

~~A~~

Cor 6.1 T is s.g. $\Leftrightarrow n_{\pm} = 0$

Lem 6.2 T sym + closed.

Then $\text{Ran}(T \pm i\lambda)$ ($\lambda \in \mathbb{R}$) is closed

and $\text{Ran}(T \pm i\lambda) = \text{Ker}(T^* \mp i\lambda)^{\perp}$

☹ $\text{Ran}(T \pm i\lambda) \ni f_n$

$(T \pm i\lambda)g_n = f_n \rightarrow f$ ($n \rightarrow \infty$) $\varepsilon \neq 0$

$$\| (T \pm i\lambda)g_n \|^2 = \| Tg_n \|^2 + |\lambda|^2 \|g_n\|^2$$

$$\therefore \| f_n - f \| = \| T(g_n - g_n) \|^2 + |\lambda|^2 \|g_n - g_n\|^2$$

\downarrow
 0

$\therefore \{g_n\}$ is Cauchy $\exists g = \lim g_n$

$$\therefore Tg_n = \underbrace{f_n \mp i\lambda g_n}_{\downarrow} \quad g_n \rightarrow g.$$

$\therefore g \in D(T)$ and $Tg = f \mp i\lambda g$

$\therefore (T \pm i\lambda)g = f \in \text{Ran}(T \pm i\lambda)$

$$\text{Ran}(T \mp i\lambda) = \text{Ker}(T^* \mp i\lambda)^\perp$$

$$\textcircled{C} \quad f = (T \mp i\lambda)g \text{ がある}$$

$$\text{このとき} \quad h \in \text{Ker}(T^* \mp i\lambda)$$

$$(f, h) = ((T \mp i\lambda)g, h) = 0.$$

$$\therefore f \in \text{Ker}(T^* \mp i\lambda)^\perp$$

$$\textcircled{D} \quad f \in \text{Ran}(T \mp i\lambda)^\perp \text{ がある}$$

$$(f, (T \mp i\lambda)h) = 0 \quad \forall h \in D(T)$$

$$\therefore ((T^* \mp i\lambda)f, h) = 0 \quad \forall h \in D(T^*)$$

$$\therefore f \in \text{Ker}(T^* \mp i\lambda)$$

$$\therefore \text{Ran}(T \mp i\lambda)^\perp \subset \text{Ker}(T^* \mp i\lambda)$$

$$\overline{\text{Ran}(T \mp i\lambda)} \supset \text{Ker}(T^* \mp i\lambda)^\perp$$

$$\text{Ran}(T \mp i\lambda)$$

\swarrow symmetric
 $\text{Cov} \quad A \text{ is s.g.} \Leftrightarrow \text{Ran}(i \mp A) = \mathcal{H}$

$$\frac{1}{1+r} = \sum_{n=0}^{\infty} (-r)^n$$

(30)

Lemma 6.3 A bdd $\Rightarrow \|A\| < 1$
 \exists a unique $(I+A)^{-1}$ bdd

$$S_n = \sum_{h=0}^n (-A)^h$$

$$\begin{aligned} (I+A) \sum_{h=0}^n (-A)^h &= \sum_{h=0}^n (-A)^h + \sum_{h=0}^n (-A)^{h+1} \\ &= \sum_{h=0}^n (-A)^h - \sum_{h=1}^{n+1} (-A)^h = -(-A)^{n+1} + I \end{aligned}$$

\downarrow bdd op \Rightarrow ok
 S_n f ok $\lim S_n f = S f$ & det $\neq 0$

$$\therefore (I+A) \sum_{h=0}^{\infty} (-A)^h f = -(-A)^{n+1} f + f \rightarrow f$$

$$\text{[2] } \sum_{h=0}^{\infty} (-A)^h (I+A) f = f$$

$\therefore S$ is $(I+A)$ inverse

$$S(I+A) = (I+A)S = I \quad \lrcorner$$

$$\therefore S = (I+A)^{-1}$$

Lemma 6.4 A s.g. then $(A \pm i)^{-1}$ bdd.
 $A \pm i$ is onto. $\exists (A \pm i)^{-1}$

$$\| (A \pm i) f \|^2 = \| A f \|^2 + \| f \|^2$$

$$\therefore \| f \|^2 = \| A (A \pm i)^{-1} f \|^2 + \| (A \pm i)^{-1} f \|^2$$

$$\therefore \frac{1}{\|A\|} \| (A \pm i)^{-1} f \|^2 \leq \| f \|^2 \quad \parallel$$

Ex. A s.g. B sym. and (Kato 1951)

$$D(A) \subset D(B) \text{ and}$$

$$\|Bf\| \leq \underbrace{a}_{0 \leq a < 1} \|Af\| + \underbrace{b}_{0 \leq b} \|f\|$$

Then $A+B$ is s.g. on $D(A)$

① $A+B$ is closed sym on $D(A)$

$$\therefore f_n \in D(A) \quad f_n \rightarrow f, \quad (A+B)f_n \rightarrow g$$

$$Af_n + Bf_n = g_n$$

$$Af_n = g_n - Bf_n$$

$$\|A(f_n - f_m)\| \leq \|g_n - g_m\| + a \|A(f_n - f_m)\| + b \|f_n - f_m\|$$

$$\therefore \|A(f_n - f_m)\| \leq \frac{1}{1-a} \{ \|g_n - g_m\| + \|f_n - f_m\| \}$$

$$\therefore \{Af_n\} \text{ is Cauchy } \therefore f \in D(A) \text{ and } Af_n \rightarrow Af$$

$$\text{同様にして } \{Bf_n\} \text{ is Cauchy } \therefore f \in D(B)$$

$$\therefore \begin{aligned} & \lim Bf_n \rightarrow h \quad \therefore f \in D(\bar{B}) \text{ and } \bar{B}f = h \\ & \text{よって } \lim \bar{B}f_n = \lim Bf_n = Bf. \quad \bar{B}f \end{aligned}$$

$\text{Re}(A + B \pm i\lambda) = \mathcal{H} \quad \dots$

~~(A+i)~~ $B + A+i = \underbrace{(B(A+i)^{-1} + I)}_{\text{onto}} (A+i)$

$\|B(A+i)^{-1}\| < 1$...

$\xi = 3$... $\|B(A+i)^{-1}f\| \leq \|A(A+i)^{-1}f\| + \|B(A+i)^{-1}f\|$

$\leq a\|f\| + \frac{1}{\lambda^2}\|f\| \leq \|f\| \quad (\lambda \gg 1)$

Finally $\dim(\lambda - T^*) = \dim(\lambda + \eta - T^*)$
 $|\eta| < |\text{Im} \lambda|$ on \mathbb{C}_I .

$\therefore \dim \mathcal{K}(\lambda + \eta - T^*) > \dim \mathcal{K}(\lambda - T^*)$
 $\Rightarrow \text{NG}$



$\therefore \dim M > \dim N$
 $\rightarrow N \cap N^c$
 $M \cap N \oplus M \cap N^c$
 $\# \quad \#$
 $M \quad \neq \emptyset$

~~$\mathcal{K}(\lambda + \eta - T^*) \cap \mathcal{K}(\lambda - T^*) = \{0\}$~~

$f \in \text{Ran}(\bar{\lambda} - T^*)$
 $\therefore f = (\bar{\lambda} - T^*)u$

$\therefore 0 = ((\lambda + \eta) - T^*)f, u) = \|(\bar{\lambda} - T^*)u\|^2 + \eta(\bar{\lambda} - T^*)u, u) - \bar{\eta}((\lambda - T^*)u, u) = |\text{Im} \lambda| \|u\|^2$

Finally $\dim \ker(\lambda - T^*)$ $\lambda \in \mathbb{C}$ 不变

$\dim \ker(\lambda + \eta - T^*) > \dim \ker(\lambda - T^*)$
 $|\eta| < |\operatorname{Im} \lambda|$ 由 No. 8.3.

证. 证 3 $\dim \ker(\lambda + \eta - T^*) \leq \dim \ker(\lambda - T^*)$

$\therefore \dim \ker(\lambda + \eta - T^*) = \dim \ker(\lambda - T^*)$

$\dim M > \dim N$

$$\frac{M \cap N}{\neq M} \oplus \frac{M \cap N^\perp}{\neq \{0\}}$$

$M \cap N^\perp = \{0\}$ 予证.

$f \in \ker(\lambda + \eta - T^*) \cap \ker(\lambda - T^*)^\perp \neq \{0\}$
 \downarrow
 $\operatorname{Ran}(\bar{\lambda} - T)$

$$f = (\bar{\lambda} - T)u$$

$$0 = ((\lambda + \eta - T^*)f, u) = (f, (\bar{\lambda} - T)u) + \bar{\eta}(f, u)$$

$$= \|(\bar{\lambda} - T)u\|^2 + \bar{\eta}(f, u)$$

$$\Rightarrow \|(\bar{\lambda} - T)u\|^2 = |\eta| \|f\| \cdot \|u\|$$

$$\star \quad \cancel{\|(\bar{\lambda} - T)u\|^2} = \|(\bar{\lambda} - T)u\| - |\eta| \|u\|$$

$$\Rightarrow |\operatorname{Im} \lambda| \|u\| - |\eta| \|u\| \geq 0$$

予证.