}}{\|\varphi\|^2}$

(1) $f \in \mathrm{Ker}\, F$ のとき

$F(f) = 0 \quad (\varphi_F, f) = 0$ $\quad$ ok

(2) $f = \alpha\varphi \quad \alpha \in \mathbb{C}$ のとき

$F(\alpha\varphi) = \alpha F(\varphi) \quad (\varphi_F, \alpha f) = \alpha \|\varphi\|^2 \dfrac{F(\varphi)}{\|\varphi\|^2}$

$= \alpha F(\varphi)$

(3) $f \in \mathcal{H}$ $\quad f = \left(f - \varphi \dfrac{F(f)}{F(\varphi)}\right) \oplus \varphi \cdot \dfrac{F(f)}{F(\varphi)} \in (\mathrm{Ker}\, F)^{\perp}$

$F(f) = \left(\varphi_F, \varphi \dfrac{F(f)}{F(\varphi)}\right)$ //

$\therefore\ F(f) = (\varphi_F, f)$

(uniquness) $F(f) = (\varphi', f) = (\varphi_F\ f)$

$\therefore\ (\varphi' - \varphi_F, f) = 0 \quad \forall f$

$\therefore\ \varphi' - \varphi_F = 0 \quad \because)\ \mathcal{H} = \mathcal{H} \oplus \mathcal{H}^{\perp}$ 〃 $\{0\}$

(bound) in the same as the example we can see that $\|F\| = \|\varphi_F\|$ 〃

$\mathcal{H}^* \ni F \longleftrightarrow \varphi_F \in \mathcal{H}$ bijective

$$\begin{array}{ccc} \xi : \mathcal{H}^* & \longrightarrow & \mathcal{H} \\ \Psi & & \Psi \\ F & \longmapsto & \varphi_F \end{array}$$

bijective isometry $\|\xi F\| = \|F\|$

$F = \varphi_F$ identification

we oftentime identify $\mathcal{H}^*$ and $\mathcal{H}$ 〃

3 Fourier transformation

$\mathcal{S}(\mathbb{R}^d) = \{ f \in C^\infty \mid \sup |x^\alpha \partial^\beta f(x)| < \infty \ \forall \alpha \beta \}$

$\alpha = (\alpha_1 \cdots \alpha_d) \in \mathbb{Z}_+^d$ multi index

$\beta = (\beta_1, \dots, \beta_d) \in \mathbb{Z}_+^d$

$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ $\quad \partial^\beta = \partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}$

We know that $\mathcal{S} \subset L^2$ dense

∵ $H_n(x) e^{-x^2} \in \mathcal{S}$

$$Ff(k) = (2\pi)^{-\frac{d}{2}} \int f(x) e^{-ik\cdot x} dx$$

$$\hat{F} f(k) = (2\pi)^{-\frac{d}{2}} \int f(x) e^{+ikx} dx$$

well-defined.

Prop. 2.4 $F : \mathcal{S} \to \mathcal{S}$

$(Ff, Fg) = (f\ g) \quad f, g \in \mathcal{S}$

$\hat{F} F f = F \hat{F} f = f$

Especially $\|Ff\|^2 = \|f\|^2 \ \forall f$.

F is bdd $\quad D(F) = \mathcal{S}$ dense

$\exists \bar{F} : L^2 \to L^2$ bijective and isometry

i.e. unitary.

invariant $e^{-\frac{|k|^2}{2}}$

Prop 2.4 ☺

$$Ff(k) = (2\pi)^{-\frac{d}{2}} \int f(x)\, e^{-ik\cdot x}\, dx$$

$$k^\alpha \partial^\beta \hat{f} = \int f(x)\, k^\alpha \partial_k^\beta e^{-ikx}\, dx$$

$$= \cancel{(-1)^{|\beta|}} \int f(x)\, k^\alpha (-i)^{|\beta|} x^\beta e^{-ikx}\, dx$$

$$= (-i)^{|\beta|} \int x^\beta f(x)\, k^\alpha e^{-ikx}\, dx$$

$$= (-i)^{|\beta|} (-i)^{|\alpha|} \int x^\beta f(x)\, \partial_x^\alpha e^{-ikx}\, dx$$

$$= (-i)^{|\alpha|+|\beta|} (-1)^{|\alpha|} \int \underline{\partial_x^\alpha x^\beta f(x)} \cdot e^{-ikx}\, dx$$

$$\sup_x \left| (1+|x|^2)^\ell\, \partial_x^\alpha x^\beta f(x) \right| < \infty \quad \text{integrable}$$

$$(Ff, \overset{e^{-\varepsilon|k|^2}}{\vee} Fg) = \int \overline{Ff(k)}\, Fg(k)\, dk$$

$$= \int dk \overset{\wedge}{\underset{e^{-\varepsilon|k|^2}}{}} \int \bar{f}(x)\, e^{+ikx} \int g(y)\, e^{-iky}\, dx\, dy$$

$$\sqrt{\varepsilon} k = \frac{1}{\sqrt{2}} \ell$$

$$= \int dx\, dy\, \bar{f}(x) g(y) \int dk\; e^{-\varepsilon|k|^2} e^{-ik(y-x)}$$

$$= \int dx\, dy\, \bar{f}(x) g(y) \int e^{-\frac{1}{2}|\ell|^2} e^{-i\frac{\ell}{\sqrt{2}\sqrt{\varepsilon}}(y-x)} (2\varepsilon)^{-\frac{d}{2}}\, d\ell$$

$$= \int dx\, dy\, \bar{f}(x) g(y)\, (2\pi)^{\frac{d}{2}}\, e^{-\frac{1}{2}\cdot\frac{1}{2\varepsilon}|y-x|^2}$$

$$(2\pi)^{\frac{d}{2}} \int dx\, dy\, \overline{f(x)}\, g(y)\, e^{-\frac{1}{4\varepsilon}|x-y|^2} \quad (2\varepsilon)^{-\frac{d}{2}}$$

$$(2\pi)^{-\frac{d}{2}} \int dx\, dy\, \bar{f}(x)\, g(y)\, e^{-\frac{1}{4\varepsilon}|x-y|^2} \quad (2\varepsilon)^{-\frac{d}{2}}$$

$$(2\pi)^{-\frac{d}{2}} \int dz\, dy\, \bar{f}(y+z)\, g(y)\, e^{-\frac{1}{4\varepsilon}|z|^2} (2\varepsilon)^{-\frac{d}{2}} \qquad x-y=z$$

$$\frac{z}{\sqrt{2}\sqrt{\varepsilon}} = u$$

$$(2\pi)^{-\frac{d}{2}} \int du\, dy\, \bar{f}(y+\varepsilon u)\, g(y)\, e^{-\frac{1}{2}u^2}$$

$$\to (2\pi)^{-\frac{d}{2}} \int \bar{f}(y)\, g(y) \int e^{-\frac{1}{2}u^2}\, du = (f, g) \quad //$$

$$\hat{F}Ff = \int e^{ikx}\, Ff(k)\, dk$$

$$= \iint e^{ikx}\, e^{-iky} f(y)\, dk\, dy$$

$$(g,\ e^{-\varepsilon|k|^2}\hat{F}Ff) \to (g, f) \qquad //$$

4. Unbounded operators

$T : \mathcal{H} \to K$ is unbdd $\Leftrightarrow$ T is "not" bdd
(densely defined)

- when T is bdd, we can extend $D(T)$ to whole Hilbert space $\mathcal{H}$. But when T is unbdd, it can not be extended.
- Domain is very important

$T \subset S \Leftrightarrow D(T) \subset D(S)$, $Tf = Sf$

$D(T+S) = D(T) \cap D(S)$

$D(TS) = \{ x \in \mathcal{H} \mid D(S) \ni x \text{ and } Sx \in D(T) \}$

Prop 4.1 (Distribution law)

$$(S_1 + S_2)T = S_1 T + S_2 T$$

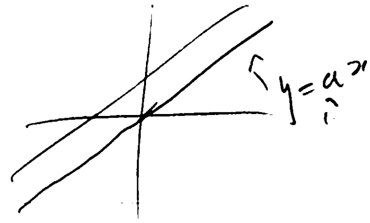
$$T(S_1 + S_2) \supset TS_1 + TS_2$$

◎ Closed op ◎ Adjoint ◎ symmetric op
◎ self-adjoint op.

T is closed $\Leftrightarrow$ $\left(\begin{array}{l} x_n \in D(T),\ x_n \to y\ \ Tx_n \to z \\ \Longrightarrow y \in D(T) \text{ and } Ty = z \end{array} \right)$

$G(T) = \{ (x, Tx) \mid x \in D(T) \}$ graph

T is closed $\Leftrightarrow$ $G(T) \subset \mathcal{H} \times K$ is closed.



Fundamental fact of graph

$G \subset \mathcal{H} \times K$ is a graph i.e. $G = G(T)$, and $(0, y) \in G$
$\Rightarrow y = 0$ Trivial!

Prop 4.2 $G \subset \mathcal{H} \times K$ linear op subspace + $(0, y) \in G \to y = 0$
$\Rightarrow {}^{\exists}T : \mathcal{H} \to K$ st $G = G(T)$.

Def T is closable $\Leftrightarrow {}^{\exists}S$ closed st $T \subset S$.

Prop 4.3 T closable $\Rightarrow {}^{\exists}\bar{T}$ st
① closed
② $G(\bar{T}) = \overline{G(T)}$
③ $\bar{T}$ is minimal closed ext.

$\bar{T}$ is said to be the closure of T.

◎ Adjoint $T : \mathcal{H} \to \mathcal{H}$ $\overline{A^{T}} = A^{*}$

$D(T^{*}) = \{ y \in \mathcal{H} \mid (Tx, y) = (x, {}^{\exists}z) \ \forall x \in D(T) \}$

$T^{*}y = z$ (densely defined なので well-def)

Prop 4.4 T, S densely defined

(1) $T \subset S \to S^{*} \subset T^{*}$

(2) $(\alpha T)^{*} = \bar{\alpha} T^{*}$

(3) $D(T+S)$ dense $\Rightarrow (T+S)^{*} \supset T^{*} + T^{*}$

(4) $D(TS)$ dense $\Rightarrow (TS)^{*} \supset S^{*}T^{*}$.

注

- Reed-Simon I, II, IV (Abstract Th; Schrödinger op; spectral anal)
- Weidman Springer

- [illegible]

Relation between adjoint and closed op

$$G(T^*) = U\left[G(T)^{\perp}\right]$$ where $U : \mathcal{H}\times\mathcal{H} \to \mathcal{H}\times\mathcal{H}$ unitary $(x,y) \to (y, -x)$
$= -(x, y)$

Prop4.5 $D(T)$ is dense

① T^* is closed
② T is closed $\to$ $D(T^*)$ dense and $(T^*)^* = T$
③ T is closable $\to$ $(\overline{T})^* = T^*$
④ T is closable $\to$ $D(T^*)$ dense $(T^*)^* = \overline{T}$.

○ Symmetric op & self-adjoint op.
① T is sym $\Leftrightarrow$ $T \subset T^*$
② T is s.a. $\Leftrightarrow$ $T = T^*$.

T sym $\to$ T closable $\to$ $\overline{T}$ sym. closed.

sym closed

$T \subset S$ s.a.
$T^* \supset S^*$

$\because)$ $(\overline{T})^* = T^* \supset T$
$\therefore (\overline{T})^* \supset \overline{T}$

$\Rightarrow$ $T \subset S^* = S \subset T^*$

$\therefore S = T^* \restriction_D$ where $\exists D \supset D(T)$.

T: Essentially self-adjoint

$\Leftrightarrow$ $\overline{T}$ is self-adjoint

$T\restriction_D$ is ess. s.a. $\Leftrightarrow$ D is a core of T

$T \subset \overline{T} \leftarrow$ s.a.

$T \subset S \leftarrow$ s.a

~~$T^* \supset S^* = S$ $\therefore$ $T \subset S \subset T^*$~~

$\overline{T} \subset S \Rightarrow S^* = S \subset (\overline{T})^* = \overline{T} \subset S$

$\therefore \quad S = \overline{T}$ 拡大は一意的

$\because$ of Prop 4.1

$D((S_1+S_2)T) = \{ y \mid y \in D(T) \text{ and } Ty \in D(S_1) \cap D(S_2) \}$

$D(S_1T + S_2T) = \{ y \mid y \in D(S_1T) \cap D(S_2T) \}$ //

$= \{ y \mid y \in D(T)$ かつ $Ty \in D(S_1) \}$

$\cap \{ y \mid y \in D(T)$ かつ $Ty \in D(S_2) \}$

$D(T(S_1+S_2)) = \{ y \mid y \in D(S_1) \cap D(S_2),\ (S_1+S_2)y \in D(T) \}$

$D(TS_1 + TS_2) = D(TS_1) \cap D(TS_2)$

$= \{ y \mid y \in D(S_1)\ S_1 y \in D(T) \}$

$\cap \{ y \mid y \in D(S_2)\ S_2 y \in D(T) \}$

necessarily

quadratic eq

discriminant

$\because$ of Prop 4.2 $\quad G \ni (x, y) \quad T : x \to y$

$G \ni (x, y) \to Tx = y$ or $Tx = z$ but
$\ni (x, z)$

$(0, y-z) \to y = z$

$\therefore T$ is well-defind. //

$Tx + Ty = T(x+y)$?

$G_1 \ni \begin{matrix} (x, z) \\ (y, w) \end{matrix} \to G \ni (x+y, z+w) \quad \therefore T(x+y) = z+w$
$= Tx + Ty$. //

Proof Prop 4.3

$T \subset S \quad \therefore \quad G(T) \subset G(S)$

$G(S)$ is closed

$\therefore \quad \overline{G(T)} \subset G(S)$

$(0, y) \in \overline{G(T)} \Rightarrow y = 0 \quad \therefore \quad \overline{G(T)} = G(\overset{\exists}{\bar{T}})$

$G(\bar{T}) \subset G(S)$

If $T \subset K \leftarrow$ closed $\Rightarrow \overline{G(T)} \subset G(K)$

$\qquad\qquad\qquad\qquad \overset{\shortparallel}{G(\bar{T})} \subset G(K) \quad \therefore \bar{T} \subset K$ //

Prop 4.4 omit

$G(T^*) = U[G(T)^\perp]$ の ☺

$= \{(x, T^*x) \mid x \in D(T^*)\}$

$U^{-1} G(T^*) = \{(-T^*x, x) \mid x \in D(T^*)\} \ni X$

$G(T) = \{(x, Tx) \mid x \in D(T)\} \ni Y$

$\langle X, Y \rangle = \langle (-T^*x, x), (y, Ty) \rangle$

$= (-T^*x, y) + (x, Ty) = 0.$

$\therefore U^{-1} G(T^*) \subset G(T)^\perp$

Let $(y, z) \in G(T)^\perp$ Hence $\langle (y, z), (x, Tx) \rangle =$

$(y, x) + (z, Tx) = 0$

$\therefore (z, Tx) = -(y, x)$

$\therefore z \in D(T^*)$ かつ $T^*z = -y$

$\therefore (y, z) = (-T^*z, z)$

$\in U^{-1} G(T^*)$

$\therefore U^{-1} G(T^*) \supset G(T)^\perp$ //

Proof Prop4.5

① $G(T^*) = \underset{\text{unitary}}{U}[\overset{\text{closed}}{G(T)^\perp}]$ ∴ closed

② $D(T^*)^\perp \ni x$; ~~$\langle x, y\rangle = 0 \ \forall y \in D(T^*)$~~

$(0, x) \in [UG(T^*)]^\perp$ ∵ $UG(T^*) \ni U(y, T^*y) = (T^*y, -y)$

$\langle (0, x), (T^*y, -y)\rangle = (0, T^*y) - (x, y) = -(x, y) = 0$

$G(T^*) = U[G(T)^\perp]$

$UG(T^*) = U^2[G(T)^\perp] = G(T)^\perp$

$[UG(T^*)]^\perp = \overline{G(T)} = G(T)$ ∴ $x = 0$.

& $G(T^{**}) = U[G(T^*)^\perp] = \cdots = G(T)$

③ $\overline{G(T)} = G(\overline{T})$

∴ $G(\overline{T}^*) = U[G(\overline{T})^\perp] = U[G(T)^\perp]$

$= G(T^*)$.

④ $D(\overline{T}^*) = D(T^*)$ dense by ②

∴ $(\overline{T}^*)^* =$ ~~[illegible]~~ $\overline{T}$ by ②

T sym. closed op $\quad T \subset T^*$

$D(T) \subset D(T^*)$

We introduce $(f,g)_{T^*} = (f,g) + (T^*f, T^*g)$

graph inner product.

$(D(T^*), (\cdot,\cdot)_{T^*})$ is a Hilbert space

& $D(T^*)$ is a closed subsp るじかい

☺ · Completeness

$\|x_n - x_m\|_{T^*} \to \ \Rightarrow \exists x \in D(T^*)$ s.t. $\|x_n - x\|_{T^*} \to 0$

· closedness is also easily proven.

$\{x_n\} \subset D(T)$ s.t. $x_n \to x$ in $\|\cdot\|_{T^*} \Rightarrow x \in D(T)$.

projection thm says $D(T^*) = D(T) \oplus_T D(T)^\perp$

Thm (von Neumann)

$D(T^*) = D(T) \oplus_T K_+ \oplus_T K_-$

where $K_\pm$ $\ker(i \mp T^*)$

Thm (von Neumann extension thm)

① $n_+ = n_- = 0 \Leftrightarrow$ T is s.a.

② $n_+ = n_- (\leq \infty) \Rightarrow$ T has s.a. ext

③ $(n_+, n_-) = (n,0)$ or $(0,n)$

$\Leftrightarrow$ T has "no" closed sym ext.

③ → ① $D(T) \subset D(T^*)$

It is enough to show $D(T) \supset D(T^*)$

Let $x \in D(T^*)$

$\therefore \ (T^* - T)x = (T+i)\overset{\exists}{y}$ $\quad \underline{y \in D(T)}$

$\therefore \ (T^* - i)(x-y) = 0$ $\quad \therefore \ x = y \in D(T)$ //

~~Example~~

<u>Prop 5.2</u> T sym. ①-③ equivalent

① T ess. s.a.

② $\mathrm{Ker}(T^* \pm i) = \{0\}$

③ $\overline{\mathrm{Ran}(T \pm i)} = \mathcal{H}$

①→② $\bar{T}$ s.a. $\quad \mathrm{Ker}((\bar{T})^* \mp i) = \{0\}$

$= \mathrm{Ker}(T^* \pm i) = \{0\}$

②→③ ok $\quad \overline{\mathrm{Ran}(T \pm i)} \oplus \mathrm{Ker}(T^* \mp i)$

③→① $\overline{\mathrm{Ran}(T \pm i)} \overset{\uparrow}{=} \mathrm{Ran}(\bar{T} \pm i) = \mathcal{H}$ //

general lemma T closable

$\overline{\mathrm{Ran}(T)} = \overline{\mathrm{Ran}(\bar{T})}$

($\subset$) ok

($\supset$) $x \in \mathrm{Ran}(\bar{T})$ $\quad x = \bar{T}\overset{\exists}{y}$

by the def of $\bar{T}$ $\ \exists\, y_n \in D(T)$ s.t. $y_n \to y$, $Ty_n \to Ty$

$\therefore \ x \in \overline{\mathrm{Ran}(T)}$

$\therefore \ \mathrm{Ran}(\bar{T}) \subset \overline{\mathrm{Ran}(T)}$ //

5. Self-adjointness

$T : \mathcal{H} \to \mathcal{H}$ sym op.

When is T s.a or ess. s.a. ?

Prop 5.1 T sym ①–③ are equivalent

① T s.a ② $\mathrm{Ker}(T^* \pm i) = \{0\}$, closed

③ $\mathrm{Ran}(T \pm i) = \mathcal{H}$

(∵) 注 (1) $Tf = af \to a \in \mathbb{R}$ ✓check
T s.a.

(2) T sym $\mathcal{H} = \overline{\mathrm{Ran}(A \pm i)} \oplus \mathrm{Ker}(A^* \mp i)$

① → ② $\mathrm{Ker}(T^* \pm i) \ni x$

$Tx = T^* x = \mp i x$ ∴ $x = 0$.

$T = T^* \leftarrow$ closed ∴ T closed

② → ③ $\overline{\mathrm{Ran}(T \mp i)} = \{0\}$

but T is closed. Then $\mathrm{Ran}(T \mp i)$ closed.

∵) $\mathrm{Ran}(T+i) \ni x_n \to x$

$(T+i) x_n \to \exists y$

$\|(T+i) x_n\|^2 = \|T x_n\|^2 - 2\mathrm{Re}(T x_n, i x_n) + \|x_n\|^2$

$\geq \|x_n\|^2$ $\{x_n\}$ Cauchy $x_n \to x$

$T \pm i$ closed $(T+i) x = y$ //

Examples $\mathcal{H} = L^2(\mathbb{R}^d)$

(1) $F : \mathbb{R}^d \to \mathbb{R}$ $F \in L^2_{loc}$

$D(M_F) = C_0^\infty$ F : real

$M_F f = Ff$ $\therefore$ sym.

$$\overline{\mathrm{Ran}(M_F + i)} = \mathcal{H}$$

$\because$) $g \in \mathrm{Ran}(M_F + i)^\perp$

$$\therefore \ (g, (M_F + i) f) = 0 \quad \forall f \in C_0^\infty$$

$$\| $$

$$\int (F+i)\bar{g} \cdot f \, dx = 0 \quad \forall f \in C_0^\infty$$

$$\boxed{\text{de Bois-Raymond lemma} \quad \int h \cdot f = 0 \ \forall f \in C_0^\infty \Rightarrow h = 0}$$

$$= (F+i)\bar{g} = 0 \quad \therefore \ \bar{g} = 0.$$

$\therefore$ M_F is ess. s.c. on C_0^∞ $\overline{M_F|_{C_0^\infty}} = M_F$

Example 2. $P = -i\frac{d}{dx}$

$P : \mathcal{S} \to \mathcal{S}$

$(\mathcal{F} P f)(k) = k \mathcal{F} f(k) \quad \therefore \mathcal{F} P \mathcal{F}^{-1} = M_k$

$P := \mathcal{F}^{-1} M_k \mathcal{F}$

$D(P) := \{ f \mid \mathcal{F} f \in D(M_k) \}$

$\mathcal{F}^* = \mathcal{F}^{-1}$

Prop $\quad P$ is s.a.

$\because \; (P^* \pm i) g = 0 \qquad g \in \mathrm{Ker}(P^* \pm i)$

$(f, (\mathcal{F}^{-1} M_k \mathcal{F})^* g) = \pm i (f, g) \qquad f \in \mathcal{S}$

$= (M_k \hat{f}, \hat{g}) = \pm i (\hat{f}, \hat{g})$

$\therefore \; \hat{g} \in \mathrm{Ker}(M_k^* \pm i) \quad \therefore \hat{g} = 0 \quad \therefore g = 0.$

closedness $\quad P f_n \to g \quad f_n \to f$

$\mathcal{F}^{-1} M_k \mathcal{F} f_n \to g \qquad \mathcal{F} f_n \to \mathcal{F} f$

$\therefore \; \mathcal{F} f \in D(M_k)$ and $g = \mathcal{F}^{-1} M_k \mathcal{F} f$

$\therefore \; P f = g \quad //$

Example 3 $\quad -\Delta \;;\; L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ —

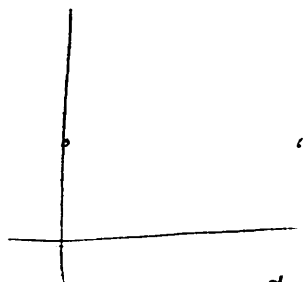
$\bar{A}'$ $\quad F(-\Delta)F^{-1} = M_{|k|^2}$

$$D(-\Delta) = \{ f \mid \hat{f} \in D(M_{|k|^2}) \}$$

$$-\Delta f = F^{-1} M_{|k|^2} F f \qquad \text{s.a.}$$

Example 4 $\quad g \in L^2_{loc}$

$$g(-i\nabla) \to F^{-1} M_{g(k)} F$$



Example 5

$$T_\alpha = i\frac{d}{dx}, \quad L^2([0,1]) \qquad f(x) = f(0) + \int_0^x g(t)\,dt$$

$$D(T_\alpha) = \{ f \mid f \text{ absolutely cont.}, \underline{f(1) = \alpha f(0)} \}$$

T_α $\quad$ <u>symmetric</u>

$$(f, T_\alpha g) = \int_0^1 \bar{f} \frac{d}{dx} g = [\bar{f} g]_0^1 + \int T_\alpha \bar{f}\, g$$

$$= \bar{f}(1)g(1) - \bar{f}(0)g(0)$$

$$(|\alpha|^2 - 1)\, \bar{f}(0)g(0)$$

$\alpha \in S^1$ $\quad$ sym.

$$\boxed{\begin{array}{lll} \operatorname{Ker}(T_\alpha^* + i) \ni \varphi & \frac{d}{dx}\varphi = -\varphi & \therefore \varphi = ce^{-x} \\ \operatorname{Ker}(T_\alpha^* - i) \ni \varphi & \frac{d}{dx}\varphi = \varphi & \therefore \varphi = ce^{+x} \end{array}}$$

No.

6. Schrödinger operators

◎ T s.a. i.e. $T = T^*$

$\|(T + i\alpha)f\|^2 = \|Tf\|^2 + |\alpha|^2 \|f\|^2$

$\therefore (T + i\alpha)f = 0 \Leftrightarrow f = 0$

$\therefore \ker(T \pm i\alpha) = 0$, $\operatorname{Ran}(T \pm i\alpha) = \mathcal{H}$

which was already see yesterday.

$T \pm i\alpha$ bijective $\exists (T \pm i\alpha)^{-1}$

$\therefore \|(T \pm i\alpha)^{-1} f\| \le \frac{1}{|\alpha|} \|f\|$ b'dd

Prop 6.1 $\alpha \in \mathbb{R}$ $T \pm i\alpha \in \mathcal{B}(\mathcal{H})$

and $\|T \pm i\alpha\| \le \frac{1}{|\alpha|}$

◎ $T : \mathcal{H} \to \mathcal{H}$ b'dd $T \in \mathcal{B}(\mathcal{H})$

$\|T\| < 1 \to \exists (T+1)^{-1} \in \mathcal{B}(\mathcal{H})$

⊙ $S_N = \sum_{n=0}^{N} (-1)^n T^n$

$S_N (T+1) f = \sum_{n=0}^{N} (-1)^n T^{n+1} f + \sum_{n=0}^{N} (-1)^n T^n f$

$= f - (-1)^{N+1} T^{N+1} f \to f \ (N \to \infty)$

$\therefore S_\infty (T+1) f = f$

Similary $(T+1) S_\infty f = f$ $\therefore S_\infty = (T+1)^{-1}$

CORE 드개 2

$\overline{T|_D}$ self-adjoint $\overset{\text{def}}{\iff}$ $T|_D$ ess. s.a.
D is called 'core' of T.

Example $\underline{F \in L^2_{loc}}$, $F: \mathbb{R}^d \to \mathbb{R}$
$M_F = \overline{M_F|_{C_0^\infty}}$. C_0^∞ is a core
$\mathcal{S}$ is a core

Example
$P = F^{-1} M_F F$ $\mathcal{S}$ is a core
(It can be proved C_0^∞ is also core)

$-\Delta = F^{-1} M_{|k|^2} F$ $\mathcal{S}$ and C_0^∞ ~~is~~ are core.

<u>Prop</u> Let D be a core of T.
Then $\forall x \in D(\overline{T|_D})$ $\exists x_n \in D$ s.t.
① $x_n \to x$ ② $Tx_n \to \overline{T|_D}\, x$

$\because$ $G(\overline{T|_D}) = \overline{G(T|_D)}$

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No.

Prop 6.2 (T. Kato 1951)

Let A be s.a. and S be sym

s.t. $D(A) \subset D(B)$ and

$\|Bf\| \le a\|Af\| + b\|f\| \quad \forall f \in D(A)$

and $a < 1$

Then $A+B$ is self-adjoint on $D(A)$.

$\because$ $\mathrm{Ran}(A+B+i\lambda)^{\perp} = \{0\}$

$$A+B \pm i\lambda = \left(B(A+i\lambda)^{-1} + 1\right)(A+i\lambda)$$

$\|B(A+i\lambda)^{-1}\| < 1$?

$\|B(A+i\lambda)^{-1}f\| \le a\|A(A+i\lambda)^{-1}f\| + b\|(A+i\lambda)^{-1}f\|$

$\|(A+i\lambda)f\|^2 = \|Af\|^2 + |\lambda|^2\|f\|^2$

1) $\|f\|^2 \ge \|A(A+i\lambda)^{-1}f\|$

2) $\|A+i\lambda\| \le \frac{1}{|\lambda|}$

$\therefore$ $\|B(A+i\lambda)^{-1}f\| \le \underbrace{\left(a + \frac{b}{|\lambda|}\right)}_{<1 \text{ if } a+\frac{b}{|\lambda|}<1}\|f\|$

$\exists$

$\therefore$ $\left(B(A+i\lambda)^{-1}+1\right)^{-1}$

Then $R(\ldots\ldots) = \mathcal{H}$

Then $R(\ldots\ldots)(A+i\lambda) = \mathcal{H}$ //

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No.

Cor. Assume the same assumption as in Prop 6.2. Suppose that D is a core of A. Then D is a core of $A+B$ i.e. $\overline{A+B|_D}$ is ess. s.a.

$\because$ $A|_D = A'$, $B|_D = B'$

B' sym $\|B'f\| \le a\|A'f\| + b\|f\|$ $\forall f \in D(A') = D$

$D(\overline{A'}) \ni f$ $\exists f_n \in D$ s.t. $f_n \to f$, $A'f_n \to \overline{A'}f$

$\therefore \{B'f_n\}$ Cauchy $\therefore B'f_n \to \overline{B'}f$

$\therefore \|\overline{B'}f\| \le a\|\overline{A'}f\| + b\|f\|$ $\forall f \in D(\overline{A'})$

$\therefore$ $\overline{A'} + \overline{B'}$ is s.a. on $D(\overline{A'})$

$\overline{A'} + \overline{B'} \supset A + B|_D$

$\overline{A'} + \overline{B'} \supset \overline{A'+B'}$ — trivial

On the other hand

$(A'+B')f_n \to \overline{A'}f + \overline{B'}f$ $\forall f \in D(\overline{A'})$

$f_n \to f$

$\to$ $\overline{A'+B'}f = \overline{A'}f + \overline{B'}f$

$\therefore$ $\overline{A'+B'} \supset \overline{A'} + \overline{B'}$ $\therefore$ $\overline{A'+B'} = \overline{A'} + \overline{B'}$. //

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No.

QM. hydrogen atom

$$i\frac{d}{dt}\varphi = -\frac{1}{2}\Delta\varphi - \frac{Z}{|x|}\varphi.$$

$$\int_A |\varphi(x,t)|^2 dx = \text{Probability of the existen of electron in } A \subset \mathbb{R}^3 \text{ at time } t.$$

$$i\frac{d}{dt}\varphi = H\varphi \qquad \underline{H = -\frac{1}{2}\Delta - \frac{Z}{|x|}}$$

$$\int_{\mathbb{R}^3} |\varphi(x,t)|^2 = 1 \quad \therefore \quad \underline{\underline{\varphi \in L^2}}$$

- Matrix case $\quad i\frac{d}{dt}v = Mv$

$$v = v_t = e^{-itM} v_0.$$

what is this?

- $\varphi(x,t) = e^{-itH}\varphi(x,0)$

No.

$-\frac{1}{2}\Delta - \frac{1}{|x|}$ in $L^2(\mathbb{R}^3)$ s.a.

$r = |x| \quad x \in \mathbb{R}^3$ $\qquad \nabla f = (f_1' \; f_2' \; f_3')$

$$\nabla(\sqrt{r}\,\varphi) = \sqrt{r}\,\nabla\varphi + \frac{1}{2}\frac{1}{r^{3/2}}\,x\varphi$$

$$|\nabla\varphi|^2 = \left|\frac{1}{\sqrt{r}}\nabla(\sqrt{r}\,\varphi) - \frac{1}{2}\frac{x}{r^2}\varphi\right|^2$$

$$\geq -\frac{1}{r^2\sqrt{r}}\nabla(\sqrt{r}\,\varphi)\cdot x\varphi + \frac{1}{4}\frac{1}{r^2}|\varphi|^2$$

$$\underline{r\frac{\partial}{\partial r}\varphi = x\cdot\nabla\varphi}$$

$$= \underline{-\frac{1}{r^{3/2}}\frac{\partial}{\partial r}(\sqrt{r}\,\varphi)} + \frac{1}{4}\frac{1}{r^2}|\varphi|^2$$

$$= -\frac{1}{2r^2}\frac{\partial}{\partial r}(r|\varphi|^2) + \frac{1}{4}\frac{1}{r^2}|\varphi|^2$$

$$\int|\nabla\varphi|^2 \geq \int\frac{1}{4r^2}|\varphi|^2 - \frac{1}{2}\int_0^\infty\frac{1}{r^2}\frac{\partial}{\partial r}(r|\varphi|^2)\,\underset{r^2\,dr}{\sin\theta\,d\theta\,d\varphi}$$

$$\geq \frac{1}{4}\int\frac{1}{r^2}|\varphi|^2$$

Prop $\displaystyle\int|\nabla\varphi|^2 \geq \frac{1}{4}\int\frac{1}{r^2}|\varphi|^2 \qquad \varphi\in C_0^\infty$

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Prop $\|V\varphi\| \le \varepsilon \|\Delta\varphi\| + b_\varepsilon \|\varphi\|$ $\varphi \in D(-\Delta) \subset D(V)$

$\frac{1}{4}\|V\varphi\|^2 \le |(\varphi, -\Delta\varphi)|$ から estimate に
最後は core に関する limiting argument.

<u>Thm</u> $-\frac{1}{2}\Delta - \frac{Z}{|x|}$ is self-adjoint
and ess. s.a. on C_0^∞.

<u>Prop</u> $V \in L^2_{loc}$, $V \ge 0$, $-\frac{1}{2}\Delta + V$ is ess. s.a. on C_0^∞

<u>Example</u> $-\frac{1}{2}\Delta + P(x)$ b'dd from below, even polynomial.