

2023/1/20

$$B_t - B_s \sim dB \sim \sqrt{dt}$$

Lemma 10.6

$$\sum_j \Delta_j^2 \rightarrow t \text{ as } n \rightarrow \infty \text{ in } L^2(\Omega)$$

where $\Delta_j = B_{t_j} - B_{t_{j-1}}$, $t_j = \frac{tj}{2^n}$ $j = 1, \dots, 2^n$.

$$\textcircled{1} \mathbb{E}[|\sum_j \Delta_j^2 - t|^2] = \mathbb{E}(\sum_j \Delta_j^2)^2 - 2t \sum_j \Delta_j^2 + t^2$$

$$= \mathbb{E}\left[\sum_{j=1}^{2^n} \Delta_j^4 + 2 \sum_{i < j} \Delta_j^2 \Delta_i^2 - 2t \sum_j \Delta_j^2 + t^2\right]$$

$$\mathbb{E}[\Delta_j^4] = 3\left(\frac{t}{2^n}\right)^2 \mathbb{E}[\Delta_j^2] = \frac{t}{2^n} \quad \text{Hence } \mathbb{E}[\dots] = \frac{t}{2^n} 3\left(\frac{t}{2^n}\right)^2 + 2 \frac{2^n(2^n-1)}{2} \left(\frac{t}{2^n}\right)^2 - 2t 2^n \frac{t}{2^n} + t^2 \rightarrow 0$$

Similarly we can see that

$$\sum_{j=1}^{2^n} \Delta_j^p \rightarrow 0 \quad (p \geq 3)$$

Lemma 10.7

$$(1) \lim_n \sum_j f(B_{t_j}) \Delta_j \rightarrow \int_0^t f(B_s) ds \quad f \in C_b(\mathbb{R})$$

$$(2) \lim_n \sum_j \frac{1}{2} (f(B_{t_j}) + f(B_{t_{j-1}})) \Delta_j \rightarrow \int_0^t f(B_s) dB_s + \frac{1}{2} \int_0^t f'(B_s) ds$$

$f \in C_b^2(\mathbb{R})$

講義で C_b^1 と f が C_b^2 十分

$$\textcircled{1} \phi_n(t) = \sum_j f(B_{t_{j-1}}) \mathbb{1}_{[t_{j-1}, t_j)}(t)$$

$$\mathbb{E} \int_0^t \left| \sum_j (f(B_{t_{j-1}}) - f(s)) \mathbb{1}_{[t_{j-1}, t_j)}(s) \right|^2 ds \quad - (*)$$

$$= \mathbb{E} \int_0^t \sum_j |f(B_{t_{j-1}}) - f(s)|^2 \mathbb{1}_{[t_{j-1}, t_j)}(s) ds$$

$|f(B_{t_{j-1}}) - f(s)| < \varepsilon$ ($\forall s \in [t_{j-1}, t_j)$) $\therefore f(B_s)$ is uniformly continuous.

$$\therefore \int_0^t \sum_j |f(B_{t_{j-1}}) - f(s)|^2 \mathbb{1}_{[t_{j-1}, t_j)}(s) ds \leq \varepsilon^2 t$$

$$\therefore (*) \leq \varepsilon^2 t \quad //$$

$$\textcircled{2} \frac{1}{2} (f(B_{t_j}) + f(B_{t_{j-1}})) \Delta_j = f(B_{t_{j-1}}) \Delta_j + \frac{1}{2} (f(B_{t_j}) - f(B_{t_{j-1}})) \Delta_j$$

$$\therefore \sum f(B_{t_{j-1}}) \Delta_j \rightarrow \int f(B_s) dB_s$$

$$\frac{1}{2} \sum (f(B_{t_j}) - f(B_{t_{j-1}})) \Delta_j \rightarrow \frac{1}{2} \int f'(s) ds \quad \text{by } \textcircled{2}$$

$$(f(B_{t_j}) - f(B_{t_{j-1}})) \Delta_j = f'(B_{t_{j-1}}) \Delta_j^2 + \frac{1}{2} f''(\xi_j) \Delta_j^3$$

$$\sum f'(B_{t_{j-1}}) \Delta_j^2 = \int_{[t_{j-1}, t_j)} f'(B_s) ds \quad - (1)$$

$$+ \sum \frac{1}{2} f''(\xi_j) \Delta_j^3 = \int f'(B_s) dB_s \quad - (2)$$

$$\textcircled{1} \mathbb{E} \left[\left(\sum |f'(B_{t_{j-1}})| |\Delta_j^2 - \delta_j| \right)^2 \right] \text{ where } \delta_j = t_j - t_{j-1}$$

$$\leq C^2 \mathbb{E} \left[\sum |\Delta_j^2 - \delta_j|^2 \right]$$

$$\left(\sum_j |\Delta_j^2 - \delta_j| \right)^2 = \sum_j |\Delta_j^2 - \delta_j|^2 + 2 \sum_{i < j} |\Delta_j^2 - \delta_j| |\Delta_i^2 - \delta_i|$$

$$= \sum_j \left(\Delta_j^4 - 2 \Delta_j^2 \delta_j + \delta_j^2 \right) + 2 \sum_{i < j} |\Delta_j^2 - \delta_j| |\Delta_i^2 - \delta_i|$$

$\therefore \mathbb{E} \sum_j \Delta_j^4 \rightarrow 0$

$$\mathbb{E} \sum_j \Delta_j^4 \rightarrow 0, \quad -2 \frac{t}{2^n} \cdot \frac{t}{2^n} \cdot 2^n, \quad \sum_j \delta_j^2 \rightarrow 0$$

$$2 \sum_{i < j} \mathbb{E} [\Delta_j^2 - \delta_j] \mathbb{E} [\Delta_i^2 - \delta_i]$$

$$= 2 \cdot \frac{2^n(2^n - 1)}{2} \cdot 0 \rightarrow 0.$$

② is Riemann int or det //

$$\int_0^t f(B_s) dB_s + \frac{1}{2} \int_0^t f'(B_s) d\langle B \rangle_s = \int_0^t f(B_s) \circ dB_s$$

is called Stratonovich integral

§11 Itô formulas

$X_t = \int_0^t f(B_s) dB_s$ is a new stochastic process.

$$X_t = X_0 + \int_0^t b(s) ds + \int_0^t f(B_s) dB_s$$

where X_0 is a ~~r.v.~~ random variable,

$\int_0^t |b(s)| ds < \infty$ a.s. for $\forall t$ and $b(s)$ is

\mathcal{F}_s -measurable for every s .

$f \in M(0, \infty)$.

We write

X_t is called a Itô process. $dX_t = g dt + f dB_t$

Now we defined another process:

$$Y_t = g(t, X_t)$$

Thm 11.1 Let $g \in C_b^3([0, \infty) \times \mathbb{R})$.

Then $Y_t = g(t, X_t)$ is an Itô process s.t.

$$dY_t = (g_t + g_x b + g_x f) dt + \frac{1}{2} g_{xx} f^2 dB_t^2,$$

實之方
$$dY_t = g_t dt + g_x dX_t + \frac{1}{2} g_{xx} (dX_t)^2$$

$$\begin{aligned} (dX_t)^2 &= g^2 (dt)^2 + 2 g dt \cdot f dB_t + f^2 (dB_t)^2 \\ &= 0 + 0 + f^2 dt. \end{aligned}$$