

2022/10/14

第2回目 8

Def 2.8 conditional expectation

$$X : (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ r.v.}$$

$$\text{and } X \in L^1, \quad \Omega \subset \mathcal{F}$$

$Y$  is called the conditional expectation of  $X$  given  $\Omega$

$\Leftrightarrow$  (1)  $Y$  is  $\Omega$ -measurable

$$(2) E[1_A X] = E[1_A Y] \quad \forall A \in \Omega$$

$$E[X | \Omega] = Y$$

Thm 2.9  $E[X | \Omega]$  exists and unique.

Note that  $E[E[X | \Omega]] = E[X]$ .

Lemma 2.10

$$(\Omega, \mathcal{F}) \xrightarrow{X} (S, \mathcal{G})$$

$$\begin{array}{ccc} Y & \downarrow & \\ & \swarrow \exists f & \\ (\mathbb{R}, \mathcal{B}(\mathbb{R})) & & \end{array}$$

Suppose that  $Y$  is  $\mathcal{G}(x)$ -measurable

Then  $\exists f : (S, \mathcal{G}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  s.t

$$Y = f \circ X$$

$$\therefore \mathcal{L} = \left\{ Y : \mathcal{G}(x)\text{-meas} \mid \exists f : S \rightarrow \mathbb{R} \text{ s.t. } Y = f \circ X \right\}$$

$$\cdot A \in \mathcal{G}(x). \quad Y = 1_A = 1_X \circ \exists f = 1_B(x) \quad \therefore 1_A \in \mathcal{L}$$

$$\cdot Y = \sum q_j 1_{A_j} \in \mathcal{L} \quad q_j \geq 0 \quad A_j \in \mathcal{G}(x)$$

$$\begin{aligned} & \cdot \mathcal{G}(x)\text{-meas } \exists f_n \text{ s.t. } f_n \uparrow Y \quad \therefore \quad \exists f_n = \exists f_n \circ X \\ & \text{so } f = \overline{\lim_n} f_n \text{ is } \mathcal{G}(x)\text{-meas.} \quad f \circ X = Y. \quad \therefore Y \in \mathcal{L} \end{aligned}$$

$\mathbb{E}[Y | \sigma(X)]$  is  $\sigma(X)$ -meas.

Hence  $\exists f : S \rightarrow \mathbb{R}$  s.t.  $\mathbb{E}[Y | \sigma(X)] = f(X) = f \circ X$

We express  $f(x)$  as

$$f(x) = \mathbb{E}[Y | X=x]$$

Example 2.11

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y | \sigma(X)]] = \mathbb{E}[f(x)]$$

$$= \int_S f(x) dP_X(x) = \int_S \mathbb{E}[Y | X=x] dP_X(x).$$

Example of conditional exp.

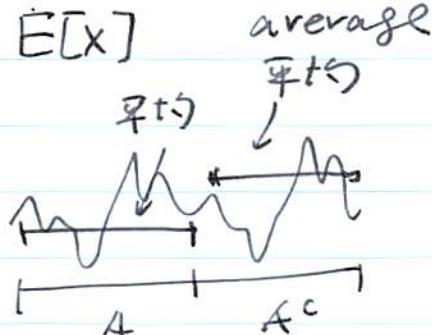
$$(1) \Omega^{\sigma} = \sigma(\{\emptyset, \Omega\}) = \{\Omega, \emptyset\}$$

$\mathbb{E}[X | \Omega^{\sigma}]$  = constant function i.e.  $E[X]$  average

$$(2) \Omega^{\sigma} = \sigma(\{A\}) = \{\emptyset, A, A^c, \Omega\}$$

$$\mathbb{E}[X | \Omega^{\sigma}] = a \mathbb{1}_A + b \mathbb{1}_{A^c}$$

$$= \frac{\mathbb{E}[\mathbb{1}_A X]}{\mathbb{E}[A]} \mathbb{1}_A + \frac{\mathbb{E}[\mathbb{1}_{A^c} X]}{\mathbb{E}[A^c]} \mathbb{1}_{A^c}$$



## § 3 Stochastic processes

$X_t : (\Omega, \mathcal{F}) \rightarrow (S, \mathcal{S})$  meas.

$(X_t)_{t \in J}$  Stochastic process

Ex  $J = [0, \infty)$ ,  $\mathbb{R}$  etc

Ex  $S = \mathbb{R}^d$ ,  $\mathcal{N} = \{0, 1, 2, \dots\}$

- (A)  $w \mapsto X_t(w)$  r.v.  $t$ : fix
- (B)  $t \mapsto X_t(w)$  path  $w$ : fix

Def 3.1 BM = Brownian Motion

$B_t : (\Omega, \mathcal{F}, P) \rightarrow (\mathbb{R}^d; \mathcal{B}(\mathbb{R}^d))$

$$(1) P(\{w \in \Omega \mid B_t(w) = x\}) = 1$$

(2)  $B_{t_j} - B_{t_{j-1}}$ ,  $0 > t_0 < t_1 < \dots < t_n$  is independent, Gaussian r.v. s.t.

$$dP_{B_t - B_s} = (2\pi(t-s))^{-d/2} \exp\left(-\frac{|x|^2}{2(t-s)}\right)$$

(3)  $t \mapsto B_t(w)$  is cont a.e.  $w \in \Omega$ .

$x=0$  standard BM.



$$\textcircled{1} \quad \mathbb{E}^x[B_t - B_s] = \int y P_{t-s}(y) dy = 0$$

$$\mathbb{E}^x[B_t] = \mathbb{E}^x[B_t - B_0] + \mathbb{E}^x[B_0] = x$$

$$\textcircled{2} \quad \mathbb{E}^x[e^{iu(B_t - B_s)}] = \int e^{iu \cdot y} P_{t-s}(y) dy = e^{-\frac{1}{2}|t-s|^2}$$

$$\textcircled{3} \quad \mathbb{E}^x[(B_t - x)(B_s - x)] = t \wedge s$$

$$\textcircled{4} \quad \mathbb{E}^x[f(B_t)] = \mathbb{E}[f(\underline{B_t - B_0 + B_0})]$$

$$= \int f(x+y) P_t(y) dy = (f * P_t)(x)$$

$$(B_t)_t : \text{BM} \rightarrow \mathbb{E}^x[f(B_t)] = (e^{-\frac{1}{2}(t-s)} f)(x)$$

$$\textcircled{5} \quad \mathbb{E}^x[f_1(B_{t_1}) f_2(B_{t_2}) \dots f_n(B_{t_n})] \quad \text{if } b \text{ dd}$$

$$= \int_{(\mathbb{R}^d)^n} \prod_{j=1}^n f_j(x_j) \prod_{j=1}^n P_{t_j-t_{j-1}}(x_j - x_{j-1}) dx_1 \dots dx_n \quad \begin{matrix} 0 \leq t_1 < \dots < t_n \\ x_0 = x \end{matrix}$$

$$\therefore \mathbb{E}^x[f_1(B_{t_1} - B_0) f_2(B_{t_2} - B_{t_1} + B_{t_1} - B_0) \dots f_n(B_{t_n} - B_{t_{n-1}} + B_{t_{n-1}} - B_{t_{n-2}} + \dots + B_{t_1} - B_0)]$$

$$B_{t_j} - B_{t_{j-1}} = X_j \quad j = 1, \dots, n$$

$$= \mathbb{E}^x[f_1(X_1) \dots f_n(X_n + \underbrace{\dots + X_1}_{+x})] \quad \begin{matrix} x_i \neq x_j \end{matrix}$$

$$= \int f_1(x_1) f_2(x_1 + x_2) \dots f_n(x_1 + \dots + x_n) P_{t_1-t_0}(x_1) \dots P_{t_n-t_{n-1}}(x_n) dx_1 \dots dx_n$$

$$= \int f_1(z_1) f_2(z_2) \dots f_n(z_n) P_{t_1-t_0}(z_1) P_{t_2-t_1}(z_2 - z_1) \dots P_{t_n-t_{n-1}}(z_n - z_{n-1})$$

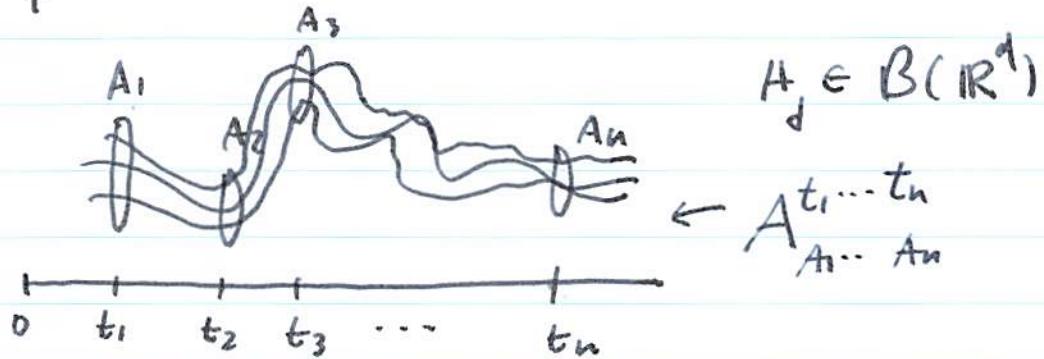
### Thm 3.2 Existence of BM

$$\mathcal{X} = C([0, \infty) : \mathbb{R}^d)$$

$$\mathcal{B} = \sigma(\mathcal{C}) \quad \mathcal{C} \text{ cylinder sets}$$

$\exists W^*$  prob. measure on  $(\mathcal{X}, \mathcal{B})$  s.t.

$$B_t(w) = w(t), \quad w \in \mathcal{X} \text{ is BM.}$$



$$A_{A_1 \dots A_n}^{t_1 \dots t_n} = \{ w \in \mathcal{X} \mid w(t_j) \in A_j \quad j=1 \dots n \}$$

$$\mathcal{C} = \left\{ A_E^{\# \Omega} \mid \Omega \in [0, \infty) \quad \#\Omega < \infty \right. \\ \left. E = E_1 \times \dots \times E_{\#\Omega} \in \mathcal{B}(\mathbb{R}^{\#\Omega}) \right\}$$

Rem  $\mathcal{X} \ni f, g$

$$d(f, g) = \frac{1}{2} \left( \sup_{0 \leq k \leq \infty} \frac{|f(x_k) - g(x_k)|}{2^k} \right) \wedge 1$$

$$f_n \rightarrow f \Leftrightarrow d(f, f_n) \rightarrow 0$$

$$(\mathcal{X}, \theta) \text{ topological space} \rightarrow \mathcal{B}(\theta) = \sigma(\mathcal{C})$$