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② ①  $B_{A^*} = [ \cup B_A ]^\perp$  is closed ..

②  $D(A^*)^\perp = \{0\}$  ええ。

Suppose that  $f \in D(A^*)^\perp$ .

Then  $\langle 0, f \rangle \in [ \cup B_{A^*} ]^\perp$  (easy)

But

$$[ \cup B_{A^*} ]^\perp = [ \cup [ \cup (B_A)^* ] ]^\perp = B_A^{**} = \overline{B_A} \\ = B_A$$

Hence  $f = A_0 = 0$ .

On the other hand

$$B_{A^{**}} = \dots = B_A \quad \text{Then } A^{**} = A.$$

$$\textcircled{3} \quad B_{A^*} = \cup [ B_A^\perp ] = \cup [ \overline{B_A}^* ]$$

$$= \cup [ \overline{B_A}^\perp ] = B_{\overline{A}}^*$$

$$\textcircled{4} \quad (\Rightarrow) \quad D(A^*) = D(\overline{A})^* \quad \text{ok}$$

( $\Leftarrow$ )  $D(\overline{A})$  is dense

Hence  $A^{**}$  is well-defined

$$B_{A^{**}} = \dots = \overline{B_A} \supset B_A$$

Then  $A^{**} \supset A \quad \therefore A$  is closable.

$\frac{11}{A}$

$A \subset A^* \Leftrightarrow A$  is sym

$A = A^* \Leftrightarrow A$  is s.a.

$\bar{A}$  is s.g  $\Leftrightarrow A$  is ess. s.g.

Lem 5.11  $A$  is sym. Then  $A$  is closable

$\because A \subset A^*$  Then  $D(A^*)$  is dense.

Hence  $A$  is closable (by Lem 5.10)

$$(\bar{A})^* = (A^{**})^* = \overline{A^*} \therefore \bar{A} \subset \overline{A^*} = (\bar{A})^*,$$

Ex  $\mathcal{H} = L^2(\mathbb{R})$ ,  $A = i \frac{d}{dx}$   $D(A) = C_0^\infty(\mathbb{R})$

$$(f, Ag) = (Af, g) \therefore A \subset A^*$$

$\overline{\frac{d}{dx}|_{C_0^\infty}} = H$  is closed & sym. //

Weierstrass approximation theorem

Let  $A$  be a closed and  $D(A) \neq \emptyset$

Lem 5.12 open mapping theorem

$T : \mathcal{H} \rightarrow K$  bdd and  $Ran T = K$

Then  $T$  is an open map i.e.

$T$  maps open sets to open sets

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Lemma 5.13 Closed graph theorem

Let  $A$  be closed and  $D(A) = \mathcal{H}$ . Then  $A$  is bdd.

$$\textcircled{(2)} \quad (D(A), \| \cdot \|_A) \quad \|f\|_A^2 = \|f\|^2 + \|Af\|^2$$

is a Banach space

$$S : D(A) \rightarrow \mathcal{H}$$

$$f \xrightarrow{\quad \quad} f$$

$$\|Sf\|_{\mathcal{H}} \leq \|f\|_A \quad \text{bdd onto}$$

$\therefore S$  is an open map.  $\therefore S^{-1}$  is cont  
 $\therefore S^{-1}$  is bdd.

$$\begin{aligned} \therefore \|Af\|_{\mathcal{H}} &\leq \|f\|_A = \|S^{-1}Af\|_{\mathcal{H}} \\ &\leq \|S^{-1}\| \|Sf\|_{\mathcal{H}} = \|S^{-1}\| \|f\|_{\mathcal{H}} \end{aligned}$$

Ex.  $[A, B] = -i$ . Then either  $A$  or  $B$  is unbdd

(2) Suppose that  $A$  and  $B$  are bdd

$$\text{Then } A B^n = n B^{n-1} A (-i) + B^n A$$

$$\therefore 2 \|A\| \|B^n\| \geq n \|B^{n-1}\|$$

$$\begin{aligned} \therefore 2 \|A\| \|B\| &\geq n \quad \forall n \\ &= \text{infty.} \end{aligned}$$

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## § 6 Self-adjoint operators

$$\{ \text{s.a.} \} \subset \{ \text{sym+closed} \} \subset \{ \text{sym} \} \subset \{ \text{dorable} \}$$

$T$  sym closed,  $T \subset S \leftarrow$  s.a. ext

$$\rightarrow T^* \supset S^* = S \supset T$$

$$\therefore S = T^* \cap_D D(T) \subset \underline{\underline{D}} \subset D(T^*)$$

Formal discussion

$$(D(T^*), \langle \cdot, \cdot \rangle_{T^*})$$

$$(f, g)_{T^*} = (f, g) + (T^* f, T^* g).$$

$$D(T^*) = D(T) \oplus \underbrace{D(T)}_{\text{if } f} \perp \underbrace{D(T)}_{\text{if } v} \leftarrow ?$$

$$\begin{aligned} \langle f, v \rangle &= (f, v) + (T^* f, T^* v) \\ &= (f, v) + (T f, T^* v) \\ &= (f, v) + (f, T^* T^* v) \end{aligned}$$

$$\therefore T^* T^* v = -v \therefore T^* v = \pm i v \quad (2 \text{ at } 2)$$

$$v \in \text{Ker}(-T^*) \quad v \in \text{Ker}(i+T^*)$$

Theorem (von Neumann)

$$D(T^*) = D(T) \oplus \text{Ker}(i-T^*) \oplus \text{Ker}(i+T^*)$$

$$M_1 = \dim(i \mp T^*) \quad \text{deficiency index}$$