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$$X_t = X_0 + \int_0^t g(s) ds + \int_0^t f(s) dB_s$$

= 4E $dx_t = g dt + f dB_t$ と表すことができる。

$$Y_t = g(x_t) \quad \text{に } \dots \text{ を } \dots \text{ する}$$

Thm $g \in C_b^3(\mathbb{R} \times \mathbb{R})$ のとき

$$\begin{aligned} dg(x_t) &= \dot{g} dt + \nabla g dx_t + \frac{1}{2} \nabla^2 g (dx_t)^2 \\ &= \dot{g} dt + \nabla g dt + \nabla g dB_t + \frac{1}{2} \nabla^2 g dt \end{aligned}$$

Examples

(1) $Y_t = f(B_t)$

$$dY_t = \nabla f dB_t + \frac{1}{2} \Delta f dt$$

$$f(B_t) - f(B_0) = \int f' dB_t + \int f'' dt = \int f' \circ dB_t$$

(2) $Y_t = f(t, B_t), \quad f(t, x) = h(t) \cdot x$

$$dY_t = \dot{f} dt + \nabla f dB_t + \frac{1}{2} \Delta f \cdot dt$$

$$= \dot{h} dt + h dB_t$$

$$\therefore \int h(\dot{B}_s) dB_s = h(t) B_t - \int \dot{h}(s) ds$$

i.e., $\int h(s) \dot{B}_s ds = h(t) B_t - \int \dot{h} B_s ds$



$$(3) \quad Y_t = e^{-\int_0^t V(B_s) ds}$$

$$\text{かつ } X_t = \int_0^t V(B_s) ds$$

$$dY_t = \nabla f dX_t + \frac{1}{2} \nabla^2 f (dX_t)^2$$

$$\text{i.s. } dX_t = V(B_t) dt$$

$$f(x) = e^{-x}$$

$$= \nabla f V dt + \frac{1}{2} \nabla^2 f \cdot 0$$

$$\therefore e^{-\int_0^t V ds} - 1 = -\int_0^t V(B_s) e^{-\int_0^s V(B_s) ds} ds$$

d次元 Brownian motion $B_t = (B_t^1, \dots, B_t^d)$

$$\int f(\omega) \cdot dB_s = \sum_{j=1}^d \int f_j(\omega) \cdot dB_s^j$$

$$\text{Then } (1) \quad \mathbb{E} \left[\int_0^t f(\omega) \cdot dB_s \right] = 0$$

$$(2) \quad \mathbb{E} \left[\int_0^t f(\omega) \cdot dB_s^i \int_0^t g(\omega) \cdot dB_s^j \right] \\ = \delta_{ij} \int_0^t \mathbb{E} [f(\omega) g(\omega)] ds.$$



$$X_t^j = X_0^j + \int b^j ds + \int \sigma^j dB_s \quad j=1, \dots, n$$

$$dX_t^j = b^j dt + \sigma^j \cdot dB_t$$

$$h = (h^1, \dots, h^m) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^m)$$

$$Y_t^h = h^k(t, X_t) \quad \text{where } X_t = (X_t^1, \dots, X_t^n)$$

このとき

$$dY_t^h = h^k dt + \nabla h^k \cdot dX_t + \frac{1}{2} \nabla^2 h^k (dX_t)^2$$

以下に証明

$$\nabla h^k \cdot dX_t = \sum_{j=1}^n \nabla_j h^k \cdot dX_t^j$$

$$\nabla^2 h^k (dX_t)^2 = \sum_{i,j=1}^n \nabla_i \nabla_j h^k dX_t^i dX_t^j \quad n \times n$$

$$dX_t^i dX_t^j = (b^i dt + \sigma^i \cdot dB_t)(b^j dt + \sigma^j \cdot dB_t)$$

$$= \sigma^i \cdot dB_t \sigma^j \cdot dB_t$$

$$= \sum_{\alpha, \beta} \sigma_\alpha^i d\beta_t^\alpha \sigma_\beta^j d\beta_t^\beta$$

$$= \sum_{\alpha} \sigma_\alpha^i \sigma_\alpha^j dt$$

$$\sigma = \left(\sigma_\alpha^i \right)_{\substack{1 \leq i \leq n \\ 1 \leq \alpha \leq d}} \quad \therefore \sum \sigma_\alpha^i \sigma_\alpha^j = (\sigma \sigma^T)_{ij}$$

$$\begin{aligned} \therefore dY_t^h &= \left(h^k + h_j^k b^j + \frac{1}{2} h_{ij}^k (\sigma \sigma^T)_{ij} \right) dt \\ &\quad + \left(h_j^k \sigma^j \cdot dB_t \right) \end{aligned}$$

$n \times n$ matrix



Cor X_t, Y_t is Itô-process とある.

2変数

$$d(X_t Y_t) = dX_t \cdot Y_t + X_t \cdot dY_t + dX_t \cdot dY_t$$

☺

$$dX_t = g_1 dt + f_1 dB_t$$

$$dY_t = g_2 dt + f_2 dB_t$$

$$\begin{aligned} d &= d \\ m &= 1 \\ n &= 2 \end{aligned}$$

$$P = P(x, y) = x \cdot y$$

$$Z_t = P(X_t, Y_t) = X_t Y_t$$

$$\begin{aligned} \therefore dZ_t &= X_t g_2 dt + \frac{1}{2} \cdot 2 \cdot f_1 f_2 dt + Y_t g_1 dt \\ &\quad + (Y_t f_1 + X_t f_2) dB_t \end{aligned}$$

$$= X_t dY_t + Y_t dX_t + dX_t \cdot dY_t //$$

Example $f(B_t) e^{-\int_0^t v(B_s) ds} = Z_t$

$$\begin{aligned} dZ_t &= df \cdot e^{-\int_0^t v} + f \cdot d e^{-\int_0^t v} + df \cdot d e^{-\int_0^t v} \\ &= (\nabla f dB_t + \frac{1}{2} \Delta f \cdot dt) e^{-\int_0^t v} \end{aligned}$$

$$+ f \cdot \left(- e^{-\int_0^t v} dt \right)$$

$$= \left(\frac{1}{2} \Delta f - \nabla f \cdot v \right) e^{-\int_0^t v} dt + \nabla f e^{-\int_0^t v} dB_t$$

$$= -Hf \quad \text{where } H = -\frac{1}{2} \sigma^2 v.$$