

Lemma 5.2 $T_1, S_1, S_2 \in \mathcal{H} \rightarrow \mathcal{H}$

$$\textcircled{1} \quad T(S_1 + S_2) \supset TS_1 + TS_2$$

$$\textcircled{2} \quad (S_1 + S_2)T = S_1T + S_2T$$

\therefore Take core domains:

$$\textcircled{1} \quad D(TS_1 + TS_2) = D(TS_1) \cap D(TS_2) \ni f$$

$$\text{i.e., } f \in D(S_1) \cap D(S_2) \text{ & } S_1f \in D(T) \\ S_2f \in D(T)$$

$$\text{Hence } S_1f + S_2f \in D(T)$$

$$\therefore f \in T(S_1 + S_2)$$

$$\textcircled{2} \quad (\subset)$$

$$D((S_1 + S_2)T) \ni f$$

$$\text{i.e., } f \in D(T) \text{ & } Tf \in D(S_1) \cap D(S_2)$$

$$\therefore f \in D(S_1T) \cap D(S_2T)$$

$$(\supset)$$

$$D(S_1T + S_2T) = D(S_1T) \cap D(S_2T) \ni f$$

$$\text{i.e., } f \in D(T) \text{ & } Tf \in D(S_1) \cap D(S_2)$$

$$\therefore f \in D((S_1 + S_2)T) \quad //$$

$$A : \mathbb{S}^{\text{odd}} \Rightarrow f_n + f \Rightarrow Af_n = Af$$

Def 5.3 A is closed $\Leftrightarrow f_n \rightarrow f$ and $Hf_n \rightarrow g$
 (von Neuman) $\rightarrow f \in D(A)$ and
 $\rightarrow g = Af$

Def 5.4 A is closable $\Leftrightarrow \exists \tilde{A}$ closed
 $\|f\| = \|f_n\| + \|Af_n\|$ complete \Leftrightarrow st $A \subset \tilde{A}$
 Let \tilde{A} be given by

$$D(\tilde{A}) = \left\{ f \in \mathcal{E} \mid \begin{array}{l} \exists f_n \in D(A) \text{ st } f_n \rightarrow f \\ Hf_n \text{ converges} \end{array} \right\}$$

$$\tilde{A}f = \lim_{n \rightarrow \infty} \tilde{A}f_n \quad f \in D(\tilde{A})$$

\tilde{A} closable

Lemma 5.5 $\vee \tilde{A}$ is minimal closed extension.

i.e. \tilde{A} is closed, $A \subset \tilde{B}$ and \tilde{B} is closed
 $\rightarrow \tilde{A} \subset \tilde{B}$.

(*) \tilde{A} is well-defined

$$\therefore f_n \rightarrow f, g_n \rightarrow g \quad f_n, g_n \in D(A)$$

$\exists S$ closed st $A \subset S$ $Sf_n \rightarrow x$
 $Sg_n \rightarrow y$

$$\therefore x = y = sf$$

* \tilde{A} is closed supposed that

✓ $\therefore f_n \in D(\tilde{A})$ st $f_n \rightarrow f, \tilde{A}f_n \rightarrow g$ as $n \rightarrow \infty$

別紙参照. $\exists f_{n_m} \in D(A)$ st $f_{n_m} \rightarrow f_n, \tilde{A}f_{n_m} \rightarrow \tilde{A}f_n$

$$\{f_{n_m}\} \ni \{f_n\} \text{ st } \tilde{A}f_{n_m} \rightarrow g$$

$$f_n \rightarrow f, Af_n \rightarrow \text{con.} = fg$$

$$\therefore f \in D(\tilde{A}) \text{ and } \tilde{A}f = g$$

\bar{A} is closed

$\therefore f_n \in D(\bar{A})$ & $\bar{A}f_n \rightarrow g$ are assumed

By the def of \bar{A} there exists f_{n_m} st

$$\bar{A}f_{n_m} \xrightarrow{\text{def}} \bar{A}f_n \quad f_{n_m} \rightarrow f_n \quad \text{as } m \rightarrow \infty$$

$$\{f_{n_m}\}_{n,m}$$

$$\|f - f_{n_m}\| \rightarrow 0 \quad \text{OK}$$

$$\begin{aligned} \|Af_{n_m} - g\| &\leq \|Af_{n_m} - Af_n + Af_n - g\| \\ &\leq \|Af_{n_m} - Af_n\| + \|Af_n - g\| \end{aligned}$$

$\Rightarrow \|n \notin \frac{1}{2}\pi \mathbb{Z} \subset S, \exists n \in m \notin \frac{1}{2}\pi \mathbb{Z} \subset S$

$\therefore Af_{n_m} \rightarrow g \quad f_{n_m} \rightarrow f \quad \therefore f \in D(\bar{A}) \&$
 $\bar{A}f = g.$

20

\bar{A} is the minimal closed ext.

Suppose that

$\therefore A \subset S$ S closed.

$$\underline{A \subset S} \quad \forall f \in \bar{A} \stackrel{\exists}{\exists} f_n \in D(A) \text{ s.t. } f_n \rightarrow f \quad \bar{A}f_n \text{ conv.}$$

$\therefore Sf_n$ conv.

$$\therefore Sf_n \rightarrow Sf \quad \therefore \bar{A}f = Sf \quad "$$

Def 5.6 $A : \mathcal{H} \rightarrow \mathcal{H}$

A^* is defined by

$$D(A^*) = \left\{ f \in \mathcal{H} \mid \begin{array}{l} (f, Ag) = (\bar{h}, g) \\ \forall g \in D(A) \end{array} \right\} \stackrel{\exists}{\exists}$$

$$A^*f = \bar{h}.$$

Since $D(A)$ is dense, $A^*f = \bar{h}$ is well-defined

But ~~$D(A)$~~ is not necessarily dense. i.e. $(A^*)^*$?

- $A \subset B \Rightarrow B^* \subset A^*$
- $(A+B)^* \supset A^* + B^*$
- $(AB)^* \supset B^* A^*$

$$B_A = \{ \langle f, Af \rangle \in \mathcal{H} \times \mathcal{H} \mid f \in D(A) \}$$

is called the graph of A .

Topology is given by

$$\begin{aligned} \langle f_n, g_n \rangle \in B_A & \quad \langle f_n, g_n \rangle \rightarrow \langle f, g \rangle \\ & \Leftrightarrow f_n \rightarrow f \\ & \qquad g_n \rightarrow g \end{aligned}$$

Lemma 5.7

H is a closed op $\Leftrightarrow B_A$ is a closed graph

$$\textcircled{(1)} \quad (\Rightarrow) \quad B_A \ni \langle f_n, Af_n \rangle \rightarrow \text{conv} = \langle f, g \rangle$$

Hence $f \in D(H)$ and $g = Af$, and $\langle f, g \rangle \in B_A$.

$$(\Leftarrow) \quad \text{If } f_n \in D(A) \text{ and } Af_n \rightarrow g.$$

This implies that $\langle f, g \rangle \in B_A$.

Then $g = Af$ $f \in D(A)$,

Lemma 5.8 A is a closable op.

$$(1) \quad \overline{B_A} = B_{\bar{A}}$$

$$(2) \quad \frac{B_A}{B_A} = B_C \Leftrightarrow \bar{A} = C$$

$$\textcircled{(2)} \quad (1) \quad B_A \subset B_{\bar{A}} \text{ then } \overline{B_A} \subset B_{\bar{A}}$$

$$B_{\bar{A}} \ni \langle f, \bar{A}f \rangle \text{ then } \langle f_n, A f_n \rangle \rightarrow \langle f, \bar{A}f \rangle$$

$$\therefore \langle f, \bar{A}f \rangle \in \overline{B_A} \quad \in B_A$$

(2) $B_C \ni \langle f, Cf \rangle$ is an accumulate point of B_A .

Then $\langle f, Cf \rangle = \lim_n \langle f_n, Af_n \rangle$
 where $Cf = \bar{A}f$ " "

Relationship between adjoint and graph

$$\overline{B_A} = B_{\bar{A}} \quad \text{ok}$$

$$B_{A^*} = ?$$

Lemma 5.9

$$B_{A^*} = U(B_A^\perp) = (UB_A)^\perp$$

$$\text{where } U \langle x, y \rangle = \langle y, -x \rangle$$



$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} \text{ rotating by } -\frac{\pi}{2}.$$

Note that $UU = -I$, $U^*U = UU^* = I$.

$$\textcircled{2} (UB_A)^\perp \subset B_{A^*} \Leftrightarrow (UB_A)^\perp \ni \langle f, g \rangle$$

$$UB_A \ni \langle Ah, -h \rangle$$

Hence $\langle Ah, -h \rangle \perp \langle f, g \rangle \quad \forall h \in D(A)$

$$\therefore (Ah f) - (h g) = 0 \quad \therefore f \in D(A^*) \text{ and}$$

$$\therefore \langle f, g \rangle \in B_{A^*} \quad g = A^*f$$

$$(UB_A)^\perp \supset B_A^* \Rightarrow B_A^* \supset \langle f, A^*f \rangle$$

$$(UB_A) \supset \langle Ah, -h \rangle \quad \text{by 2.12}$$

$$(\langle f, A^*f \rangle, \langle Ah, -h \rangle)$$

$$= (f, Ah) - (A^*f, h) = 0$$

$$\therefore \langle f, A^*f \rangle \in (UB_A)^\perp \quad ,$$

$$(UB_A)^\perp = \cup(B_A^\perp) \quad \text{ok}$$

We assume that

Lemma 5.10 $\vee D(A)$ is dense

- ① A^* is closed (we do not know that A^* is densely df.)
- ② A is closed
 $\Rightarrow D(A^*)$ is dense and $A^{**} = A$
- ③ A is closable $\Rightarrow (\bar{A})^* = A^*$
- ④ A is closable $\Leftrightarrow D(A^*)$ is dense.