Schrödinger Operators with random point interactions

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We consider the Schrödinger operator with point interactions in $\mathbb{R}^d$ ($d = 1, 2, 3$), defined as

$$\begin{aligned} &H_{\Gamma,\alpha} u = -\Delta|_{\mathbb{R}^d \setminus \Gamma} u \quad (u \in D(H_{\Gamma,\alpha})), \\ &D(H_{\Gamma,\alpha}) = \{u \in L^2(\mathbb{R}^d) \cap H^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma) \mid -\Delta|_{\mathbb{R}^d\setminus\Gamma} u \in L^2(\mathbb{R}^d), \\ &\qquad u \text{ satisfies } (BC)_\gamma \text{ for every } \gamma \in \Gamma\}, \end{aligned}$$

where Γ is a locally finite subset of $\mathbb{R}^d$, and $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma}$ is a sequence of real numbers. The boundary conditions $(BC)_\gamma$ are defined as follows:

$\boxed{d=1}$ $u(\gamma+) = u(\gamma-) = u(\gamma)$, $u'(\gamma+) - u'(\gamma-) = \alpha_\gamma u(\gamma)$.

$\boxed{d=2}$ $u(x) = u_{\gamma,0} \log|x-\gamma| + u_{\gamma,1} + o(1)$ as $x \to \gamma$, and $2\pi\alpha_\gamma u_{\gamma,0} + u_{\gamma,1} = 0$.

$\boxed{d=3}$ $u(x) = u_{\gamma,0}|x-\gamma|^{-1} + u_{\gamma,1} + o(1)$ as $x \to \gamma$, and $-4\pi\alpha_\gamma u_{\gamma,0} + u_{\gamma,1} = 0$.

It is well-known that $H_{\Gamma,\alpha}$ is a self-adjoint operator on $L^2(\mathbb{R}^d)$ under the condition

$$\inf_{\gamma,\gamma'\in\Gamma,\ \gamma\neq\gamma'} |\gamma - \gamma'| > 0,$$

for example, when $\Gamma = \mathbb{Z}^d$ (see e.g. [1]). The self-adjointness of $H_{\Gamma,\alpha}$ is proved under more general assumption; see e.g. Kostenko–Malamud [3] for $d = 1$, and Kaminaga–M–Nakano [2] for $d = 2, 3$.

There are numerous results about the spectral or scattering properties of $H_{\Gamma,\alpha}$, and most of the results up to 2004 are summarized in Albeverio et al. [1]. There are also many results in the case that the set Γ or the sequence α is random (some of them will be introduced in the lecture). However, there were no results when $d = 2, 3$ and Γ_ω is the Poisson configuration, up to 2019. In 2020, Kaminaga–M–Nakano [2] give the following result.

Assumption 1. *(i) Γ_ω is the Poisson configuration with intensity measure ρdx for some constant $\rho > 0$.*

(ii) The coefficients $\alpha_\omega = (\alpha_{\omega,\gamma})_{\gamma\in\Gamma_\omega}$ are real-valued i.i.d. random variables with common distribution measure ν on $\mathbb{R}$. Moreover, $(\alpha_{\omega,\gamma})_{\gamma\in\Gamma_\omega}$ are independent of Γ_ω.

Theorem 2 (Kaminaga–M–Nakano 2020)**.** *Let $d = 2, 3$, and suppose $(\Gamma_\omega, \alpha_\omega)$ satisfies Assumption 1. Put $H_\omega = H_{\Gamma_\omega,\alpha_\omega}$. Then,*

(i) H_ω is self-adjoint a.s.

(ii) $\sigma(H_\omega) = \mathbb{R}$ a.s.

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Next, we shall introduce the *integrated density of states (IDS)* $N(\lambda)$ for negative energies λ as follows:

$$N(\lambda) = \lim_{L\to\infty} \frac{N_{Q_L}(\lambda)}{L^d} \quad (\lambda < 0), \tag{1}$$

where $Q_L = (0, L)^d$ and

$$N_{Q_L}(\lambda) = \#\{\mu \le \lambda \mid \mu \text{ is e.v. of } H_{Q_L\cap\Gamma_\omega,\alpha_\omega|_{Q_L\cap\Gamma_\omega}}\}.$$

It can be proved that RHS of (1) exists a.s. and independent of ω, under Assumption 1.

In this talk, we shall give the asymptotics of $N(\lambda)$ as $\lambda \to -\infty$, as follows.

Theorem 3. *Let $d = 3$. Suppose $(\Gamma_\omega, \alpha_\omega)$ satisfies Assumption 1, and assume* $\operatorname{supp}\nu$ *is a bounded set in $\mathbb{R}$. Let t_0 be the unique positive solution of $t = e^{-t}$. Then, for sufficiently small $\epsilon > 0$, there exists a constant $\lambda_0 < 0$ dependent on ρ, ϵ and* $\operatorname{supp}\nu$ *such that*

$$\left(\frac{2\pi}{3}t_0^3\rho^2 - \epsilon\right)|\lambda|^{-3/2} \le N(\lambda) \le \left(\frac{2\pi}{3}t_0^3\rho^2 + \epsilon\right)|\lambda|^{-3/2}$$

for any $\lambda \le \lambda_0$. In other words,

$$\lim_{\lambda\to-\infty} \frac{N(\lambda)}{|\lambda|^{-3/2}} = \frac{2\pi}{3}t_0^3\rho^2. \tag{2}$$

Notice that the limit value (2) is independent of $\operatorname{supp}\nu$, though the value λ_0 depends on $\operatorname{supp}\nu$. The result (2) is completely different from the correspondent result for the Schrödinger operator with negative random *scalar* potential of the Poisson type

$$H_\omega = -\Delta + \sum_{\gamma\in\Gamma_\omega} V_0(x-\gamma),$$

where $V_0 \in C_0^\infty(\mathbb{R}^d)$, $V_0(x) \le 0$, and $V_0(0)$ is the non-degenerate global minimum of V_0. In this case, $N(\lambda)$ decays super exponentially as $\lambda \to -\infty$ (see e.g. Pastur–Figotin [4]).

References

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