Semi-classical analysis and Wigner measure for non-relativistic QED

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Lamb shift

►We consider a system of QM coupled to QFT (QED or Strong interaction).

►In the Dirac theory, the energy levels $2S_{1/2}$ and $2P_{1/2}$ of a hydrogen atom coincides: $2S_{1/2} = 2P_{1/2}$. It has been however observed in 1947 by Lamb and Retherford that $2S_{1/2} < 2P_{1/2}$

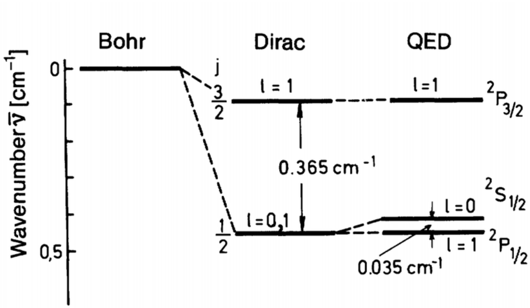


Figure: Lamb shift

►To show the Lamb shift we have to consider an interaction between electrons and a quantized radiation field. This has been established by H. A. Bethe in 1947.

►We consider a system describing an interaction between particles and a quantum field, and derive a classical equation (e.g., Newton-Maxwell equation) due to a semi-classical limit and Wigner measures.

►(Finite dimension) P. L. Lions and T. Paul (1993).

►(Infinite dimension) Z. Ammari and F. Nier (2008,2011).

►(Interaction system) Z. Ammari. M. Falconi (2014),

Z. Ammari. M. Falconi and FH (2019)<—todays talk

Symplectic structures

►$\hbar = h/2\pi = 1.054\,571\,817... \times 10^{-34}\,\mathrm{J\,s}$

►$\hat{P} = -i\hbar D_x$ and $\hat{Q} = x \to$ CCR: $[\hat{P}, \hat{Q}] = -i\hbar$

►$X = (q, p) \in T^*\mathbb{R}^d = \mathbb{R}^d \oplus \mathbb{R}^d \cong \mathbb{C}^d \qquad (q, p) \cong q + ip$

►phase translation: $X = (q, p) \in T^*\mathbb{R}^d$

$$T(X) = \exp\left(i(p\hat{Q} - q\hat{P})\right)$$

►symplectic structure: $X = q + ip, X' = q' + ip'$

$$\sigma(X, X') = qp' - pq' = \mathrm{Im}(X, X')_{\mathbb{C}^d}$$

►Algebraic relations

$$T(X)T(X') = e^{-i\hbar\sigma(X,X')/2}\,T(X + X')$$

Wick symbol

►Gaussian function $\phi_0(x) = (\pi\hbar)^{-d/4}\exp(-\frac{|x|^2}{2\hbar})$

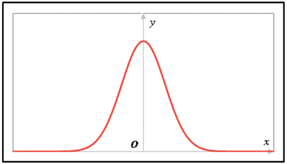


Figure: $\phi_0(x) \to \delta(x)$ as $\hbar \to 0$

►$X = (q,p) \in T^*\mathbb{R}^d$. Baker-Campbel-Huasdorff-formula

$$T(X) = e^{-i\hbar qp/2} e^{ip\hat{Q}} e^{-iq\hat{P}}$$

►$T(X)f(x) = e^{-i\hbar qp/2} e^{ipx} f(x - \hbar q)$

►For $X = (q,p) \in T^*\mathbb{R}^d$ we define $\phi_X = T(\frac{\sqrt{2}}{i\hbar}X)\phi_0$

$$\phi_X \to e^{iqx}\delta(x - \sqrt{2}p) \quad (\hbar \to 0)$$

▶coherent state

$$|\phi_X\rangle\langle\phi_X| \quad X \in T^*\mathbb{R}^d$$

▶We have

$$\int_{T^*\mathbb{R}^d} |\phi_X\rangle\langle\phi_X| \frac{dX}{(2\pi\hbar)^d} = \mathbb{1}_{L^2(\mathbb{R}^d)}.$$

▶$B : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$

▶$\sigma^{Wick}(B)(\cdot) : T^*\mathbb{R}^d \to \mathbb{C}$ is defined by

$$\sigma^{Wick}(B)(X) = (\phi_X, B\phi_X)$$

Lemma

Let B be trace class. Then $\sigma^{Wick}(B)(\cdot) \in L^1(T^*\mathbb{R}^d)$ and

$$\mathrm{Tr}[B] = \int_{T^*\mathbb{R}^d} \sigma^{Wick}(B)(X) \frac{dX}{(2\pi\hbar)^d}$$

If $\mathrm{Tr}[B] = 1$, then $\sigma^{Wick}(B)(X)\frac{dX}{(2\pi\hbar)^d}$ is a prob measure on $T^*\mathbb{R}^d$.

Semi-classical measures = Wigner measures

►Let $(\rho_\hbar)_\hbar$ be the family of non-negative trace classes such that $\mathrm{Tr}[\rho_\hbar] = 1$ for $\hbar > 0$. Then

$$\sigma^{Wick}(\rho_\hbar)(X)\frac{dX}{(2\pi\hbar)^d}$$

is a prob measure on $T^*\mathbb{R}^d$

Definition (Wigner measure)

The weak-$*$ limit of $\sigma^{Wick}(\rho_{\hbar_n})(X)\frac{dX}{(2\pi\hbar_n)^d}$ as $\hbar_n \downarrow 0$ on $T^*\mathbb{R}^d$:

$$\mu = \lim_{\hbar_n \to 0} \sigma^{Wick}(\rho_{\hbar_n})(X)\frac{dX}{(2\pi\hbar_n)^d}$$

is called the Wigner measure. M_W denotes the set of Wigner measures associated with $(\rho_\hbar)_{\hbar\in(0,1)}$

Remark: μ depends on the choice of subsequence $\{\hbar_n\}$.

Characterizaion of Wigner measures

►Let $b \in \mathscr{S}'(T^*\mathbb{R}^d)$.

►The Weyl quantized operator $b^W(x, \hbar D_x) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is given by its kernel

$$b^W(x,y) = \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}\xi\cdot(x-y)} b(\frac{x+y}{2}, \xi) \frac{d\xi}{(2\pi\hbar)^d}.$$

Lemma

Let $\mu \in M_W$. Then there exists a sequence $(\hbar_n)$ such that $\hbar_n \to 0$ and

$$\lim_{n\to\infty} \mathrm{Tr}[b^W(x, \hbar_n D_x)\rho_{\hbar_n}] = \int_{T^*\mathbb{R}^d} b(X) d\mu(X)$$

for $b \in C_0^\infty(T^*\mathbb{R}^d)$.

Lemma

Let N be a harmonic oscillator. Suppose that $\exists \nu > 0$ *such that* $\mathrm{Tr}[N^\nu \rho_\hbar] \leq C_\nu$*. Then*

$$\rho_{\hbar_n} \to \mu \in M_W \Longleftrightarrow \lim_{n\to\infty} \mathrm{Tr}[T(\sqrt{2}\pi Z)\rho_{\hbar_n}] = F^{-1}[\mu](Z)$$

►Fourier transform

$$F^{-1}[\mu](Z) = \int_{\mathbb{C}^d} e^{+2i\pi \mathrm{Re}(\xi,z)} d\mu(\xi)$$

Example 1

►$\rho_\hbar(X) = |\phi_X\rangle\langle\phi_X|$ is trace class for $X \in T^*\mathbb{R}^d$

►It is easy to see that

$$\mathrm{Tr}[\rho_\hbar(X)T(\sqrt{2}\pi Z)] = (\phi_X, T(\sqrt{2}\pi Z)\phi_X) \stackrel{\hbar\to 0}{\to} e^{2\pi i \mathrm{Re}(Z,X)} = F^{-1}[\delta_X](Z)$$

and $M_W = \{\delta_X\}$. I.e., $\# M_W = 1$

►Semi-classical limit of coherent state $|\phi_X\rangle\langle\phi_X|$ is the delta measure δ_X with mass at X on $\mathbb{R}^d \oplus \mathbb{R}^d$

Example 2

►Let $d=1$ and we consider the αth Hermite function ψ_α.
►$\psi_\alpha=(\hbar^\alpha\alpha!)^{-1/2}(\hat{Q}-i\hat{P})^\alpha\phi_0(x)$. Then

$$(-\hbar^2\Delta+x^2-\hbar)\psi_\alpha=\alpha\psi_\alpha$$

►The Wigner measure associated with $\rho_\hbar=|\psi_{[a/\hbar]}\rangle\langle\psi_{[a/\hbar]}|$ is the delta measure with mass on $aS^1\subset\mathbb{R}\oplus\mathbb{R}$:

$$\mu=\frac{1}{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}a}d\theta=\delta_{aS^1}$$

►$d\geq 2$ it follows that

$$\mu=\delta_{a_1S^1}\times\cdots\times\delta_{a_nS^1}$$

Hamilton equation and time evolutions

►Hamiltonian $H = -\hbar^2\Delta + V$ (V is smooth).
►classical Hamiltonian $H(q,p) = p^2 + V(q)$, $(q,p) \in T^*\mathbb{R}^d$
►Hamilton equation

$$\begin{cases} \dot{q}_t = \frac{\partial H}{\partial p} \\ \dot{p}_t = -\frac{\partial H}{\partial q} \end{cases}$$

Hence

$$\begin{cases} \dot{q}_t = 2p_t \\ \dot{p}_t = -\nabla V(q_t) \end{cases}$$

►The flow $\Phi_t : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ is defined by

$$\Phi_t(q,p) = (q_t, p_t).$$

Hamilton equation and time evolutions

Definition (Pure state)

Let $M_W = M_W(\rho_\hbar, \hbar \in (0,1))$ be the set of Wigner measures associated with $(\rho_\hbar)_\hbar$. $(\rho_\hbar)_\hbar$ is pure if and only if $\# M_W = 1$.

Theorem (Lions and Paul 93)

Suppose that $(\rho_\hbar)_\hbar$ is pure and $M_W = \{\mu_0\}$.
(1) Then $\rho_\hbar(t) = e^{-i\frac{t}{\hbar}H}\rho_\hbar e^{i\frac{t}{\hbar}H}$, $\hbar \in (0,1)$ is also pure.

$$\lim_{\hbar\to\infty} \mathrm{Tr}[T(\sqrt{2}\pi\xi)\rho_\hbar(t)] = F^{-1}[\mu_t](\xi)$$

(2) By a flow $\exists\Phi_t : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$, it is represented as

$$\mu_t = \Phi_t \# \mu_0$$

and $\Phi_t(q_0, p_0) = (q_t, p_t)$ satisfies that $\left\{ \begin{array}{l} \dot{q}_t = 2p_t \\ \dot{p}_t = -\nabla V(q_t) \end{array} \right.$

Fock space

►$\mathscr{H}$: Hilbert space e.g. $\mathscr{H} = L^2(\mathbb{R}^d)$

►$\mathscr{F} = \oplus_{n=0}^{\infty}[\otimes_s^n \mathscr{H}]$

►$\Omega = 1 \oplus 0 \oplus 0 \oplus, \dots, \in \mathscr{F}$ Fock vacuum

►annihilation op. $a(f)$ and creation op. $a^\dagger(g)$ are defined by

$$a(f) : \otimes_s^n \mathscr{H} \to \otimes_s^{n-1} \mathscr{H}$$
$$a^\dagger(f) : \otimes_s^n \mathscr{H} \to \otimes_s^{n+1} \mathscr{H}$$

►CCR: $[a(f), a^\dagger(g)] = \hbar(\bar{f}, g)$

►$a^\sharp \sim \sqrt{\hbar}$

►$\{a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega\} \subset \mathscr{F}$ dense

2nd quantization

►2nd quantization of T:

$$\Gamma(T)a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega = a^\dagger(Tf_1)\cdots a^\dagger(Tf_n)\Omega$$

►$\Gamma(e^{itK}) = e^{itd\Gamma(K)}$, $t \in \mathbb{R}$
►$\mathrm{d}\Gamma(K)a^\dagger(f_1)\cdots a^\dagger(f_n)\Omega = \sum_{j=1}^n a^\dagger(f_1)\cdots a^\dagger(Kf_j)\cdots a^\dagger(f_n)\Omega$
►The number op. $N = \mathrm{d}\Gamma(\mathbb{1})$

$$N\Omega = 0, \quad Na^\dagger(f_1)\cdots a^\dagger(f_n)\Omega = na^\dagger(f_1)\cdots a^\dagger(f_n)\Omega$$

►$d\Gamma(K) \sim \hbar$

Coherent vectors

►Field operator :

$$\phi(f) = \frac{1}{\sqrt{2}}(a^\dagger(f) + a(\bar{f})) \sim \sqrt{\hbar}$$
$$\pi(f) = \frac{i}{\sqrt{2}}(a^\dagger(f) - a(\bar{f})) \sim \sqrt{\hbar}$$

►$W(f) = \exp(i\phi(f))$

►Weyl relation

$$W(f)W(g) = e^{-i\frac{\hbar}{2}\mathrm{Im}(f,g)}W(f+g)$$

►coherent vectors $W(\frac{f}{i\hbar})\Omega = e^{-\frac{i}{\hbar}\pi(f)}\Omega$

Weyl symbol

►Let $L^2(\mathbb{R}^d) = \mathscr{Z}$ for simplicity.

►$\mathbb{P}$ denotes the set of all finite rank orthogonal projection on $\mathscr{Z}$.

►For $p \in \mathbb{P}$ we identify $p\mathscr{Z}$ with $\mathbb{C}^d \cong T^*\mathbb{R}^d \cong \mathbb{R}^{2d}$.

►$f : \mathscr{Z} \to \mathbb{C}$ is cylindrical $\Longleftrightarrow \exists p \in \mathbb{P}$ and $\exists$ a function g on $p\mathscr{Z}$ st

$$f(z) = g(pz)$$

►$f \in \mathscr{S}_{cyl}(\mathscr{Z}) \Longleftrightarrow \exists p \in \mathbb{P}, \exists g \in \mathscr{S}(p\mathscr{Z})$ st $f(z) = g(pz)$.

►Let $b \in \mathscr{S}_{cyl}(\mathscr{Z})$.

$$b^{Weyl} = \int_{p\mathscr{Z}} F[b](z) W(\sqrt{2}\pi z) dz$$

Theorem (Ammari-Nier 2008 (Thm.6.2))

Let $(\rho_\hbar)_\hbar$ be a normal state on $\mathscr{F}$. Let N be the number operator. Suppose that $\exists\delta$ such that

$$\sup_{\hbar\in(0,1)} \mathrm{Tr}[\rho_\hbar N^\delta] \leq C_\delta$$

Then for any sequence $\{\hbar_n\}$ conversing to 0, there exists a subsequence $\{\hbar_{n_k}\}$ and a measure μ on $L^2(\mathbb{R}^d)$such that

$$\lim_{k\to\infty} \mathrm{Tr}[\rho_{\hbar_{n_k}} b^{Weyl}] = \int_{L^2(\mathbb{R}^d)} b(z)d\mu(z)$$

for any $b \in \cup_{p\in\mathbb{P}} F^{-1}(M_b(p\mathscr{Z}))$

Ex. $\rho_\hbar = |W(\frac{f}{i\hbar})\Omega\rangle\langle W(\frac{f}{i\hbar})\Omega|$ is pure and $M_W = \{\delta_f\}$.

Wick symbol

Let $p, q \in \mathbb{N}$. $\mathbb{P}_{p,q}(\mathscr{Z})$ denotes the set of (p,q)-homogenous polynomial functions on $L^2(\mathbb{R}^d)$ such that

$$b(z) = (\otimes^q z, \tilde{b} \otimes^p z)$$

with some $\tilde{b} \in B(\otimes_s^p \mathscr{Z}, \otimes_s^q \mathscr{Z})$. For $b \in \mathbb{P}_{p,q}(\mathscr{Z})$ we define $b^{Wick} : \mathscr{F}_{\mathrm{fin}} \to \mathscr{F}_{\mathrm{fin}}$ by

$$b^{Wick} \lceil_{\otimes_s^n \mathscr{Z}} = \begin{cases} 0 & n < p \\ \frac{\sqrt{(m+p)!(m+q)!}}{m!} \hbar^{\frac{p+q}{2}} (\tilde{b} \otimes_s \mathbb{1}_{\otimes_s^m L^2(\mathbb{R}^d)}) & n = m+p \end{cases}$$

b^{Wick} is formally written as

$$b^{Wick} = \int_{\mathbb{R}_y^{dq} \times \mathbb{R}_x^{dp}} \tilde{b}(y,x) a^\dagger(y_1) \cdots a^\dagger(y_q) a(x_1) \cdots a(x_p) dx dy.$$

Characterizaion of Wigner measures

Lemma (Ammari-Nier 2008)

Let $(\rho_\hbar)$ be the set of normal states st

$$\sum_{k=0}^{\infty} C^k Tr[N^k \rho_\hbar]/[k/2]! < K$$

uniformly in $\hbar \in (0,1)$. Suppose that $\lim_{\hbar\to 0} Tr[b^{Wick}\rho_\hbar] = \int_{\mathscr{Z}} b(z)d\mu(z)$ for any $b \in \oplus_{p,q}^{alg}\mathbb{P}_{p,q}^{\infty}(\mathscr{Z})$. Then M_W is pure.

$$M_W = \{\mu\} \Longleftrightarrow \lim_{\hbar\to\infty} \mathrm{Tr}[W(\sqrt{2}\pi\xi)\rho_\hbar] = F^{-1}[\mu](\xi)$$

►Fourier transform

$$F^{-1}[\mu](\xi) = \int_{\mathscr{Z}} e^{+2i\pi \mathrm{Re}(\xi,z)} d\mu(z)$$

Time evolution

- Hamiltonian

$$H = \underbrace{\hbar \int \omega(k) a^\dagger(k) a(k) dk}_{H_{\rm f}} + \sqrt{\hbar} \frac{g}{\sqrt{2}} (a^\dagger(\bar{f}) + a(f))$$

- Classical Hamiltonian

$$H(\bar{z}, z) = \int \omega(k) \bar{z}(k) z(k) dk + \frac{g}{\sqrt{2}} \int (\bar{z}(k) \bar{f}(k) + z(k) f(k)) dk$$

- $$i\dot{z}_t(k) = \frac{\partial H}{\partial \bar{z}} = \omega(k) z_t(k) + \frac{g}{\sqrt{2}} \bar{f}(k)$$

Hamilton equation and time evolutions

Theorem (Ammari-Nier 2011)

(1) Suppose that $(\rho_\hbar)_\hbar$ *is pure and* $M_W = \{\mu_0\}$*. Then*

$$\rho_\hbar(t) = e^{-i\frac{t}{\hbar}H}\rho_\hbar e^{i\frac{t}{\hbar}H} \quad \hbar \in (0,1)$$

is also pure:

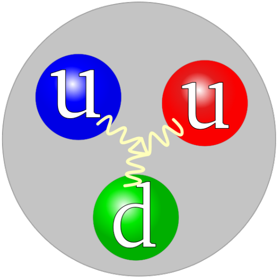
$$\lim_{\hbar\to\infty} \mathrm{Tr}[W(\sqrt{2}\pi\xi)\rho_\hbar(t)] = F^{-1}[\mu_t](\xi)$$

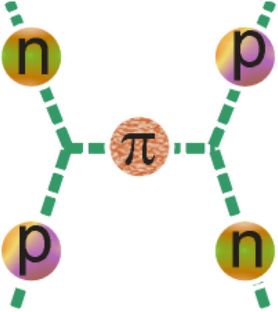
(2) By $\exists\Phi_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$*, it is represented as*

$$\mu_t = \Phi_t \# \mu_0$$

and $\Phi_t(z_0) = z_t$ *satisfies Hamilton equation:*

$$i\dot{z}_t(k) = \frac{\partial H}{\partial \bar{z}} = \omega(k)z_t(k) + \frac{g}{\sqrt{2}}\bar{f}(k)$$





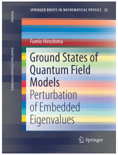
Yukawa's Model

It is known that in quantum field theory there exist 4 interactions: weak interaction, strong interaction, QED and gravity: We demonstrate the strong interaction. The strong interactions are the interaction between quarks due to gluons or the interaction between nucleons due to pions

Interaction model

Feynman-Kac formula $\rightarrow$ 

Ground state $\rightarrow$



▶Nelson model

$$H = (-\frac{\hbar^2}{2m}\Delta + V)\otimes 1\!\!1 + \sqrt{\hbar}\phi + \hbar 1\!\!1 \otimes H_{\mathrm{f}}.$$

is defined on $L^2(\mathbb{R}^d)\otimes\mathscr{F}(L^2(\mathbb{R}^d))$.

▶interaction term ϕ:

$$\phi = \frac{1}{\sqrt{2}}\int\left(a^\dagger(k)\frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}}e^{-ikx} + a(k)\frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}}e^{-ikx}\right)dk$$

▶free Hamiltonian $H_{\mathrm{f}} = \mathrm{d}\Gamma(\omega) = \int\omega(k)a^\dagger(k)a(k)dk$. Here ω is the

►Classical Hamiltonian is give by

$$H = H(q,p,z,\bar{z}) = \frac{p^2}{2m} + V(q) + \int \omega(k)|z(k)|^2 dk$$
$$+ \frac{1}{\sqrt{2}} \int \left(\frac{e^{-ikq}\hat{\varphi}(k)}{\sqrt{\omega(k)}} \bar{z}(k) + \frac{e^{ikq}\hat{\varphi}(-k)}{\sqrt{\omega(k)}} z(k) \right) dk$$

whee $(q,p,z) \in T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d)$

►Hamilton equation

$$(S-KG) \begin{cases} \dot{q}_t = \frac{\delta H}{\delta p_t} & = \frac{p_t}{m} \\ \dot{p}_t = -\frac{\delta H}{\delta q_t} & = -\nabla V(q_t) - \nabla W(q_t) \\ i\dot{z}_t(k) = \frac{\delta H}{\delta \bar{z}_t} & = \omega(k) u_t(k) + \frac{e^{-ikq_t}\hat{\varphi}(k)}{\sqrt{\omega(k)}} \end{cases}$$

►

$$\nabla W(X) = \frac{1}{\sqrt{2}} \int -ik \frac{e^{-ikq_t}\hat{\varphi}(k)}{\sqrt{\omega}} \bar{u}_t(k) + ik \frac{e^{ikq_t}\hat{\varphi}(-k)}{\sqrt{\omega}} u_t(k) dk$$

►Let

$$\rho_\hbar(t) = e^{-i\frac{t}{\hbar}H}\rho_\hbar e^{i\frac{t}{\hbar}H}$$

►In the same way as Schrodinger operator and QFT we can construct a Wigner measure μ_t on the phase space $T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d)$:

$$\lim_{\hbar\to\infty} \mathrm{Tr}[T(\sqrt{2}\pi X)\otimes W(\sqrt{2}\pi z)\rho_\hbar(t)] = F^{-1}[\mu_t](X,z)$$

for $(X,z) \in T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d)$
►the flow $\Phi_t : T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d) \to T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d)$ is defined by

$$\mu_t = \Phi_t \# \mu_0$$

►$\Phi_t(q,p,z) = (q_t,p_t,z_t)$ satisfies (S-KG).

Non-relativistic QED

▶Hamiltonian

$$H = \frac{1}{2}(-i\hbar\nabla \otimes \mathbb{1} - \sqrt{\hbar}A(x))^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes \hbar H_{\mathrm{f}}$$

▶Classical Hamiltonian

$$H(z,\bar{z},q,p) = \frac{1}{2}(p - A(q))^2 + V(q) + \sum_{j=1}^{d-1}\int \omega(k)|z_j(k)|^2 dk,$$

where

$$A(q) = \frac{1}{\sqrt{2}}\sum_{j=1}^{d-1}\int e_\mu(k,j)\bar{z}_j(k)e^{-ikq}\hat{\varphi}(k) + z_j(k)e^{ikq}\hat{\varphi}(-k)dk$$

▶phase space is $T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d;\mathbb{C}^{d-1})$

▶Hamilton equation:

$$(N-M)\begin{cases} \dot{q}_t = \frac{\partial H}{\partial p} = p_t - A(q_t) \\ \dot{p}_t = -\frac{\partial H}{\partial q}(p_t - A(q_t)) \cdot \nabla A(q_t) \\ i\dot{z}_j = \frac{\partial H}{\partial \bar{z}} = (p_t - A(q_t)) \cdot e(k,j)e^{-ikq_t}\hat{\varphi}(k) + \omega(k)z_j(t) \end{cases}$$

►Let

$$\rho_\hbar(t) = e^{-i\frac{t}{\hbar}H}\rho_\hbar e^{i\frac{t}{\hbar}H}$$

►We can construct a Wigner measure μ_t on the phase space $T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d;\mathbb{C}^{d-1})$

$$\lim_{\hbar\to\infty} \mathrm{Tr}[T(\sqrt{2}\pi X)\otimes W(\sqrt{2}\pi z)\rho_\hbar(t)] = F^{-1}[\mu_t](X,z)$$

for $(X,z) \in T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d;\mathbb{C}^{d-1})$
►the flow $\Phi_t : T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d;\mathbb{C}^{d-1}) \to T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d;\mathbb{C}^{d-1})$ is defined by

$$\mu_t = \Phi_t \# \mu_0$$

►$\Phi_t(q,p,z) = (q_t,p_t,z_t)$ satisfies (N-M).

Technical ingredients

►(Invariance) $e^{-itH}D(N) \subset D(N)$ for all t.

►(Self-adjointness) Let $V_+ \in L^1_{loc}(\mathbb{R}^d)$ and V_- be relatively form bounded wrt $-\Delta$ with a relative bound < 1. H is self-adjoint on $D(-\frac{1}{2}\Delta \dot{+} V_+ \dot{-} V_-) \cap D(H_{\rm f})$ (FH00, Hasler-Herbst11, Falconi15, Matte17)

Summary

► Hamiltonian $H = H_\hbar$

► Classical Hamiltonian $H = H(z, \bar{z}, q, p)$ for $(q, p, z) \in T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d; \mathbb{C}^{d-1})$

► Hamilton equation

$$(N-M)\begin{cases} \dot{q}_t = \frac{\partial H}{\partial p} \\ \dot{p}_t = -\frac{\partial H}{\partial q} \\ i\dot{z}_j = \frac{\partial H}{\partial \bar{z}_j} \end{cases}$$

► $\rho_\hbar(t) = e^{-i\frac{t}{\hbar}H}\rho_\hbar e^{i\frac{t}{\hbar}H}$

► $\lim_{\hbar\to\infty} \mathrm{Tr}[T(\sqrt{2}\pi X) \otimes W(\sqrt{2}\pi z)\rho_\hbar(t)] = F^{-1}[\mu_t](X, z)$

► $\Phi_t : T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d) \to T^*\mathbb{R}^d \oplus L^2(\mathbb{R}^d)$ is defined by

$$\mu_t = \Phi_t \# \mu_0$$

► $\Phi_t(q, p, z) = (q_t, p_t, z_t)$ satisfies (N-M).