La Sapienza seminar 2017/2/15 (Thu.)

$\mathcal{H}$ sep. Hilbert sp / $\mathbb{C}$

$\{A_1 \cdots A_d, B_1 \cdots B_d\}$ liner op s.t.

$[A_j, B_k] = -i \mathbb{1} \delta_{jk}$ on $D \subset \bigcap_{jk} D(A_j B_k) \cap D(B_k A_j)$

$(\mathcal{H}, D, \{A_1 \cdots, B_1 \cdots\})$ rep of CCR.

① $\dim \mathcal{H} = +\infty$ ② $[A, B] = -i \mathbb{1}$

$\Rightarrow$ one of A and B unbdd.

Question $\exists$ H s.a. $[H, T] = -i \mathbb{1}$

$(\mathcal{H}, D, \{H, T\})$ rep CCR. 1933. Pauli

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ultra strong t.o $\subset$ strong t.o. $\subset$ t.o. $\subset$ weak t.o $\subset$ ultra weak t.o

① us t.o. $e^{-isA_j} e^{-itB_k} = e^{ist} e^{-itB_k} e^{-isA_j}$

Weyl rel. $\{A_j, B_k\}$ satisfies Weyl rel.

$\{H, T\}$ Weyl $\Leftrightarrow$ T is ultra strong time.

Ex $\mathcal{H} = L^2$, $P_j = -i \frac{\partial}{dx_j}$, $Q_j = x_j \times$

$\{P_j, Q_k\}$ Weyl rel.

Weyl rel $\Rightarrow$ CCR / CCR $\not\Rightarrow$ Weyl rel.

[von Neumann $\{A, B\}$ Weyl irreducible

$A \cong P$, $B \cong Q$.]

$\{H, T\}$ Weyl $\Rightarrow$ $H \not\cong P$, $T \cong Q$

② Strong time op

$B e^{-itA}\varphi = e^{-itA}(B+t)\varphi$

$e^{-itA} D(B) \subset D(B) \quad \forall f \in D(B)$

$\{A, B\}$ weak Weyl rel.

$\{H, T\}$ 〃 T strong time op

lem. (Arai) (1) $\bar{T}$ is also s.t.o.

(2) $H > -\infty \Rightarrow \bar{T}$ is not s.a.

(3) $\sigma(H) = \sigma_{ac}(H)$

Ex 1. $H = \frac{1}{2m}p^2$ on $L^2(\mathbb{R}^1)$

$\rightarrow T_{AB} = \frac{m}{2}(Qp^{-1} + p^{-1}Q)$ Aharonov-Bohm op.

Ex 2. Scattering th. $H' = H + V \quad T.$

① $\exists W_\pm = \text{s-}\lim_{t \to \pm\infty} e^{itH'} J e^{-itH} P_{ac}(H).$

② $\lim_{t \to \pm\infty} \| J e^{-itH} P_{ac}(H)\varphi\| = \|P_{ac}(H)\varphi\|$

③ $\mathrm{Ran}(W_\pm) = P_{ac}(H')\mathcal{H}$

asymptotic completeness

$H'_{ac} = W_\pm H_{ac} W_\pm^{-1}$ on $P_{ac}(H')\mathcal{H}$

$\Rightarrow T' := W_\pm T W_\pm^{-1}$ strong time op of H'_{ac}

Ex 3. $[H, T] = -i\mathbb{1} \sim T = i\frac{d}{dH}$ $\left(\begin{array}{l} f \in C^2(\mathbb{R}\setminus K) \\ |\{\lambda \mid f'(\lambda) = 0\}| = 0 \end{array} \right.$

$[T, f(H)] = i f'(H)$

$T' = \frac{1}{2}\left(T f'(H)^{-1} + f'(H)^{-1} T\right)$ is

strong time op of $f(H)$ H. + Kuribayashi + Matsuzawa.

Weyl → w-Weyl → CCR

$\quad$ |

us top $\subset$ s.t.o $\subset$ t.o.

③ time op.

Lem (Arai, Galapon) $\sigma(H) = \{E_n\}_{n=1}^{\infty}$ $= \sigma_{disc}(H) =$

① E_n simple ② $E_n \to \infty$ ③ $\sum \frac{1}{E_n^2} < \infty$.

$\Rightarrow$ $\exists T$ s.t $[H, T]\varphi = -i\varphi$ $\exists \mathcal{E} \ni \varphi$

$$T\phi = i \sum_n \left(\sum_{m \neq n} \frac{(e_m, \phi)}{E_n - E_m} \right) e_n$$

$\mathcal{E} = LH\{e_n - e_m\}$. / $[H, T]e_n = -ie_n$

$(H - E_n)Te_n = -ie_n$

$e_n \notin D(T)$.

Cor $\sigma(H) = \{E_n\}$

① E_n simple ② $E_n \to 0$ ③ $\sum E_n^2 < \infty$ ④ $0 \notin \sigma_p(H)$.

$\Rightarrow$ $\exists T'$ s.t $[\frac{1}{H}, T']\varphi = -i\varphi$ on $\mathcal{E}$.

Thm (Arai + H) $\sigma(H) = \{E_n\} = \sigma_{disc}(H)$ $\sum E_n^2 \le \int |V(x)|^{\frac{d}{2}+2} < \infty$

(1) $E_n \to \infty$. $\Rightarrow$ $\exists T$ on $\mathcal{E}$ H.

(2) $E_n \to 0$, $0 \notin \sigma(H)$ $\Rightarrow$ T on $\mathcal{E}$ $\frac{1}{H}$

④ Ultra weak time op.

- T sym. $(H\phi, T\psi) - (T\phi, H\psi) = -i(\phi, \psi)$

$\mathcal{H}$ $\mathcal{H}$ $\Leftrightarrow$ T is weak time

- $t : D_1 \times D_2 \to \mathbb{C}$, $\mathcal{E}, D \subset D_1 \cap D_2$

① $\mathcal{E} \subset D(H) \cap D$

② $t[\varphi, \psi] = t[\psi, \varphi]^*$ ③ $t[H\phi, \psi] - t[H\psi, \phi] = -i[\phi, \psi]$ $\forall \phi, \psi \in \mathcal{E}$

$\Leftrightarrow$ T is ultra weak time

Idea

$\sigma(H) = \sigma_{dis}(H) = \{E_n\}$ $\quad 0 \notin \sigma_p(H)$, $E_n \to 0$

$H = (H^{-1})^{-1} \quad f(x) = x^{-1} \quad \therefore \quad H = f(H^{-1})$

$-\frac{1}{2}\left(T_{-1} H^{-2} + H^{-2} T_{-1}\right) = A$

$t[\varphi, \psi] = -\frac{1}{2}\left\{ (H^{-2}\varphi, T_{-1}\psi) + (T_{-1}\psi, H^{-2}\varphi) \right\}$

Thm (Arai + H) $\sigma(H) = \sigma_{dis}(H)$

$\exists$ ultra mech time t on $H^{-1}\mathcal{E}$.

⑤ Main theorem.

Thm (AH) (1) $\sigma_{sc}(H) \neq \emptyset$ (2) $\sigma_{ac}(H) = [0, \infty)$

(3) $\sigma_p(H) = \sigma_{disc}(H) = \{E_n\}$ $E_n \to 0$ $0 \notin \sigma_p(H)$

(4) $\exists T_{ac}$ (time op) of H_{ac} exists.

$\Rightarrow \exists t$ ultra mech time of H

$\because$ $\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_p$

$H = \underline{H_{ac}} \oplus \underline{H_p}$

$t[\phi, \psi] = (\phi, T_{ac}\psi_1) + t_p[\phi_2, \psi_2]$. ⌟

<u>Example</u> $H = -\frac{1}{2}\Delta + V$ $(d=3)$ where

$V(x) = \dfrac{U(x)}{(1+|x|^2)^{\frac{1}{2}+\varepsilon}}$ (1) $U(x) < 0$ cont spherically sym.

Agmon potential (2) $U(x) \sim -\frac{1}{|x|^{\alpha}}$.

(3) $2\varepsilon + \alpha < 1$

Example $H = -\frac{1}{2}\Delta - \frac{\gamma}{|x|}$ $d=3$. $\left\{-\frac{\gamma^2}{2n^2} \,\middle|\, n\right\}$

Example $\sin(2\pi\beta H)$ $\quad \beta \notin \left\{ \frac{k}{2E_n} \,\middle|\, \begin{matrix} k \in \mathbb{Z} \\ n \in \mathbb{N} \end{matrix} \right\}$

Summary

ultra st C   st C   true op C   weak C   ultra w
W   w-W   CCR   w-CCR   sesqui