

## §12 Schrödinger operators with vector potentials

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$$H(a) = \frac{1}{2}(-i\nabla - a)^2 + V, \text{ where } a = (a_1, \dots, a_d) \in (C^\infty_0)^d$$

$$\begin{aligned} (Qf)(x) &= E^x \left[ e^{-\int_0^t V(B_s) ds} e^{-i \int_0^t a(B_s) \cdot dB_s} f(B_t) \right] \\ &= E^0 \left[ e^{-\int_0^t V(B_s+x) ds} e^{-i \int_0^t a(B_s+x) \cdot dB_s} f(B_t+x) \right] \end{aligned}$$

$$Q_t = e^{-t^3 K} \quad \Sigma \text{ Fr. } C t^{2/3}, \quad \begin{array}{l} \text{Suppose that } V \text{ is bdd} \\ \text{and } V \geq 0 \end{array}$$

Lemma 12.1  $Q_t$  is bdd.

$$\textcircled{1} \quad \int |(Qf)(x)|^2 dx = \int |E^x[f(B_t)]|^2 dx = \|e^{-t^3 h} f\|^2$$

where  $h = -\frac{1}{2} \Delta$ . Then  $\|Q_t f\| \leq \|f\|$ ,

Lemma 12.2

$$(f, Q_t g) = (Q_t f, g) \quad \text{i.e., } Q_t^* = Q_{t+}$$

$$\textcircled{2} \quad \textcircled{2} = \int dx E^0 \left[ \overline{f(x)} e^{-\int_0^t V(B_s+x) ds} e^{-i \int_0^t a(B_s+x) \cdot dB_s} g(B_t+x) \right]$$

$\dot{B}_s = B_{t-s} - B_t$  is also BM.

( $0 \leq s \leq t$ )

$$= \int dx E^0 \left[ \overline{f(x)} e^{-\int_0^t V(\dot{B}_s+x) ds} e^{-i \int_0^t a(\dot{B}_s+x) \cdot d\dot{B}_s} g(\dot{B}_t+x) \right]$$

$$\dot{B}_t + \chi = y$$

$$= \int dy \mathbb{E}^0 \left[ f(y) e^{-\int_0^t V(B_s + y) ds} - i \int_0^t a(B_s + y) \circ dB_s g(y) \right]$$

$\cancel{y - \dot{B}_t}$

$$= \int dy \mathbb{E}^0 \left[ f(y + \dot{B}_t) e^{-\int_0^t V(B_s + y) ds} + i \int_0^t a(B_s + y) \circ dB_s g(y) \right]$$

$$\textcircled{*} = \int_0^t a(B_s + \chi) \circ dB_s$$

$$= \lim_n \frac{1}{2} \sum_{j=0}^{2^n-1} \{a(\dot{B}_{t_j+n} + \chi) + a(\dot{B}_{t_j+n} + \chi)\} (\dot{B}_{t_{j+1}} - \dot{B}_{t_j})$$

$$\text{where } t_j = \frac{t_j}{2^n} \quad j = 0, \dots, 2^n$$

$$= \lim_n \frac{1}{2} \sum_{j=0}^{2^n-1} \{a(B_{t-t_{j+1}} - B_t + \chi) + a(B_{t-t_j} - B_t + \chi)\} \\ \cdot \{B_{t-t_{j+1}} - B_{t-t_j}\}$$

$$\text{変数変換 } \dot{B}_t + \chi = \chi - \dot{B}_t = y$$

$$\rightarrow \lim_n \frac{1}{2} \sum_{j=0}^{2^n-1} \{a(B_{t-t_{j+1}} + y) + a(B_{t-t_j} + y)\} \\ \cdot (B_{t-t_{j+1}} - B_{t-t_j})$$

$$= - \int_0^t a(B_s + y) \circ dB_s$$

$$= \int dy \overline{\mathbb{E}^0 \left[ e^{-\int_0^t V(B_s + y) ds} - i \int_0^t a(B_s + y) \circ dB_s \right]} f(y + \dot{B}_t) g(y)$$

= (Q\_t f, g) //

Lemma 12.3  $Q_t Q_s = Q_{t+s}$

$$\text{Since } A_0^t(x) = \int_0^t V(B_s+x)ds + i \int_0^t a(B_s+x)odB_s.$$

$$(Q_t f)(x) = \mathbb{E}^x \left[ e^{-A_0^t(x)} f(B_t) \right].$$

$$\text{Hence } Q_t Q_s f(x) = \mathbb{E}^x \left[ e^{-A_0^t(x)} \mathbb{E}^{B_t} \left[ e^{-A_0^s(x)} f(B_s) \right] \right]$$

By the Markov property of  $(B_t)$ , we have

$$= \mathbb{E}^x \left[ e^{-A_0^t(x)} \mathbb{E}^x \left[ e^{-A_t^{t+s}(x)} f(B_{t+s}) \mid \mathcal{F}_t \right] \right]$$

Since  $A_0^t(x)$  is  $\mathcal{F}_t$ -measurable,

$$= \mathbb{E}^x \left[ \mathbb{E}^x \left[ e^{-A_0^t(x)} e^{-A_t^{t+s}(x)} f(B_{t+s}) \mid \mathcal{F}_t \right] \right]$$

$$= \mathbb{E}^x \left[ e^{-A_0^{t+s}(x)} f(B_{t+s}) \right] = (Q_t Q_s f)(x) //$$

Lemma 12.4  $t \mapsto Q_t f$  is cont in  $L^2$ .

$\because$  Since  $Q_t Q_s = Q_{t+s}$ , it is enough to show that  $Q_t f$  is cont at  $t=0$ .

$$(Q_t f - Q_0 f) = \mathbb{E}^x \left[ e^{-A_0^t(x)} f(B_t) - f(B_0) \right]$$

- $f, g \in C_0^\infty(\mathbb{R}^d)$  とす

$$(f, Q_t g - Q_0 g)$$

$$= \int_{\mathbb{R}^d} \bar{f}(x) \mathbb{E}^x \left[ \left( e^{-t \Delta_0} - 1 \right) g(B_t) \right] dx - A$$

$$+ \int_{\mathbb{R}^d} \bar{f}(x) \mathbb{E}^x \left[ g(B_t) - g(B_0) \right] dx - B$$

$\lim_{t \rightarrow 0} A = 0 \quad (\because \text{Lebesgue dominated conv.})$

$\lim_{t \rightarrow 0} B = \lim_{t \rightarrow 0} (f, e^{-th} g - g) = 0 \quad \text{when } h = -\frac{1}{2}\Delta$

- Since  $\|Q_t\| \leq 1$ , by an approximation

$$(f, Q_t g - Q_0 g) \rightarrow 0 \quad \text{for } f, g \in L^2$$

$$\begin{aligned} \|Q_t f - Q_0 f\|^2 &= \|Q_t f\|^2 - 2 \operatorname{Re}(Q_t f, Q_0 f) \\ &\quad + \|Q_0 f\|^2 \end{aligned}$$

$$= (f, Q_0 f) - 2 \operatorname{Re}(Q_t f, f) + \|f\|^2$$

$$\rightarrow 0 \quad \text{as } t \rightarrow 0$$

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By Lemmas 12.1~12.4.

$$\exists \text{ s.t. } K \text{ s.t. } Q_t = e^{-tK} \quad \forall t \geq 0.$$

~~suppose  $f \in C_0^\infty$~~

$$Y_t = f(B_t) e^{-X_t}, \quad \text{where}$$

$$X_t = \int_0^t V(B_s) ds + i \int_0^t a(B_s) \circ dB_s$$

$$= \int_0^t V(B_s) ds + i \int_0^t a(B_s) \cdot dB_s + i \frac{1}{2} \int_0^t (\nabla \cdot a)(B_s) ds$$

$$\therefore dX_t = \left( V + \frac{1}{2} i (\nabla \cdot a) \right) dt + i a \cdot dB_t$$

$$dY_t = df \cdot e^{-X_t} + f \cdot d\bar{e}^{-X_t} + df \cdot d\bar{e}^{-X_t}$$

$$df = \nabla f \cdot dB_t + \frac{1}{2} \Delta f dt$$

$$d\bar{e}^{-X_t} = -\bar{e}^{-X_t} dX_t + \frac{1}{2} \bar{e}^{-X_t} (dX_t)^2$$

$$= \bar{e}^{-X_t} \left( -dX_t + \frac{1}{2} (dX_t)^2 \right)$$

$$= \bar{e}^{-X_t} \left( (-V - \frac{1}{2} i \nabla \cdot a) dt - i a \cdot dB_t + \frac{1}{2} (-a^2) dt \right)$$

$$= \bar{e}^{-X_t} \left\{ \left( -V - \frac{1}{2} i \nabla \cdot a - \frac{1}{2} |a|^2 \right) dt - i a \cdot dB_t \right\}$$

$$dY_t = e^{-X_t} \left( \frac{1}{2} \Delta f dt + \nabla f \cdot dB_t \right)$$

$$+ e^{-X_t} f \left\{ \left( -V - \frac{1}{2} i D a - \frac{1}{2} |a|^2 \right) dt - i a \cdot dB_t \right\}$$

$$+ e^{-X_t} (-i a \cdot \nabla f) dt$$

$$\begin{aligned} \therefore \mathbb{E}[dY_t] &= \mathbb{E} \left[ \int_0^t e^{-X_s} \left[ \frac{1}{2} \Delta f(B_s) - V(B_s) f(B_s) \right. \right. \\ &\quad \left. \left. - \frac{1}{2} i (D \cdot a)(B_s) f(B_s) - \frac{1}{2} |a(B_s)|^2 f(B_s) \right. \right. \\ &\quad \left. \left. - i a(B_s) \cdot \nabla f(B_s) \right] ds \right] \end{aligned}$$

$$\begin{aligned} H(a)f &= \left[ \frac{1}{2} (P-a)^2 + V \right] f = \left( \frac{1}{2} P^2 - \frac{1}{2} Pa - \frac{1}{2} aP + \frac{1}{2} a^2 + V \right) f \\ &= \left[ -\frac{1}{2} \Delta - a(-iD) + \frac{1}{2} i(D \cdot a) + \frac{1}{2} |a|^2 + V \right] f \end{aligned}$$

$$\therefore \mathbb{E}[dY_t] = \mathbb{E} \left[ \int_0^t e^{-X_s} - (H(a)f)(B_s) ds \right]$$

$\therefore Y_t = Y_0 + \int_0^t dY_s$

$$\frac{1}{t} \mathbb{E}[Y_t - Y_0] = \frac{1}{t} \mathbb{E} \left[ \int_0^t e^{-X_s} - (H(a)f)(B_s) ds \right]$$

$$\begin{aligned} & \because (g, Q_t f - f) \frac{1}{t} \\ &= \frac{1}{t} \int g(x) \mathbb{E}^x [Y_t - Y_0] dx \end{aligned}$$

$$= \frac{1}{t} \int g(x) \mathbb{E}^x \left[ \int_0^t -H(a)f(B_s) e^{-X_s} ds \right] dx$$

$$\rightarrow (g, -H(a)f) \text{ as } t \rightarrow 0.$$

On the other hand  $\frac{1}{t}(Q_t f - f) = \frac{1}{t} \{ e^{tK} f - f \}$   
 $\rightarrow -Kf \text{ as } t \rightarrow 0$

$$\therefore K = -H(a) \text{ on } C_0^\infty(\mathbb{R}^d)$$

Since  $H(a)$  is ess s.g. on  $C_0^\infty(\mathbb{R}^d)$ ,

$K = H(a)$  as a self-adjoint operator

### Theorem 12.5

$$(f, \bar{e}^{-tH(a)} g) = \int dx \bar{g}(x) \mathbb{E}^x \left[ \bar{e}^{-\int_0^t \nu(B_s) ds} - \bar{e}^{-\int_0^t a \cdot dB_s} f(B_t) \right]$$